Oral Presaentation
HISTORY IN MATHEMATICS ACCORDING TO ANDRÉ WEIL

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André Weil (1906-1999) was one of the most famous mathematicians of the XXth century, working mainly in arithmetic and algebraic geometry, and he worked also on the history of mathematics, especially history of number theory. He had a very specific way to give sense to mathematical activity, in narrow relation with its history, which is useful for mathematicians at first, and perhaps also for teachers and students. Our question here is to understand, what really are history and mathematics – history linked to mathematical activity – and how each one is useful for the other, in Weil's conception ; so in this perspective we want to clarify the interest of history in teaching of mathematics.

1. SUMMER 1914: FROM CARLO BOURLET ...

In his *Apprenticeship of a Mathematician* (Weil, 1992, 22), André Weil wrote:

> At that time, the textbooks used in secondary education in France were very good ones, products of the "new programs" of 1905. We tend to forget that the reforms of that period were not less profound, and far more fruitful, than the gospel (supposedly inspired by Bourbaki) preached by the reform of our day. It all began with Hadamard's Elementary geometry and J. Tannery's Arithmetic, but these remarkable works, theoretically intended for use in «elementary mathematics» (known as *math.elem.)* course during the final year of secondary school, were suitable only for the teachers and best students: this is especially true for Hadamard's. In contrast, Emile Borel's textbooks, and later those by Carlo Bourlet, comprised a complete course of mathematics for the secondary school level. I no longer recall which one of these fell into my hands in that summer of 1914, but I still have an algebra text by Bourlet for third, second, and first form instruction, which was given to me in Menton, in the spring of 1915. Leafing through it now, I see it is not without its defects; but it must be said that this is where I derived my taste for Mathematics.
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> From 1914 to 1916 [...] As for mathematics I had for the time no need of anyone: I was passionately addicted to it (Weil, 1992, 23).
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> In the taupe [this is the name commonly given to courses preparing students for entrance examinations of the Ecole Polytechnique and the scientific section of the Ecole Normale], of course, the student acquires or at any rates acquired at that time – a facility with algebraic manipulation, something a serious mathematician is hard put to do without, whatever some might say to the contrary (Weil, 1992, 31).

These facts are important for us, because we think that the perspective in which Weil linked history and mathematics later, is surely related to the first influences he was subjected to, those of its professor at the lycée Saint-Louis, Auguste-Clément Grévy, and then the mathematicians Jacques Hadamard and Elie Cartan, and the books of Emile Borel and Carlo Bourlet, as we saw above.

We claim that we cannot really understand what someone says about mathematics, and eventually about its history, if we don't know who, from the beginnings, teached him what are mathematics.

2. ... TO READ ORIGINAL TEXTS …

Weil was student at the École Normale Supérieure in Paris from 1922 to 1925, where his best friends were Henri Cartan, Jean Delsarte, Claude Chevalley. In the 1920's, the first determination of "history" for André Weil (Weil, 1992) was his willing to read the masters – Bernhard Riemann, read by Weil since 1923, and then Pierre de Fermat – as the best source of inspiration, before any reading of the "auteurs à la mode du jour". In 1926, he was one of the first French mathematicians to go in Germany (Audin, 2012). He sustained the beginning of his own activity by fruitful relations (conversations) with colleagues out of France: for example, with Carl Siegel at Göttingen and with Francesco Severi (Weil, 1979, I-524-525), with Mittag-Leffler (Weil, 1992, 54). But throughout his life he travelled and met many colleagues: to know the full network of these relations, the best is to look at the index of names in *The Apprenticeship of a Mathematician* (Weil,1992, 193-197).

3. … DOING MATHEMATICS AND HISTORY OF MATHEMATICS TODAYS

A second step, after 1926 [Weil, 1979, III-400] is related to his frequenting the Institute of Mathematics of Francfort (with Max Dehn, Ernst Hellinger, Carl Siegel, Otto Toeplitz, Paul Epstein, Otto Szàsz), and the *Dehn's seminar* on History of mathematics (started in 1922).

