Oral Presentation

HISTORICAL EPISTEMOLOGY: PROFESSIONAL KNOWLEDGE AND PROTO-MATHEMATICS IN EARLY CIVILISATIONS

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This paper has been prompted by the work of Hoyrup (2004) and Netz (2002) that we should be cautious of the conceptual categories we use to investigate the past. Lloyd (1990) showed that a 'mentality' ascribed to a cultural group was untenable, and both Hoyrup and Netz discuss the action of using a tool as extended from physical to cognitive and theoretical objects. Regarding historical epistemology as a process of investigating the dynamics of proto-scientific activity led to considering the Vygotskian (1978) concept of tools-in-action leading to mental functions and Gibson's (1977) Theory of Affordances to consider early people as active agents embedded in a multi-layered socio-ecological environment. Some brief notes on pre-classical civilisations indicate particular points of focus.

INTRODUCTION

Pre-Classical Contemporary Historiography

Studies in pre-classical mathematics were often regarded as 'primitive' efforts toward the more sophisticated and elegant mathematics of Greece, rather than valid explorations of ideas within specific socio-cultural contexts. However, from 1959, a series of papers by Abraham Seidenberg on the ritual origins of mathematics appeared (1959, 1962a, 1962b,) which, while dismissed by some researchers as largely fanciful and unreliable, contained references to aspects of the so-called primitive world that were based largely upon cultural and religious studies, using translations of Sanskrit and other ancient texts,[i] these being the only 'non-mathematical' evidence available at the time. More recently, the critiques led by Unguru (1975) Hoyrup (1994) and Netz (1999) and followed by more consolidated accounts of Babylonian, Indian, and Egyptian mathematics (Robson, 2008; Plofker, 2007; Imhausen, 2003a) show that some of Seidenberg's opinions may have been justified, and contemporary historiography now addresses wider contexts through extended forensic interpretation of ancient texts and artefacts, using techniques from areas such as archaeology, social anthropology and linguistics. Combined with sensitive and broad knowledge of the development of past mathematics the mode of historiographic research has changed considerably in the last twenty years, offering us deeper insights into the pre-classical past.

Historical Epistemology

Epistemology makes a distinction between methods of discovery and methods of justification: that is, the way one discovers a property of a situation or a relation between objects and how a conjecture becomes a *mathematical* truth may be quite different from how it is later justified, accepted by a community, and established as a truth. We address questions like, 'what are the methods and grounds for such discoveries;' 'what is the role insight plays in these discoveries;' and 'how do interconnections between mathematical concepts lead to discoveries'? Because there are no meta-mathematical discussions in the texts made by the actors about what they are doing and why, one answer is to regard historical epistemology as investigating the *dynamics* of proto-scientific developments, insofar as they can be extracted from an analysis of texts and practices. Any analysis of the development of mathematical ideas necessarily calls for a serious approach to the social and cultural contexts and the physical environments in which the knowledge in question was generated, and should attempt to answer questions about the motivations and means (practical and theoretical) available to the agents involved. Hitherto, researchers have been looking at *mathematical*[ii] texts; namely those considered to have been written to teach or learn mathematics, but they were looking at a group of texts *already categorized in a particular way* – because they 'look like' contemporary mathematics (or parts of it) they were identified as somehow the 'same' (e.g. as equations), but neglected texts that speak *about* mathematics, or show practices being developed in some way might suggests a 'mathematical – like' activity.

Hoyrup, (1994) suggests that *sub-scientific* knowledge[iii] arises out of cultural practices, rather than taking 'already-known' knowledge being 'applied' to a problem, this knowledge can be found in craft skills like making tools, in rituals, emergent astronomy, astrology, or knowledge arising in the making of, or building significant objects by a particular group or individual.

Mathematics has not always been the 'same' because, at different periods, different kinds of mathematics were possible. The contexts, tools, and motivations, were different, and many of the problems were concerned with the immediate needs; ritual, social, and economic of the people involved. Much of this happened before there was any need for recording, since societies developed ways of handing on ideas in myths, storytelling, poetic algorithms, and practices not recognised as 'scientific' by former historians.

