Oral Presentation

KNOWLEDGE ACQUISITION AND MATHEMATICAL REASONING

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Mathematical and logical reasoning can be understood as being tautologous which makes the reasoning, informationally, empty. Mathematical and logical truths are valid, i.e., true in every possible world. That is, mathematical and logical truths do not exclude any possibilities, and contradictory statements exclude all of them. To understand how mathematics increase our knowledge, it is important to analyze concrete mathematical reasoning. In geometry, the essential element is the constructivity of the entire reasoning process. A key notion in understanding mathematical knowledge acquisition is the notion of constructivity, which is closely connected to the methodology and epistemology of mathematics. However, at the same, the constructivity allows us to understand the applicability of mathematical reasoning to experimental and empirical reasoning. The strategies of experimental and mathematical reasoning are parallel.

INTRODUCTION

The notion of reasoning, as well as the notion of mathematical reasoning, is used in everyday language. However, it is not obvious what this everyday notion is intended to mean; maybe it is, as everyday notions usually are, ambiguous. Moreover, in scientific usage, the notion of reasoning seems to be a very flexible notion. Even in the philosophy of science, there is no consensus on the meaning of the notion of scientific reasoning (Niiniluoto, 1999). In mathematics and in logic, there are different philosophical approaches that interpret the mathematical and logical reasoning in different ways (Benacerraf & Putnam, 1989).

The notion of reasoning is connected to the notion of learning: all learning is, in one sense or another, reasoning. So far, so good. However, the meaning of the notion is, once again, ambiguous; the learner learns by reasoning, but not all reasoning need be learning. Sometimes reasoning is just an explication of what we already know. There are interesting degrees of knowledge, ranging from (full) knowledge to (full) ignorance (Hintikka, 1989).

There are different kinds of reasoning, for example, Peirce characterized three kinds of reasoning, namely deductive, inductive, and abductive reasoning (Peirce, 1955). We come across deductive reasoning in logic and in mathematics, and we meet inductive reasoning in (ordinary) empirical scientific reasoning; for example, normal statistical reasoning is inductive. Abductive reasoning is more problematic, and it is met in discovery processes (Hintikka, 1998). Deductive reasoning is truth preserving, which implies that deductive reasoning does not increase our knowledge. Inductive and abductive reasoning increase our knowledge, which makes these modes of reasoning very problematic, and there are no generally accepted inference rules for inductive or abductive logic (Kelly, 1996).

Mathematical and logical, reasoning and all truth preserving reasoning that can be characterized as being tautologous (see Tractatus 6.1231). This tautologousness means that the reasoning is, informationally, empty which in terms of information theory means that mathematical and logical truths do not exclude any possibilities, that is, they are true in every possible world. On the contrary, contradictions are, informationally, full, since they exclude all the possibilities, that is, they are false in every possible world. Both logical truths and contradictions seem to be useless in any real communication; they cannot be used in conveying any factual and meaningful information.

However, the informational emptiness is not the whole story. It is true that logic and mathematics are tautologous and, hence, "“useless” in real communication, but then several questions arise: Why study mathematics? Why is mathematics so difficult to study? Can mathematics increase, in any reasonable sense, our knowledge? Why can mathematics be applied in so many fields of sciences? These questions are interesting as such, but they are closely connected to each other. As formal sciences, mathematics and logic are, informationally, empty, but this makes it possible to apply them to different fields of sciences. At the same time, as formal and abstract sciences, they are not easy to grasp.

Mathematics and logic, even if they are formal sciences, evoke emotions and passions. We have to understand that there is no pure mathematics or pure philosophy of mathematics in a sense that it would be explicit, explicitly presented and have a lack of “unintentional meanings” or “unintentional connotations”. The philosophical views are built from heteronomous sources, some ideas increasing, some decreasing. The heteronomity is a permanent condition, which has to be kept in mind while formulating a philosophy of mathematics; in particular, this heteronomity has to be recognized in mathematics and logic teaching.

It is hard to see any single fundamental opinion which could be seen as prevailing, and it is not an easy task to build a coherent picture. In a sense, the kind of practical attitude given by Beta in Lakatos (1989; 54) may seem to be the final opinion: “Whatever the case, I am fed up with all this inconclusive verbal quibble. I want to do mathematics and I am not interested in the philosophical difficulties of justifying its foundations. Even if reason fails to provide such justification, my natural instinct reassures me.” In textbooks of logic and mathematics, the emphasis has been on teaching inference rules, but not on teaching the strategic aspects of the whole reasoning process (Hintikka, 1996; 2007; Detlefsen, 1996). Teaching strategic aspects supposes that the teacher has in his or her mind a holistic picture, which he or she is intending to convey to students. However, the very nature of mathematics and mathematical reasoning is still a problem to be solved. Independently, whether we solve the problem consciously or unconsciously, we have a philosophy of mathematics. This philosophy affects the way we think about, teach, or do the mathematics. So, it is better that the philosophy of mathematics is explicit.