Weil wrote: "I have met two men in my life who make me think of Socrates: Max Dehn and Brice Parain" (Weil, 1992, 52). Max Dehn wrote: "Mathematics is the only instructional material that can be presented in an entirely undogmatic way". It would be interesting to know how in the mind of Dehn, and possibly of Weil, it was related to instruction with historical sources. Perhaps that, as a kind of paradox, undogmatic attitude is possible only if we respect our source text, because effective respect of source puts in perspective any dogma.

On the link between mathematics and history, a comparison between the Weil's approach and the genetic method of Toeplitz would be useful (Toeplitz, 1964) – written in the 1930's, and started in a paper in 1927, from a talk in 1926 (Toeplitz, 1927). As quoted by Gottfried Köthe in its foreword to the German edition of 1949:

> Toeplitz does not want to have his method labelled "historical": "The historian – the mathematical historian as well – must be record all that has been, whether good or bad. I, on the contrary, want to select and utilize from mathematical history only the origins of

> those ideas which came to prove their value. Nothing, indeed, is further from me than to give a course on the history of infinitesimal calculus. I, myself, as a student, made escape from a course of that kind. It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts, proofs" (Toeplitz, 2007, xi).

Toeplitz was convinced that "historical considerations could be useful in bridging the gap between mathematics at the Gymnasium level and at the universities" (Folkerts, 2002, 136) ; or, as said by Köthe, he was "convinced that the genetic approach is best suited to build the bridge between the level of mathematics taught in secondary schools and that of colleges courses" (Toeplitz, 2007, xi).

In 1979, Weil explained that in the Dehn's seminar, they read one classical original paper, and, if necessary, they used contemporary authors and witness:

> On ne croyait pas devoir feindre d'ignorer ce que l'auteur n'avait pas su ; au contraire, on s'en servait pour mettre en lumière les intuitions qu'il n'avait pas été en mesure d'exprimer clairement.[…]. Je ne conçois pas de plus saine méthode historique que celle-ci (Weil, 1979, III, 460).

Nonetheless he precised that "Dehn showed how this text [Cavalieri, read in 1926] should be read from the viewpoint of the author [...]" (Weil, 1992, 52).

Toeplitz assumed to modify the "objective complete history", in order to construct a tool for teaching (his genetic method). Weil also assumed to disrupt the "true historical data" – with the introduction of nowadays knowledge – but he rather did that in order to get a better reading of the history itself. Furthermore, his target was not teaching, but the development of an historical foundation or motivation for new research, for him and for any mathematician at work.

History according to Weil will be a history of mathematics for mathematicians, almost a part of mathematics. Weil was scrupulously honest with ancient texts, but he worked with the help of modern knowledge and notations, because he wanted to use past to validate mathematics of today. So he allowed this goal to historical data. In our conclusive section we will give an example of an historical analysis by Weil, where past is taken as a mirror in which we admirated our own understanding and the way in which today the things seem to be clarified. We read the past at the light of what we know today (and the history is closed because of this goal).

It is interesting, but another different way to do history of mathematics for mathematicians at work today, would be in the first instance to try to understand old texts on their own, in their own time, and in the second place, with such an understanding in hands, to try to understand some mathematical piece of today. In this case our supposition is that we do not yet understand what we are doing, and consequently history can act on our free future works.

4. THIRD STEP: HISTORICAL NOTES IN BOURBAKI'S "ÉLEMENTS DE MATHEMATIQUE"

André Weil was one of the founders of the Bourbaki group, in fact its unquestionable leader. The startìng point of Bourbaki was a conversation between André Weil and Henri Cartan, about their teaching in Strasbourg. Weil said: "Why don't we get together and settle such matters once for all, and you won't plague me with your questions any more ?" (Weil, 1992, 100).

From the beginning of Bourbaki, officially founded in 1935 (the first Bourbaki congress was held in July), he suggested to include historical notes "in order to put in a right perspective some too much dogmatical expositions". For him, this suggestion was natural, as a consequence of his experience in the Dehn's seminar:

> Ayant eu le bénéfice d"une pareille expérience, je me trouvais naturellement porté, lorsque Bourbaki commença ses travaux, à proposer d"y faire figurer des commentaires historiques pour replacer dans une juste perspective des exposés qui risquaient de tomber dans un dogmatisme excessif (Weil, 1979, III, 460).