PRE-CLASSICAL CULTURAL STUDIES

Egypt

The work of Annette Imhausen (2007, 2003a,b) shows that original mathematical texts on papyrus, leather, or other materials are extremely rare[iv] and so it is very difficult to make an overall assessment of the Egyptian mathematical corpus. Earlier accounts of Egyptian mathematics (Gillings, 1971; Clagget, 1999) contain similar subject matter presented in contemporary mathematical terms, and consequently their commentaries have been limited by their preconceptions.[v] Apart from the more substantial Rhind and Moscow Papyrus and the Leather Roll, and some fragments of mathematical problems, there is scarce little else. However, other Egyptian texts contain a variety of everyday contexts, and their problems of calculating rations, granary volumes, the daily baking, brewing and herding activities are recorded on a variety of documents, not all obviously mathematical.[vi] Difficulties lie in the technical terms that give clues to different kinds of concepts, so that establishing the social contexts of the texts involved is problematic. (2003b: 371-373). There are administration texts that show the application of techniques, and tomb representations of everyday tasks where the actual procedure is ambiguous, but knowing the administrative, economic, and practical contexts helps to understand the problems that draw on the professional lives of scribes and use their terminology, and techniques. Imhausen shows how difficult it is to translate everyday colloquial Egyptian (2003b: 374) where not all processes describe the same steps, and although these steps may be followed, the sense of the problem is obscure because we lack information about how individual objects are actually made and we do not fully understand Egyptian scribes' own conceptions of their mathematical world. It is difficult to find out what parts of a procedure were 'sub-scientific' or 'proto-mathematical' since the true social context is not clearly known. What is needed is more information about the role of mathematics in Egyptian culture, and the satirical text *Papyrus Anastasi* (Imhausen 2007: 10-11) while not a *mathematical* text, is discussed in detail, showing how important mathematics is for Egyptian scribes[vii]. The mathematical problems found in *Anastasi* do not contain all the data needed to solve the exercises, but if they come from a well-known body of scribal mathematical tasks, the account of the type of problem was enough for an Egyptian reader to know the relevant group of problems.

There are some 14 diagrams in the ancient Egyptian *mathematical* corpus, but as yet, there are no systematic instructions for constructing them. We do not know their representational conventions, nor how those conventions relate to other aspects of Egyptian culture. There are tomb paintings representing a ritual from the Book of the Dead (Taylor, 2010) showing the use of a simple beam balance for weighing the heart of the dead person against the 'feather of truth'. If the balance reaches equilibrium, the dead person can enter Paradise. Clearly, the balance has to be accurate, and the technique employed in this ritual has to be managed by the priests, so that the outcome is beneficial (Seidenberg & Casey, 1980). On the other hand, it is apparent from a few problems (Rhind 24-29) that proportional reasoning was used in the arithmetic, and while much of the text is written as arithmetic, the conceptual background may well have been mechanical or geometrical. Having realized that other sources are relevant to a fuller understanding of the mathematics, it will take some time to collate the evidence and relevance of wider aspects of ancient Egyptian life.

Mesopotamia

This was an area where early translations regarded many of the mathematical problems to be 'the same'; or at least similar enough for researchers like Thureau-Dangin, Bruins and Neugebauer[viii] to recognise elements in the texts that enabled them to translate the problems into contemporary mathematical terms. Since then, excavations in Mesopotamia have been more extensive in chronological period and cultural areas, and new methods used to investigate not only the *mathematical* texts, but much more extensive research into the social and cultural contexts of the people living in that area. These results are now found in Nissen, Damerow and Englund (1993), Hoyrup (1994, 2010), and in Robson (2008) so that we now have extensive and detailed descriptions of different sources supporting the development and use of mathematical concepts and procedures, as well as much deeper knowledge of the languages employed.