Mathematics and logic are understood as being formal tools that can be used in different fields of sciences. However, the notion of a tool is not as innocent as is sometimes assumed. Mathematics and logic are cultural constructs, hence mathematical and logical notions, similarly to material objects, like a hammer, carry their cultural history. Mathematics and logic are not merely tools, but part and parcel of the methodology of natural sciences; they are built into the knowledge acquisition processes (Hintikka, 2007).

In the following we are not intending to give a conclusive characterization of mathematics and logic. We are not intending to remove the multifaceted nature of mathematics and logic. The intention is to characterize one possible view which does justice to mathematical and logical reasoning. We will connect the expressed approach to some other approaches which give a richer view of the topic.

ABOUT THE PHILOSOPHY OF MATHEMATICS

The fundamental questions of the philosophy of mathematics and of logic – such as “What is mathematics?” and “What is logic?” – are open questions which do not have well characterized conclusive answers (Hintikka, 1976). Still, they are worth asking. There are several different kinds of answers in the history of philosophy, mathematics and logic. In the introduction of *The Principles of Mathematics*, Russell says that “The present work has two main objects. One of these, the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles …” (Russell, 1903, p. v). The characterization is easy to accept: mathematics is a deductive science, which is based on some fundamental statements usually called axioms and on some set of rules of inference. The Russellian approach has its philosophical roots in the emergence of new mathematical logic, which “tends to identify mathematics with its formal axiomatic abstraction (...) as the formalist school” (Lakatos, 1989, p. 1). Russell and Frege can be seen as founders of the modern mathematical logic.

The late 19th century and early 20th century formed a “golden age” for modern formal logic. There is no single logic, but it has seen several different kinds of objectives. Logic has been understood, for example, as “laws of thought”, a universal language or general natural science, which all have different interpretations. So, as laws of thought, logic describes how humans think (psychologism in logic) or logic tells us how to reason correctly, not how human actually or usually reason (normatism).

Mathematics and logic can be understood as just a formal study of (uninterpreted) symbols. The expressions “formal logic” or “symbolic logic” may suggest such an interpretation, which is a very problematic interpretation (Haack, 1995, p. 3). Of course, in logic, the manipulation of symbols, according to inference rules, is a central task. This manipulation is not the central content of mathematics and logic: they are rich in content and, hence, no simple idea captures their whole meaning.

The present day approach – in which logic is a field of mathematics – is compatible with the normative interpretation. Even if we do not understand logic or mathematics as part of philosophy, they are rich in content. There is no need to assume any “philosophical logic”, besides the “mathematical logic” (Hintikka, 1973, ch I). Carnap, in his early publications, emphasized the formal aspects of logic and philosophy. The notion of syntax was central for him (Carnap, 2000), and Carnap’s notion of syntax is reminiscent of Wittgenstein’s notion of grammar, which is a fundamental notion of his philosophy of language.

The fundamental idea that interconnected late 19th and early 20th century logic was formulated in logicism, which was the study of the foundations of mathematical reasoning. The basic intention was to reduce mathematics to logic. Russell was very optimistic when he said that it is possible to reduce mathematical propositions “to certain fundamental notions of logic” (Russell, 1903, p. 4). Nowadays, we may say that the fundamental idea was wrong: mathematics cannot be reduced to logic. Still, we can say that the logicist approach was very fruitful: the approach inspired research and brought together different kinds of researchers.

The more general idea behind the development of logic was the Leibnizian idea of universal language (*lingua characterica*), which was shared among the logicians of the “golden age”. The “golden age” of logic was a proper golden age; the development of logic and mathematics was something remarkable. The names like Frege, Hilbert, Russell, Carnap, Gödel, Tarski, and Genzen give an impression how rich the development in logic and in mathematics was at that time, and Frege and Russell can be seen as the founders of the modern logic.