He did the job himself for General Topology and for Infinitesimal Calculus. Then others collaborators (as Jean Dieudonné, mainly) continued the job: "The point is, for each theory, to clarify completely the directive ideas, how these ideas are developed, and how they interact each one on the others" (Bourbaki, 1984, 5).

According to Weil, the history of mathematics is a kind of "natural history", in a world of living interactive entities named "mathematical ideas": original technical gestures, formulations, methods, "theories".

> The Bourbaki's collaborators do have a conception of their collective work as attached to a tradition, represented by Poincaré and [Elie] Cartan in France, by Dedekind and Hilbert in Germany. The "Eléments de mathématiques" had been written to provide a solid foundation and an easy access to this type of research [in this tradition], and in a sufficiently general form to be applicable in many possible contexts (Dieudonné, 1977, XI).

Thus, historical notes have to be understood in relation with this tradition, in order to confirm the formal foundation by an underlying historical background, and to expose in advance a large set of possible extensions.

> In history of mathematics we very often see that a theory does start by a very specific problem, and efforts to solve it. [..] There are theories which are fruitful and still very much alive (as for examples: theory of Lie groups, algebraic topology) […] almost each idea in such theories has repercussions on other theories (Dieudonné, 1977).

A problem is important if it generates a method or a fruitful and alive general theory (as analytic theory of numbers, theory of finite Groups; but these theories are not in the Bourbaki scope).

As observed by Halmos (Halmos, 1979), in the Dieudonné's Panorama, every chapter ends with a list of *initiators*: creators of principal ideas or substantial contributors.

The introduction of history in this way, via fruitful problems and initiators, is completely in the Weil's style, including of course the perturbation of history by our today's understandings, and a somewhat naive belief in progress.

In "Two lectures on number theory, past and present" (Weil, 1979, III, 280), we read that Weil knew, or seemed to know, what are "a perfectly good and valid subject", and "perfectly good mathematics". Weil and Bourbaki developed history under this assumption that there are good mathematics, good scientists and geniuses, and today we know what these good mathematics are, and who these geniuses are. This implies a specific choice, and according to that choice, the purpose of history is to put mathematical ideas in the "right perspective".

5. STRUCTURES AND HISTORY

Through the historical notes, Bourbaki constructed a tool for teaching mathematics, although the *Éléments de mathématiques* do not constitute a manual. But this work is also of an historical nature in another aspect, as a story of structures.

Bourbaki planed to reformulate the *Éléments de mathématique*, through an axiomatic development of mathematical structures. Bourbaki took ideas on *axiomatics* from Euclid or Hilbert (but their ideas of "axiomatics" are different), and he used the logico-set theoretical system – induced by the Cantor approach's within set theory, and the research in logic at the end of XIXth century –, which is an available tool at this time (the 1930's). The axiomatics in the algebraic context is implicitely supported by the works of Emmy Nœther and Emil Artin on the natural typical types of rings in geometry. And on the side of functional spaces and analysis, it is related to the works of Maurice Fréchet or Felix Hausdorff. In Geometry or Analysis again, it is related to Elie Cartan or Jacques Hadamard works, two mathematician that Weil met (and admired) at the beginning of the 1920's.

André Weil explained that, he probably imported the notion of *structure* from the field of linguistics (Russian linguists, Roman Jakobson), that he knew by his friends or relations Claude Levy-Strauss, Brice Parain, or Emile Benveniste:

> In establishing the tasks to be undertaken by Bourbaki, significant progress was made with the adoption of the notion of structure, and of the related notion of isomorphism. Retrospectively these two concepts seem ordinary and rather short on mathematical content, unless the notions of morphism and category are added. At the time of our early work these notions cast new light upon subject which were still shrouded in confusion: even the meaning of the term of "isomorphism" varied from one theory to another. That there were simple structures of group, of topological space, etc., and then more complex structures, from rings to fields, had not to my knowledge been said by aznyone before Bourbaki, an dit was something that needed to be said. As far for the choice of the word "structure", my memory fails me: but at that time, I believe, it had already enterd the working vocabulary of linguists, a milieu with which I had maintained ties (in particular with Emile Benveniste). Perhaps there was more here than a mere coincidence (Weil, 1992, 114).