India

The Vedic people who entered North West India were responsible for the earliest extant texts known as the Vedas, the oldest scriptures of Hinduism that became a recognised corpus of Sanskrit literature before the middle of the first millennium BCE. These texts contain hymns, formulas, and spells for rituals, and part of these were the *Śulba Sūtras* that provide the 'cord-rules' for the construction of sacrificial fire altars (Seidenberg 1962a). They give the instructions for building brick altars used in ritual sacrifice. Most mathematical problems considered in the *Śulba Sūtras* spring from a single ritual requirement; namely that of constructing altars that have different shapes but occupy the same area. (Plofker 2009: 13-28). How this ritual geometry became integrated into the process of sacrificial offerings is unknown, the rules may have emerged through trying to represent cosmic entities physically and spatially, or perhaps existing geometric knowledge was incorporated into ritual to symbolise some universal truth about spatial relationships. Whatever their deep motivations, the basic tools were simply 'peg and cord' to make arcs of a circle. The origins of this geometry can be seen in the use of the shadow-stick gnomon to set the East-West equinoxes and record the daily passage of the sun (Keller, O. 2006: 28-41). Many of the instructions contain transformation rules for preserving areas, such as changing a rectangle into a square of the same size, and one of the elementary perceptions involved are the proportional properties of the right angled triangle and the division of a rectangle by a diagonal. (Keller, 2006: 125-166) (Plofker, 2009:13-42) From the instructions in the texts, the following diagrams (Figs. 1a and 1b) show that removing the small red square from the large blue square produces the equal areas of the blue rectangle and the green square.

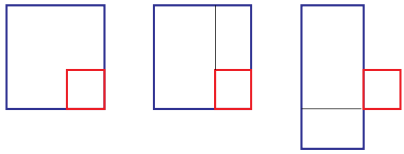

Fig 1a

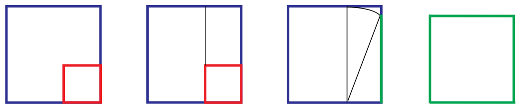

Fig 1b

The ritual is preserved in a film made by Frits Staal, now available on the internet[ix].

China

Divine origins of arithmetic and geometry are repeatedly stressed in the earliest records we have of Chinese history. The earliest Canonical[x] account of the past is the creation of a socio-political order by a human ruler in the *Shu Jing*, *The Book of Documents* (c. 500 BCE) traditionally compiled by Confucius from earlier sources where the legendary Emperor Yao commissioned two star-clerks Xi and He, to "accord reverently with august Heaven and its successive phenomena, with the sun, and the moon, and the stellar markers, and thus respectfully to bestow the seasons upon the people." (Cullen 1996: 3) The times of the summer and winter solstices and the spring and autumn equinoxes were recorded, eventually establishing a solar calendar of about 366 ¼ days. Chinese mathematical astronomy appeared as a functioning system in the Han Dynasty (c. 207BCE - 220CE) where the use of a carpenter's square, a compass, and other astronomical tools were already well practiced. Chinese astronomers had to deal with a luni-solar calendar, having to work with months that followed lunations quite closely, as well as keeping a civil year of a whole number of months in step with the seasons, and months in step with a cycle of a 'week' of 10 days (Cullen 1996: 7-26). The calendar had an important *Ritual function* because the Chinese emperor was responsible for looking after the people, as well as the world order, and disorder in nature was a sign of a malfunction of the human order thereby causing criticism of the emperor's rule. The astronomer's task was to reduce as many phenomena as possible to rules and thus to predictability, so that state rituals should be carried out at the proper times, and if mistimed or not performed correctly, harm could come to the population. It was therefore expected that the motions of the visible planets should be tabulated in detail, and that lunar eclipses, and other phenomena should be predicted. Irregular events could be ominous; comets, meteor showers, and novae could predict disaster. Almanacs were published to detail these events, and virtually all activities had to be considered in terms of the calendar. As yet, there is very little in translation about mathematical techniques in the centuries before the Han, but the use of the *Gnomon* and plumb line as essential tools must have appeared well before the written data, and the 'out-in' principle in geometry appeared as a very powerful method of using the idea of equivalence of areas.[xi] The diagram (Fig. 2) where the upper and lower areas are equal is visually 'obvious' was well established in the *Zhou bi suan ji*ng in the Zhou dynasty (1046 - 256 BCE) (Cullen, 1996).