Russell was a foundational researcher in the emerging modern logical theory. He knew exactly the ethos of modern empirical philosophy, and his logic and philosophy also had a foundational role in the emergence of this new empirical philosophy. However, at the same, Russell was anchored in the old philosophical tradition, his philosophical roots in the (criticism of) Kantian philosophy. His philosophical orientation can be seen very clearly in *The Principles of Mathematics*, which are very clear from the structure of the book[i]. Russell sees logic as a certain kind of natural science: “Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world, just as truly as zoology, though with its more abstract and general features” (Russell, 1929, p. 169).

Frege’s philosophical roots are in the tradition of universal language. Logic was for him the language “in the sense that, for him, something could be said if, and only if, it could be said in that very language" (Haaparanta, 1986, p. 159). His two-dimensional logical notation was pictorial and, hence, intuitively very attractive. However, the notation is very unpractical: it becomes very difficult to *see* when we consider longer sentences (see Frege, 1979). The linear notation introduced by Peano became the prevailing notion, and was used by Russell and Whithead in *Principia Mathematica*. Even if Frege never developed an explicit theory of semantics, his (semantical) analysis of language, based on his analysis on the notions of *Sinn* and *Bedeutung*, is extremely deep. The rejection of the possibility of the explicit theory of semantics is based on his opinion that it is not possible for us to look at the language outside of the language. (Haaparanta, 1986, 41). This opinion was later shared, for example, by Wittgenstein. Moreover, Russell's theory of definite descriptions is syntactic, but the intention is semantic (Hintikka & Kulas 1985, pp. 33-34). So, it is possible to agree with Wheeler (2013, p. 293), when he said: "One would be hard pressed to overestimate Frege's impact. His term logic and the invention of the predicate calculus (1879; 1893) revealed a rich, yet unified structure behind complex, quantified sentences of mathematics, and this breakthrough in logic opened the way to rigorously analyzing the meaning of mathematical statements and mathematical proof."

There was a great deal of belief in the possibilities and the power of growing logic. Gödel (1931) proved his famous and shocking incompleteness theorem for first order logic. The paper in which the theorem was proved is extremely important; it introduces several new and essential mathematical notions. For example, the method of Gödel numbering made it possible to speak about mathematics within mathematics, i.e., it made the metamathematics part of mathematics itself. The proof constructs a sentence which says that it is true but not provable. The proof clearly shows in which sense mathematical proofs can be constructive, and moreover, the theorem was something unexpected: it crushed Hilbert's original program (Nagel & Newman 1989; Hintikka, 2000).

After Gödel's result, logicians managed to formalize the notion of computability. In the 1930s, several different formalizations of the notion emerged, namely *recursivity* (Gödel, Kleene, Herbrand), *λ-definability* (Kleene, Church, Rosser), and *Turing machine computability* (Turing, Post). It was especially interesting was that all these were proved to be coextensive, which has been the basis for the *Church's thesis*, which says that an intuitive notion of computability can be identified with the notion of recursivity. Church's thesis cannot be proved, since it interconnects a nonlogical notion of intuitive computability and a logical notion of recursivity. However, the notion of computability allowed for logical proofs that prove something not-computable. In fact, the class of non-computable functions has proven to be an extremely interesting area of study (Mutanen, 2004).

The semantical or model theoretical approach has been developed extensively since the 1930s, with Carnap becoming one of the founders of the model theoretical approach. Tarski, in his papers 1933 and 1944, formulated a logico-mathematical notion of truth, which was intended to explicate the Aristotelian notion of truth. The Tarskian notion is, nowadays, known as an explication of the correspondence theory of truth (Hodges, 1986). The history of the model theoretical approach can be seen as anchored in independence and definability results in logic and in mathematics. Padoa's principle states that a predicate is not definable in a theory, if it is possible to give two different interpretations to the predicate, while all the other non-logical constants of the theory have the same interpretation. The explication of non-Euclidean geometry was a similar model theoretic proof that parallel axiom is independent of the other axioms of geometry. The modern model theoretical approach has been developed by researchers like Carnap, Tarski, but also by Löwenheim, Skolem, Henkin, and Beth. However, there is no proper disagreement between proof theoretical and model theoretical methods within first-order logic: Gödel's completeness theorem shows that a sentence is provable if, and only if, it is valid.

The difference between syntactical (proof theoretical) and semantical (model theoretical) methods is very important to keep in mind. Even if in school teaching calculating, and hence syntactical methods, are emphasized, model theoretical methods are also introduced. Maybe it could be reasonable to highlight the methodological approaches more systematically. This could enrich the conceptual understanding of mathematics and logic. The approach we are formulating in this paper gives an example of such an enrichment.