For him, in fact, the notion of structure is really a mathematical notion, or more exactly has to be mathematically determined. In any event, the notion of structure has to be mathematically defined with respect to mathematical activity, and to be recognized through the history of mathematics. When he read Fermat, he observed the invention of gestures, as “the method of descent”, or as more elementary “tricks”, and these things are in some sense “structures”, or “mathematical ideas”.

In Bourbaki, a very special *grosso modo* determination of a structure is a set equiped with constants, functional symbols, relational symbols, axioms. As mathematical objects are specified in Model theory or in Universal Algebra. But more generally, structures are also any mathematical algorithm, mathematical process, mathematical idea. Furthermore, in the Bourbaki view, as specified by Claude Chevalley (Aczel, 2009, 124), each mathematical fact has to be explained, and the result of a computation is not enough for this purpose: the true explanation is the discovery of the natural structure under a given fact.

We can say that, for André Weil, the history of mathematics is the history of emergence of structures, of mathematical delimitations of mathematical ideas. And consequently, the *Éléments de mathématique* by Bourbaki are not a treatise of mathematics, but a fictional articulation, a kind of history, made of successive treatises on chosen mathematical subjects. The story started with an “hygienic” background on sets and logics; then, some “structures-mères” were exposed, in a rational way. But the mathematical necessity of their choice and the order of these “structures-mères” is not proved, it is based on a feeling of history, and on the initial formation of the concerned mathematicians in those days.

However, in two different periods, in the Bourbaki group, two mathematicians were very attracted – in two different ways – by the question of structures in mathematics, and the research of natural structures: Charles Ehresmann and Alexander Grothendieck did not share the Weil’s perspective. For Ehresmann, it was essential to incorporate the “local structures” to the “structures-mères”, and, for Grothendieck the question was to reach the level of “categories”.

But these ideas were new, and Weil seemed very careful with new things, because he believed in the historically known facts as being the deep roots of mathematical development. The difference was between the historically constraint of mathematical meaning and the Cantor’s proclamation of a basic pure freedom in mathematics.

A history of mathematics based on problems is possible, but a history based only on our today’s view of old problems (that is to say on what we consider as convenient solutions) is too restrictive. We have to understand how in their own time problems were posed and solved. This will be a good tool for the comprehension of what we do, and what we will do.

6. LETTER TO SIMONE WEIL: MATHEMATICS AS AN ART

In the difficult context of the beginning of the war in 1940, André Weil used history to explain the meaning of his mathematical work to his sister, the philosopher Simone Weil. He did that in two letters in 1940 (Weil, 1979, I), (Weil, S., 2012). There exists an English translation by Martin H. Krieger (Krieger, 2005). It is important to notice that, there is a kind of fusional relationship between Simone and André.

In the first letter he wrote that the mathematician is an artist, similar to a sculptor, working in a very hard matter, namely the strict mathematical culture, where constraints are previous theories and problems. He suggested to examine history of mathematics from this point of view (as the history of an art).

In the second letter he adds:

> When I invented (I say invented, and not discovered) uniform spaces, I did not have the impression of working with resistant material, but rather the impression that a professional sculptor must have when he plays by making a snowman.

For Weil, it is impossible to explain the mathematical research to the layman (that means a deaf person related to mathematics), and what about history of mathematical research? Probably it is impossible too, and this explains why he only developed a history for mathematicians. He wrote:

> Quant à parler à des non-spécialistes de mes recherches ou de toute autre recherche mathématique, autant vaudrait, il me semble, expliquer une symphonie à un sourd [A. Weil, CP, I, p.255. Lettre à Simone weil] La mathématique […] n'est pas autre chose qu'un art, une espèce de sculpture dans une matière extrêmement dure et résistante […] l'œuvre qui se fait est une œuvre d'art, et par là même inexplicable […] l'histoire de l'art est peut-être possible: et l'on n'a jamais, que je sache, examiné l'histoire des Mathématiques de ce point de vue (à l'exception de Dehn, autrefois à Francfort […]). Et il est tout à fait vain de se lancer là-dedans sans une étude approfondie des textes. […] J'ai dit une fois à Cavaillès qu'il y aurait lieu d'étudier les débuts des fonctions elliptiques […] (Weil, 1979, I, p.255).