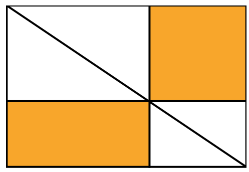

Fig. 2.

Earlier practical examples of these ideas have so far not been found, yet in the earliest known bamboo text *Suan shu shu – A Book on numbers and computation* put together in the Qin dynasty, (221-186 BCE) which is a collection of problems showing examples of the use of fractions, distribution of goods using proportion, and problems on areas of fields, shapes and volumes. (Cullen, 2004). Applications and developments of these algorithms are found in more detail later in the *Nine Chapters* (Chemla & Shuchun 2004) where, in particular, we find the dissection of a cube into three square pyramids (388-406, Diagram 5.15). Fig 3

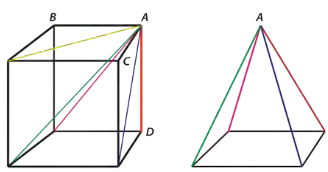

Fig 3

We have learnt to be much more cautious in our interpretations of ancient evidence, but also realise that many of the ideas found in the well-documented activities in Mesopotamia, India, Egypt and China indicate that at the most basic level; namely ideas of ratio and proportion, dissection and rearrangement, and area conservation, that appeared as professional knowledge of the scribe, priest, or shaman, be it 'sub-scientific' or 'proto-mathematical', in some of the earliest activities in human development.[xii]

INVESTIGATING MATHEMATICS PAST AND PRESENT

Jens Hoyrup: Canons and Taboos (2004)

Hoyrup (2004) addresses a perennial problem: were historical concepts really different, and the historical actors unable to think or express themselves in our terms, or is everything just a question of terminology and notation? Citing the example of

Unguru's (1975) paper on rewriting Greek mathematics showed that historians of mathematics, being mathematicians, tacitly assumed that mathematical entities were Platonic, sharing the same ideal forms, having no connection with the thought of the individual or the historico-cultural context into which the historical writer's ideas were being broadcast. Hoyrup points out that this kind of debate is unduly simplistic, and that more careful reading of early sources indicates that early mathematical writers might have other reasons than inadequate conceptual capacity or unhelpful terminology to be able to express themselves in ways different from our own.

By deconstructing the idea of a "mode of thought" as intangible[xiii] that "does not in itself assist us in understanding whether, why or in which respect this mathematics differed from ours". Hoyrup suggests that talking about the *mathematical concepts* of a culture is less elusive, but we should not identify a concept with the words that are used to describe it. He describes a mathematical concept as a *mental tool* that is being used for specific operations, together with the connected network of concepts and their properties, characterises a particular 'mode of thought' (Hoyrup 2004: 131)

Hoyrup's metaphor is very similar to the ideas of Vygotski who proposed that just as physical tools extend our physical abilities, mental tools extend our mental abilities, enabling us more able to solve problems. Before we learn to use mental tools, learning is largely controlled by the environment; being a matter of reaction to various stimuli. Once we learn to master mental tools, we become learners, who by attending and remembering in an intentional and purposeful way, can transform cognitive behaviors, and also use other mental tools to transform our physical, social, and emotional behaviors. These mental tools can also transform our minds, leading to the emergence of *higher mental functions*. This idea first appeared in Vygotski's *Thought and Language* (1962) and was further developed in his *Mind in Society* (1978). Mental functions are cognitive processes acquired through learning and teaching within a system of practices common to a specific culture. Hoyrup points out that structures of mathematical operations emerge from operations with physical tools or particular cultural practices: (for example, bamboo sticks, or tokens on a counting board, using a dust abacus, or practicing routines for accounting, or solving equations). These practices are never identical with the abstracted mathematical structure because *the mathematical structure is essentially abstract.*

> "it cannot be excluded that mathematical conceptual structures that are fairly congruent with something *we* know grow out of manipulations of tools that are quite different from those from which we are now accustomed to see them evolve. Identifying underlying tools that differ from ours does not prove that the corresponding concepts were also fundamentally different." (Hoyrup 2004: 133)