Mathematics and logic are heterogeneous disciplines in which there are several different kinds of approaches present. To get a better picture we have to consider mathematics and logic "from outside". However, this task is not so straight-forward, because it leads us to one central mathematical and logical problem: the character of metamathematics. This leads us to the lines of thought that are central for the argumentation in this paper.

LOGIC AS CALCULUS AND LOGIC AS LANGUAGE

The formal character is present in modern mathematical and logical theory, which can be seen from the works and journals of logic and mathematics. Even if logic and mathematics are expressed in different kinds of formalisms, logic and mathematics are not merely a formal game of the symbols on paper. Hilbert's famous characterization of mathematics, as a mere game played by simple rules with meaningless symbols on paper, must be understood within his more general philosophical view of mathematics. Hilbert was interested in problems of metamathematics, and his intentions were almost the converse to that of Wittgenstein.

Wittgenstein imbedded the problem of mathematics in his more general philosophy of language, when he asked the question: "Is mathematics about signs on paper?" The answer he gives is "No more than chess is about wooden pieces." (Wittgenstein, 1988, p. 290) According to Wittgenstein, mathematics is a certain kind of activity or a certain game to be played. It is not possible to take a look at the fundamentals of the game, that is, there is no metamathematics which could tell us about what mathematics really is, and it is not possible to look at the mathematical game outside of the game itself; we are bound just to play the game. That is, the only way to get to know mathematics is just to do mathematics. The meaning of the mathematical notion cannot be found from the result, but rather to understand the meaning, one must look at the proof, "the calculation actually going on in the proof" (Wittgenstein, 1988, pp. 369–370).

To get a better grasp let us consider the following distinction made by van Heijenoort (1967): [1] logic and mathematics as calculus and [2] logic and mathematics as language. The very idea is that if we understood logic and mathematics as calculus then it would appear to be interpretable and reinterpretable over and over again. The possibility of interpreting over and over again provides a great deal of practical freedom: a mathematician or a logician can decide which kind of interpretation he or she chooses, and this interpretation is developed systematically in model theory. On the contrary, mathematics and logic can be understood as language, that is, as a language with a fixed interpretation. Thus, logic and mathematics as language are languages which speak about the reality, as Russell characterized mathematics to be above. In fact, Hilbert's characterization of mathematics as a game is a game in the sense of the calculus; and for Wittgenstein, the game is in the sense of language.

The taxonomy given by van Heijenoort can be generalized as a whole language as Kusch (1989) demonstrates. The taxonomy is based on very fundamental philosophical presuppositions, which are not easily recognized. In particular, the philosophical presuppositions behind mathematics and logic are extremely difficult to recognize. Moreover, as fundamental philosophical presuppositions, they are orientating principles rather than explicit statements or norms (Hintikka, 1996).

Independently on the philosophical orientation, as Wittgenstein said, "calling arithmetic a game is no more and no less wrong than calling moving chessmen, according to chess-rules, a game" (Wittgenstein, 1989, p. 292). Wittgenstein interconnects mathematical and chess games, but at the same time, he brilliantly separates mathematics and chess from a game of billiards: "A billiards problem is a physical problem (although its solution may be an application of mathematics). (..) a chess problem is a mathematical problem" (Wittgenstein, 1989, pp. 292-293). The characterization of mathematics as a game does justice to mathematics as a dynamic computation process, which was explicated in Turing's formulation of computation (Turing, 1936).

In Turing machine computation, the starting point is a known (and usually solvable) problem, for example, what is the sum of given numbers. However, in mathematical and logical reasoning, we do not merely consider these kinds of well-defined and answerable problems; even if they seem to be over-represented in school mathematics. Mathematics is, essentially, something more than mere computation or merely following given rules. These rules allow us to formulate constructive proofs and this constructiveness is related to the demonstrativity of mathematical and logical reasoning. The strategies of mathematical and logical reasoning are the most important things to learn, in order to understand mathematical and logical reasoning. Moreover, strategic aspects are central, when mathematics and logic are applied in different fields of sciences (Hintikka & Kulas, 1985, ch III p. 17).

CONSTRUCTIVE METHODS

The philosophical background of constructive philosophy is very deep. In *Meno,* Plato demonstrates a dialectical method, which is a marvelous example of epistemic construction in which dialog proceeds via questions and answers. These questions and answers build up the knowledge of the learner (the answerer) in a factual manner. The teacher (questioner) has a strategic map of the learning situation. The dialog is extremely rich and one can find all the central aspects of constructive learning and teaching from the text. The discussion in philosophy and in pedagogy, based on *Meno,* is still going strong.