Later Weil accomplished such a study on elliptical functions (Weil, 1976).

Weil considered mathematical research as an art, and therefore as an inexplicable activity. For him, it is as a kind of sculpture, in a very hard marble or porphyry. But he thought that the history of art is possible, and after Dehn, that the history of mathematics could be done in this way. He insisted on starting with a deep study of an original text. So this type of history is reserved to mathematician, and the questions are again: Who could read it? Who can do it?

In fact, André Weil could not really discuss his mathematical works with his sister Simone, but he discussed with Simone (who was not a mathematician) on the historical and philosophical subjects of antique mathematics, as the pythagorean ideas (Weil S., 1999, 2012). For Simone, this discussion was incorporated in her teaching of philosophy of sciences. She admired her brother unconditionally.

7. LETTER TO SIMONE WEIL: DEVELOPPING ANALOGIES

In the second letter to Simone, he explained the meaning of his own work, informally but with a lot of details. Mainly he claimed to develop and to construct a triple analogy, between three mathematical domains in progress:

- the theory of numbers and *fields of numbers*,
- the *Riemanian theory of algebraic functions* on complex numbers,
- the theory of (algebraic) *functions on finite fields* (Galois fields).

These theories are described from an historical point of view. He considered that he constructed a kind of trilingual dictionary, in order to decipher a trilingual text made of desultory fragments, trying to construct mathematical analogies – see also the Weil's paper of 1960 "De la métaphysique aux mathématiques" (Weil, 1979, II). It is related to the process of "changement de cadre" (Douady, 1984). It is a case of what we call a "mathematical pulsation" (Guitart, 1999, 2008). More recently, Gérard Laumon introduced this question of analogy in his "Allocution de Réception à l'Académie des sciences" (Laumon, 2005).

8. HISTORICAL REFERENCES IN HIS MATHEMATICAL WORK

In his mathematical works, André Weil used of "historical insights" to motivate and possibly to start his mathematical gestures, and also to increase the prestige of his results (beside the main point which is that these results do solve a problem). The legitimacy of this process is clear in the area of mathematics, in the "creative phase", when we do invent – or discover – our problematics. We have to know that such a "history" is only a tool for doing mathematics.

We give one example, expressed by two quotations, from two papers of Weil.

The first quotation comes from "Sur les fonctions algébriques à corps de constantes finis" written in 1940 (Weil, 1979, I, 257):

> Les travaux de Hasse et de ses élèves; comme ils l'ont entrevu, la théorie des correspondances donne la clef de ces problèmes ; mais la théorie algébrique des correspondances, qui est due à Severi, n'y suffit point, et il faut étendre à ces fonctions la théorie transcendantale de Hurwitz.

And the second quotation comes from "On the Riemann hypothesis in function-fields" written or edited in 1941 (Weil, 1979, I, 277):

> I have now found that my proof of the two last-mentioned results is independent of this transcendental theory, and depends only upon the algebraic theory of correspondences on algebraic curves, as due to Severi.

With such observations, we understand that Weil invented a genealogy of his work for future readers, which inserts it in the great flow of the history of mathematics; and, simultanuously, objectively he gave some interesting mathematical explanations.

9. THE THREE PRINCES AND THE QUEEN, THE PROBLEMS: AN ENCHANTING STORY FOR MATHEMATICIANS

André Weil wrote in "L'avenir des mathématiques" in 1947:

> Mais si la logique est l'hygiène du mathématicien, ce n'est pas elle qui lui fournit sa nourriture ; le pain quotidien dont il vit, ce sont les grands problèmes. "Une branche de la science est pleine de vie, disait Hilbert, tant qu'elle offre des problèmes en abondance" (Weil, 1979, I, 361).