After offering a variety of examples from Egyptian and Old Babylonian culture, he concludes,

> … that much of what the texts do not say or do not do must be explained, not from what their authors *could not think* but instead in terms either of *what they did not*

> *find it professionally fitting to say, or what they found it incoherent to say.* (Hoyrup 2004: 142)

Practices can be built up by a powerful economic or political clique and defended to exclude others, or to preserve a practice in the face of modification or opposition. Thus the role and social status of the scribe in ancient Egypt, as evidenced by the Egyptian *Papyrus Anastasi,* or the concentration of the Chinese astronomers on prediction of events to defend their status, or the priests maintaining the rituals of the *Sulbautras,* all provide examples where certain practices may be built up and maintained for reasons other than immediately useful or 'scientific', so that

> … the absence of such conceptualizations from ancient sources as a modern mathematical reader might expect to find there does not prove that the ancient authors *were not able* to think more or less in our patterns – it may also be due to an explicit rejection of this way of thinking, either because of the existence of some canon or because they deemed it conceptually incoherent. Only close analysis of the sources at large will, in the best of cases, allow us to distinguish between cognitive divergence and cognitive proscription." (Hoyrup 2004:144-45)

Reviel Netz: It's not that they couldn't.

In this paper, Reviel Netz (2002) takes up the discussion by demonstrating that transforming an old piece of mathematics into its *contemporary equivalent* is misleading, because it conceals the idiosyncratic features of the old mathematics that prevented it from becoming contemporary mathematics. He insists that it is not just a matter of notation.

It is difficult to see what is meant by 'similar' or 'equivalent' here.

> "The standard example - the equivalence of Euclid's *Elements* II with algebraic equations - seems to suggest a meaning of 'equivalence' along the following lines: historians of mathematics often take two theorems to be equivalent when, from the perspective of the modern mathematician, the proof of any of the theorems serves to show, simultaneously, the truth of the other." (Netz 2002: 264 footnote)

In my opinion, this means at the very least, being able to make the mental transformation from one medium to another, and 'seeing the algebra in the geometry', which comes from ignoring the true socio-cultural context of the work.

Netz (2002: 265) also cites Unguru's (1979) response to the opposition led by van der Waerden (1976) involved wide-ranging historiographical and philosophical comments that conclusively settled the argument. At the heart of Unguru's reply lies his critique of Jacob Klein's (1968) *Greek Mathematical Thought and the Origins of Algebra,* where Unguru claimed that Greek mathematics could not be interpreted to be the same as modern mathematics, because the Greeks did not possess the right kind of concepts: for algebra, *one needs 'second-order' concepts that refer to other concepts, but the Greeks had only first-order concepts, referring directly to reality.* (Italics, mine)[xiv] Thus, modern mathematics, in the Greek context, was *conceptually impossible*. In Klein's work, concepts constitute the 'mode of thought' of a group of individuals, without any historical account of why their mental possibilities should be limited in a particular way. So the historiography of conceptual structures is no more than a version of the history of mentalities. The idea that Greek science could be explained as expressing an abstract mentality was attacked by Lloyd (1990), by showing the contradictions and inconsistencies when particular mentalities are assigned to an individual, a group, and even to a whole nation.

Netz draws on several studies of Greek mathematics where exceptions to the rules appear, claiming differences in mathematical *practice*[xv]. While examples may be rare, they exist, and taken together, they show the inadequacy of the argument from conceptual impossibility. Furthermore, Netz points out that there are extra-mathematical concerns influencing the definitions of unit and number (Euclid, VII. 1-2). Definitions interact with the intellectual world where they serve philosophical goals; so Netz claims that readers of *Elements I* would feel that a discussion of what'points' were, was philosophically necessary, so Euclid put the definition, 'a point is that which has no part', at the beginning of his work. Furthermore when, in the *Sand Reckoner* Archimedes introduced a new numerical system for a non-mathematical audience, it was necessary to name, in a natural language sense, extremely large numbers, Netz (2002: 277)[xvi].