It is not obvious in what sense mathematical and logical reasoning are constructive. Carnap (1969, p. 152) says that “The basic language of the constructional system is the symbolic language of logistics. It alone gives the proper and precise expression for the constructions; the other languages serve only as more comprehensible auxiliary languages.” For Carnap, the foundation of constructions is in phenomenalism: “In this book, I was concerned with the indicated thesis, namely that it is, in principle, possible to reduce all concepts to the immediately given.” (Carnap 1969, p. vi) However, constructive philosophy does not presuppose commitment to phenomenalism or to any other ism.

Perhaps the best example of construction in mathematics can be found in geometry. Elementary geometry is known to be decidable, which means that there is a (computable) decision method for the geometry. This does not imply that it would be a trivial or a mechanical task for generating proofs in elementary geometry. One excellent example in which the geometrical constructions and their knowledge-providing character becomes evident is the slave boy example in Plato’s dialogue *Meno*.[ii] In the dialog, Socrates directs the reasoning process of a slave boy by his questioning method. The reasoning is based on the drawings made on the ground during the process. The dialog shows how these drawings increase the slave boy’s knowledge. These drawings, together with general geometrical knowledge, construct the intended result. This knowledge construction process is essential in all mathematical and logical reasoning.

The conclusion of the reasoning in *Meno* is a geometrical theorem. The proof of the theorem is a strategic search for the information needed in the proof. The strategy is realized by the Socratic questioning method. However, in the end, anyone who has followed the construction *sees* the result; that is, he or she understands the theorem and, hence, *sees* the truth of it. In fact, the Socratic method used in the dialog demonstrates a general pedagogical paradigm which can be used – and has been used – in any teaching.

The Socratic questioning method brings up the strategic level of mathematical reasoning. The questions Socrates asks are motivated by a strategy that directs the reasoning towards the intended conclusion. What knowledge is needed to lead such a process? How do such processes take into consideration the learner's level of knowledge? The process is a step-by-step process, in which each step is made obvious by giving the information needed – the questioning-answering method is designed to guarantee the success. What about the teacher's knowledge? The Socratic irony refers to idea that Socrates, in fact, knew, but he feigned being unknowing. However, there is no need to have full knowledge before the teacher leads the reasoning process; what he or she has to have is good a methodological knowledge of the problem setting. Hence, methodological knowledge is a solution to Meno's paradox (Hintikka, 2007; Kelly 1996). The explicit presentation of the reasoning process with the pictures and formulas makes the reasoning process observable. Hence, the entire audience can follow the reasoning and infer the same conclusion for himself or herself. That is, mathematically reasoned knowledge will become transmissible by such an explicit and public process (Hendricks, 2001; 2010).

The increase of geometrical knowledge in the example in *Meno* can be related to a more general problem of knowledge transmissibility. In fact, the argument shows that such geometrical knowledge is a paradigmatic example of transmissible knowledge. The reason is methodological: geometrical knowledge is constructed during the reasoning process, in a step-by-step manner. In fact, this observation can be generalized to all mathematical and even certain kinds of empirical reasoning. The pedagogical aspect of the dialog is that Socrates asks the question in a way that allows the slave boy to understand the questions and find the answers himself. So, all the steps become constructively known by the slave boy. This kind of explicit knowledge acquisition process can be followed and reproduced. Moreover, as Hendricks (2010) shows, knowledge transmissibility is closely related to public announcement that explicitly take place in the strategically led discussions like Socrates and the slave boy had in Plato's *Meno*.

The idea of the constructions is to take more and more new individuals into consideration and look at their relations to other individuals. In intuitionism, the constructive method has been an essential part of logical reasoning: "In practice, the most important requirement of the program of constructive proof is that no existential statement shall be admitted in mathematics, unless it can be demonstrated by the production of instance." (Kneale & Kneale, 1962, 675) The observation was generalized to a geometrical method of analysis and synthesis by Hintikka and Remes (1974). They characterize geometrical analysis as follows:

"Speaking first in intuitive terms referring to geometrical figures, an analysis can only succeed if, besides assuming the truth of the desired theorem, we have carried out a sufficient number of auxiliary constructions in the figure in terms of which the proof is to be carried out. (…) This indispensability of constructions in analysis is a reflection of the fact that in elementary geometry, an auxiliary construction, a *kataskeue* (…), which goes beyond the *ekthesis* (…) or the 'setting-out' of the theorem in terms of a figure, must often be assumed to have been carried out before a theorem can be proved." (Hintikka & Remes 1974, p. 2)