Weil thought that "logic is only the hygiene of mathematics", but the real foods for mathematics are problems. As Hilbert said: "A science is alive as long as it as abundance of problems".

So the history of mathematics does start with problems, rather than with logical foundations. The question of logical foundation is nothing else than one special problem, for "hygiene".

Hence again, we have the question of initiators, of progression of good ideas, and good problems.

> In each new special theory there are initiators, and then good contributors (Dieudonné, 1977).

> No mathematician ever attained such a position of undisputed leadership […] as Euler did […]. In 1745 his old teacher Johann Bernouilli, not a modest man as a rule, addressed him as "mathematicorum princeps" (Weil, 1984, 169).

> In 1775 he [Euler] clearly felt ready to pass the title to Lagrange "the most outstanding geometer of this century" […] In the next century the title of "mathematicorum princeps" was bestowed upon Gauss by the unanimous consent of his countrymen. It has not been in use since (Weil, 1984, 309).

"The Arithmetics is the Queen of mathematics" (Gauss). In the Weil's style, the historian knows a priori who are the geniuses or inspired initiators (e.g. Riemann, Fermat, i.e. the Princes), those creating good new theories; and then, the historian writes an informal but mathematical explanation of links between main ideas in theories.

He does or writes a kind of mathematical story telling of the living world of mathematical problems, ideas and theories; it is also a travel story of a mathematician through mathematical ideas and analogies.

The good reader for that type of history has to be himself a mathematician; in this case, such a history is useful, the reader could find a true mathematical clarification of some notions. From our point of view, the decisive point is that it is not a fairy tale (arbitrary), but an enchanting story of the real (presupposed) growing of mathematical knowledge, a story of its own mathematical clarification. We can consider that such a story is still a part of mathematics.

For Weil, it is a construction of the meaning of mathematics, and because of that, it is enchanting for a mathematician, it is the reason for which he admires mathematics. A mathematician is such an admirer.

André Weil is a specialist of Number Theory, and, furthermore he wrote five books in order to teach Number Theory.

His very basic book, written with the collaboration of Maxwell Rosenlicht, *Number Theory for Beginners* is an elementary manual, and it introduces the very basic operations of arithmetic, without historical information, only by Definitions / Theorems / Proofs / Exercises (Weil and Rosenlicht, 1979).

In *Basic number theory* (Weil, 1967), a more advanced study, Weil exposed local field, adele, class-field theory, and there he added a chronological list of initiators:

> a chronological table […] as a partial substitute for an historical survey of a chronological list of the mathematicians who seem to have made the most significant contributions to the topics treated in this volume.

Fermat (1601-1665)	Riemann (1826-1866)
Euler (1707-1783)	Dedekind (1831-1916)
Lagrange (1736-1813)	H. Weber (1842-1913)
Legendre (1752-1833)	Hensel (1861-1941)
Gauss (1777-1855)	Hilbert (1862-1943)
Dirichlet (1805-1859)	Takagi (1875-1960)
Kummer (1810-1893)	Hecke (1887-1947)
Hermite (1822-1901)	Artin (1898-1962)
Eisenstein (1823-1891)	Hasse (1898-) [1979]
Kronecker (1823-1891)	Chevalley (1909-) [1984]

We notice the name of Emile Artin, which is essential for the history of the quadratic reciprocity law in the 1920's ; this is important, because, for André Weil, the modern history of Number theory turns around this law.

In fact the subject of Adeles is also introduced by Weil in another book, *Adeles and Algebraic Groups* (Weil, 1982). In the foreword, Weil said that "it is based on lectures, which were nothing but a commentary on various aspects of Siegel's work". It is as data recorded for future historians, but also in itself it is a piece of new mathematics. We can say the same thing of the encyclopedic collection of the *Séminaires Bourbaki*. This shows us how much the different aspects of his work (teacher, researcher, author, historian) are very closely related.

The two other books are completely different, and they mix deep mathematics and deep history, they are explicitly exercises of historical reading for mathematicians.