Netz refers to Hoyrup's use of concepts as tools and maintains (2002: 282) that it is clear that as people produce artifacts, and have recourse to several tools that are culturally available, so the *action of using the tool* can be extended from material objects, to cognitive and indeed theoretical objects.[xvii] The possibilities opened up by a tool, whether physical or cognitive are considerable, and unpredictable. The unsuspected possibilities of applying a tool to a task set up an interaction between the tool and the task itself, with often unpredictable results. Many values influence any particular activity, and depending on the different value, different practices might arise. For the sake of efficient calculation, typographic representations of numbers are preferable; for the sake of proximity to natural language, verbal representations are preferable. Practices are determined not by the *totality* of values brought to bear, but by the *most important* of such values: the value of efficient calculation was important to Archimedes, but in the context of a literary treatise, it has an even more important value for a non-mathematical audience, that of proximity to natural language. So Netz puts forward the following explanation of the non-arithmetical nature of Greek mathematics:

> Greek literary production is marked by a hierarchy of values always related to a certain 'literary' or 'verbal' preference: literature is ranked above science, inside science philosophy is ranked above mathematics; persuasion (to the Greeks, the central verbal art) is ranked above precision, and natural language above other symbolic domains. Hence it is easy to understand Euclid's deference to philosophy in his definition of number. More significant, inside Greek mathematical writings, the qualitative statements of geometrical demonstration become the norm against

> which arithmetical representations of the same object come to be seen as marked. (Netz 2002: 287)

Thus reinforcing Hoyrup's point about other external influences from practices built up by tradition or a powerful economic or political clique that are defended to exclude others, or to preserve a practice in the face of modification or opposition.

ASPECTS OF HUMAN COGNITION

Gibson's Theory of Affordances.

Gibson's principal idea is that cognition is not isolated from all the other attributes that may influence a learning agent at a particular time, place, or context. It is a way of looking at cognition that considers *an active agent embedded in a multi-layered socio-ecological environment.* Gibson first applied his theory to psychology as 'A way to understand how learning takes place through perception of, and interaction with, an environment' (Gibson, 1977), where he conceived the individual actor (animal or human) in a general ecological environment, and considered the options available in terms of possible awarenesses, perceptions and their consequent actions.

The *Affordances* of an environment are what it *offers* the agent, what it *provides* or *furnishes;* the consequences of which could be good or bad. As an affordance of support for a species of animal, they have to be measured *relative to the animal*. They are unique for that animal or agent, and not just abstract physical properties. Affordances have a unique unity relative to the nature, physical attributes, and behavior of the agent being considered. Affordances are "action possibilities" latent in the environment, objectively measurable, and independent of the individual's ability to recognize them, but always *in relation to the actor* and therefore dependent on their capabilities. *Constraints* may be actual, physical, environmental, or perceptual, depending on the context and the abilities of the agent. Knowledge emerges through the *primary agent's bodily engagement with the environment*, rather than being simply determined by and dependent upon either pre-existent situations or personal construals. Greeno (1994) took up Gibson's agent-situation interactions in ecological psychology in his 'situated cognition' research because its holistic approach rejected the assumptions of individual 'factors' in current psychology. This perspective focused on 'perception-action' instead of memory and retrieval. A perceiving-acting agent is coupled with a developing-adapting environment and what matters is how the two interact. Greeno also suggested that affordances are "preconditions for activity," and that while they do not determine behavior, they increase the likelihood that a certain action or behavior will occur. These ideas continue to be developed in an active school of ecological psychology, where Harry Heft (2003) writes,

> At a basic, pre-reflective level of awareness, prior to the abstractions (e.g. categorization, analysis) all humans so readily perform on immediate experience, we perceive our everyday environment as a place of functionally meaningful objects