Geometrical constructions bring new geometrical objects into the reasoning process, and these new objects increase the information used in reasoning. This can be generalized into logical reasoning by observing that the geometrical objects behave similarly to individuals in logical reasoning. These new individuals increase the information, which can be precisely defined and even measured. The definition of the increased information is based on the number of interconnected individuals in the reasoning. The number tells us the depth of the argumentation, and it can be shown that an increase of the depth increases the logical information. Hintikka (1973) based his definition of surface tautology and depth tautology on this measure:

"Depth information is the totality of information that we can extract from a sentence by all the means that logic puts to our disposal. Surface information, on the contrary, is only that part of the total information which the sentence gives us explicitly. It may be increased by logical operations. In fact, this notion of surface information seems to give us, for the first time, a clear-cut sense in which a valid logical or mathematical argument is not tautological, but may increase the information we have. In first-order logic, valid logical inferences must be depth tautologies, but they are not all surface tautologies." (Hintikka, 1973, p. 22)

EMPIRICAL REASONING

Logical reasoning is theoretical in the sense that it can be done by paper and pencil. The results of such reasoning are statements. Such reasoning should be separated from empirical reasoning: "This means that all talk about construction, including the construction postulates, is inappropriate, for it is about *doing* things, whereas, in fact, geometry is a *theoretical* discipline that treats eternal things. Since, what Plato criticizes is just the "language" of geometers, it does not mean that all the geometer's concern with construction problems could be excluded from geometry as a science, rather, they should be reinterpreted as *theoretical* statements." (Stenius, 1989, p. 78)

If we use Hintikka's (1973) notions, we can say that this kind of theoretical reasoning is part of the indoor games. However, there is need for the logical analysis of empirical and experimental reasoning. To carry out this task, Hintikka (1973) introduces outdoor games, which are games of seeking and finding in reality. There is no essential methodological difference between indoor and outdoor games. In fact, this close interconnection is already recognized by Newton: "It is the use of the method of analysis as a model of experimental procedure of the great modern scientists, notably by Newton." (Hintikka & Remes, 1974, p. xvii)

The analysis of experimental reasoning shows that this kind of model-related logic can be a realistic reconstruction of experimental (and empirical) reasoning. Thus, we can understand why Hintikka and Remes (1974) say that:

"We do believe that in a very deep sense, Newton really practiced what he preached, and that his methodological pronouncements present an interesting general model of the experimental method at large. We have come to realize that both these claims, also the historical one, need further argument and further evidence, before we are prepared to rest our case. (…) In the case at hand, the need and unpredictability of auxiliary constructions in analysis shows once and for all that in spite of its heuristic merits, the method of analysis just cannot serve as a foolproof discovery procedure." (Hintikka & Remes, 1974, p. xvii)

In a similar manner to how geometrical reasoning can be generalized as logical reasoning, this Newtonian reasoning can be generalized as general experimental reasoning. The theoretical foundation is the interrogative model of inquiry developed by Hintikka. The interrogative model of inquiry shows how the usual experimental reasoning can be constructive, just as logical, mathematical, and geometrical reasoning are, whilst the constructive aspects connect the interrogative model to present day discussions about causality (Woodward, 2003). The difference between the interrogative model and logical theory is in the character of the forthcoming information. With both logical and experimental reasoning, the intention is not to find singular facts or to generalize universal laws from given sets of data, but to understand the mechanisms underlying the phenomena (Hintikka & Kulas, 1985, p. x). However, the underlying logic is the same usual logic and, in particular, logical and experimental reasoning are strategically parallel (Hintikka, Halonen & Mutanen 2002).

CONCLUDING REMARKS

We have seen that there are several different interpretations of mathematics and logic which are not compatible. This is not something that should be denied or avoided. Rather, it is a symptom of the richness of the content of mathematics and logic. The heterogeneity of mathematics and logic cause polysemy into the field, which may occur in practical problem solving situations. This is challenging for teachers and researchers, but the challenging situation makes mathematics and logic extremely interesting topics to study. As we have seen, it is possible to find rich interpretations which formulate a holistic picture of the field of study and which allow for open discussion together with other interpretations.

NOTES

1. To fully comprehend this, please take a look at the table of contents of the book.

2. To see more detailed analysis of the example, see Hintikka & Bachman 1991 pp. 20-28.

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