> and events. In their immediacy, the "things" of our everyday environment have perceivable psychological value for us in terms of the possibilities they offer for our actions and, more broadly, for our intentions. This aboriginal mode of awareness runs through the flow of our ongoing perceiving and acting, constituting its experiential bedrock. ……. Perceiving the affordances of our environment is, if you will, a first-order experience that is manifested in the flow of our ongoing perceiving and acting. By *first-order experience* I mean experience that is direct and unmediated; it is the experiencing of *x,* in contrast to experiencing *x* through the intercession of *y* or *z*." (Heft, 2003: 151)

In this sense it is the *intuitive and unmediated experience* that gave rise to the Vygotskian perceptions of the emergence of activities and use of physical tools leading to the development of mental functions. Gibson is clear that the environment offers affordances and constraints (physical, social, or mental) that may act upon or be acted upon by the agent, thus illustrating the variety of possible outcomes. From what we now know of ancient cultures, we can recognise individuals in a *milieu* of affordances and constraints available in their environments, ecological, social, and cognitive, acting upon individuals and groups that resulted in the artefacts, products, and writings that have been (and continue to be) discovered, analysed and debated.

Visualisation and Diagrammatic Reasoning

The brief survey by Hanna and Sidoli (2007) shows how recent interest in visualisation has grown in both mathematics, philosophy of mathematics, and mathematics education, and while they refer to Mancosu's chapter on visualisation in (2005: 13-30) which is valuable itself as an investigation into relatively recent mathematics, Mancosu there considers the *re-emergence* in modern terms of what I regard as an *ancient cultural-historical human ability*. After the denigration of visual evidence in mathematical proof in the nineteenth century, he sees the recent interest in visualisation as a change in mathematical style (2005: 17). On the other hand Giaquinto (2007: 35-49) addresses what he sees as the acquisition of "… basic geometrical knowledge …not acquired by inference from something already known or some external authority …" and furthermore, in (262-263) he describes different aspects of visualized motion that contribute to our ability to act upon and transform our mental images.

In the contexts explored above we see that archaeologically recovered materials from Egypt and Mesopotamia, India and China provide some of the earliest written sources of astronomy and mathematics known to us today. By the middle of the first millennium BCE the cultures discussed here had reached a high level of socio-economic organization and technical expertise, developed from the use of simple tools for plotting objects in the sky or measuring the ground for both practical and ritual purposes. The use of simple practical tools inspiring many developments were motivated by a variety of purposes that depended on the affordances that the agents perceived in their environments. The manipulation of constructed objects like manufactured bricks or wooden frameworks was transferred to local media (sand table, clay, papyrus, painted surfaces) that may be used for demonstrating a particular practice or the transmission of technical knowledge, as a visual record to be passed on to others.

In all of these representations it is inevitable that some tacitness remains and unarticulated aspects are 'taken-for-granted' by the actors. The observation of and contemplation upon the dynamic effects of manipulation of the 'object-image' affords the possibility of using these properties in a different context as a new tool to solve what may be a totally unrelated problem. Access and conviction grew from 'hands-on' practical activities, manipulation of actual objects and their transformations by developing the use of representations of these objects, mentally, and in terms of locally available media as iconic likenesses so that operations on these representations led to the 'dissection and re-arrangement' of indexical (Peircean) fluidity of the icon. But while it seems clear that at some stage the contextual affordances gave rise to the comparison and reinforcement of intuitive properties of, for example, right-angled triangles and the emergence of proportional relations, some important questions remain.

If, as I have stated at the beginning of this paper we regard historical epistemology as investigating the *dynamics* of professional knowledge and proto-scientific developments, insofar as they can be extracted from an analysis of texts and practices, to what extent can I justify the transfer of practical knowledge to theoretical knowledge and higher mental functions as evidenced by the texts we already have?
Both Hoyrup and Netz allow the development of a *functional dynamic* of mental operations with uncertainties about the outcome: Hoyrup considers the "metaphor that a mathematical concept is *a tool*: a mental tool, but a tool only by being a tool for operations. The shared properties and conditions of the whole network of connected mathematical concepts with participating operations then characterize the corresponding mode of thought." (Hoyrup 2004:131)

And Netz agrees that outcomes cannot be pre-determined: "There is always a grey area of what a tool can do, depending on which task you put it to: grey area which is not fully determined by the tool itself-so that a dialectic of tools and tasks ensues." (Netz 2002: 282-283)

Mental tools contain knowledge representation structures that allow for drawing inferences from prior experiences about complex objects and processes even when only incomplete information on them is available, and so the epistemic function of visualisation in mathematics can go beyond the merely heuristic one and become a means of discovery of new ideas - and even become belief-forming dispositions. (Giaquinto, 2007: 35-49). Allied to the affordances, we can recruit a conception of an emergent *community of learning* which emphasizes various processes of socialization, involving communities and their values with not only the acquisition of skills and participation in activities, but a third stage where individual and collective learning goes beyond mere information given, and advances knowledge and understanding by a collaborative, systematic development of common objects of activity into shared *knowledge-creation*. (Sami and Hakkarainen 2005)[xviii]

Why should we restrict the creativity of our ancestors with an attitude restricting the possibilities of what was available to them? Is it not better to allow that "we come up with an account where mathematics is not always the same, while people are: which forms, I believe, the historian's intuition." (Netz 2002: 288) After all, I could remark that the field is open, and "absence of evidence is not evidence of absence." (Sagan 1995: 221).

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NOTES

i "text" here applies (as in linguistics) to inscriptions of any kind, on any kind of object.

ii The word *mathematical* here may be pre-emptive, determining the 'mathematical' context in advance

iii See Hoyrup 1994: 25. "… specialists' knowledge that (at least as a corpus) is acquired and transmitted *in view of its applicability*. Even sub-scientific knowledge is thus knowledge beyond the level of common understanding…"

iv It appears that no new mathematical sources have been discovered in the last 70 years.

v Clagett (1999) for example, sees arithmetic and geometric progressions, and geometric problems, while Gillings (1972) sees equations of the first and second degree.

vi An overview of the individual problems and their classification can be found in the introduction of Imhausen 2003.

vii The well-known *Satire of the Trades* compares the position of the scribe to other professions where a scribe has pleasant work and a higher place in society, referring to mathematics and tax collection as part of the scribal profession (Lichtheim 1976). See UCL website http://www.ucl.ac.uk/museums-static/digitalegypt/literature/satiretransl.html

viii Much of this work was not easily accessible. For details see Robson 2008 Chapter 1. 1-26.

ix Seidenburg & Casey (1980: 336 footnote 53) states that Frits Staal made the film of the *Atiratra Agnicayana* ritual, now available at https://www.youtube.com/watch?v=UnbqnMhbB44 x By *Canonical*, is meant the officially accepted 'traditional' account.

xi The 'in-out' principle is also applied in the Nine Chapters (Chemla & Shuchun 2004: 661-693).

xii Of course, whether or not there was any transfer of ideas between these cultures remains an open question.

xiii There are deeply embedded influences from de Tocqueville, Levy-Bruhl and others that attribute styles of thinking as a characteristic of different social or national groups.

xiv There is a problem with the labeling of and attributing 'second order concepts' to writers in the past. How do we know what 'first order concepts' they had? These categories are all of our own making.

xv See Euclid I,4 for 'superposition' as part of the proof that triangles are congruent which is used again in I, 8 and III, 24 but rarely found elsewhere. 'Superposition' may be intuitively obvious, implies a physical action on an ideal object.

xvi In Vedic literature there are names for *each* of the powers of 10 up to 10^{62}. In the Buddhist tradition, the *Lalitavistara Sutra* recounts a competition between the mathematician Arjuna and the Buddha for naming very large numbers. Today, the words *lakh* and *crore* referring to 100,000 and 10,000,000, respectively, are in common use in newspapers and among English-speaking Indians.

xvii The role played by culture and language in human development is an essential aspect of the Vygotskian framework which examines the relation between learning and mental development through (a) social sources of individual development, (b) semiotic mediation, and (c) genetic (developmental) analysis.

xviii Scandinavian Cultural-Historical Activity Theory developed from Vygotski's (1978) Cultural-Historical Psychology and Leontev's (1977) Activity Theory. See Engestrom, Yrjo (1999).