The book on elliptic functions, *Elliptic functions according to Eisenstein and Kronecker* (Weil, 1976) is appreciated by the mathematical community: "this text undoubtedly contributes notably to the history of our science, it is also of great value to contemporary mathematical research" (P. Hilton, Chairman, Editorial board, Ergebnisse der mathematik)

The last book *Number Theory An approach through history from Hammurapi to Legendre* (Weil, 1984) leads to the reciprocity law, through a history account of technical gestures in number theory. Let us give two short explanations on this very original last book.

Starting from the works of the three princes (Euler, Lagrange, Gauss) and other initiators, the history of mathematics is at first the story of the life of the Queen (arithmetics), the mathematical comprehension of the life of its problems.

> Our main task will be to take the reader, so far as practicable, into the workshop of our authors, watch them at work, share their successes and perceive their failures (Weil, 1984. IX).
>
> In a Seminar at the Institute for Advanced Study in Princeton, Weil said that "he knew 50 proofs of the law of quadratic reciprocity, and that for each he had seen there were two others he had not » (Gerdstenhaber) "It can be said that *everything* which has been done in arithmetic from Gauss to these last years consists of variations on the law of reciprocity: one started with Gauss's law and arrived, thereby crowning all the works of Kummer, Dedekind ansd Hilbert, at Artin's reciprocity law, and *it is the same"* (Weil, "Une lettre et un extrait de lettre à Simone Weil" (Weil, 1979, I), (Lemmermeyer, 2000, v-vi, xi.).

The list of the proofs (246 proofs) and the bibliography on the quadratic reciprocity law, as given by Lemmermeyer, can be considered as an effect of the practice of Weil with history of mathematics for mathematicians.

10. HISTORY OF MATHEMATICS: WHY AND HOW (1978)

In 1978, André Weil wrote a paper "History of mathematics: why and how?" (Weil, 1979, III), explaining his conception of the practise of history of mathematics in a remarkable concise and clear way. At first he considered that we have good historians as Moritz Cantor, Gustav Eneström, Paul Tannery, and that we can discuss of their methods, with respect to Leibniz's conception:

> Leibniz wanted the historian of science to write in the first place for creative or would-be creative scientists. Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples (Leibniz, Math. Schr., ed. C.1. Gerhardt, t.V, p. 392.).

At one moment he wrote: "A mathematician will find it appropriate to …". This sentence shows that he is doing a history *as* a mathematician, and *for* mathematicians. Also he observed that "Eisenstein fell in love with maths at only an early age by reading Euler and Lagrange": so let us read the masters (as also Weil did); so history is "some guidance to go back in mathematical readings".

For mathematics, as well as for history of mathematics, it is useful to distinguish between tactic and strategy. Tactic is the day-to-day handling of the tools of the period (with competent teachers and contemporary works). Strategy is the art to recognize the main problems, the pertinent structures, etc.

From Eneström and Tannery, history consists in following the evolution of ideas over long periods, to follow

> the filiation of idea, and the concatenation of discovery (Tannery)

to be able

> to look beyond the everyday practice of his craft (Weil).

So we get the question: "What is and what is not a mathematical idea?".

For the determination of history, according to Weil we quote the three following sentences by Weil in this paper:

> History and philosophy of maths: it is hard to me to imagine what these two subjects can have in common

> Mathematical ideas are the true objects of history of mathematics.

> Large part of the art of discovery consists in getting a firm grasp on the vague ideas which are "in the air".

Coming back to the question of anachronism or attribution of our conception to an ancient author, Weil underscored that this default is different from the use by the historian of our modern knowledge. As a comment on this opinion, we have, by Tannery:

> The greater his talent as a scientist, the better his historical work is likely to be.

11. AN EXAMPLE OF HISTORY "A LA WEIL": THE DEBATE ABOUT TANGENTS BETWEEN DESCARTES AND FERMAT

Weil concluded his paper "History of mathematics: why and how ?" (Weil, 1979, III) with a discussion on the debate about tangents between Descartes and Fermat.

The discussion lays on two ideas of Descartes and Fermat, isolated and formulated by Weil, and three observations of Weil, as below:

Idea 1 (Descartes): a variable curve [for example a circle] intersecting a given one C at a point P, becomes tangent to C at P when the equation for their intersection acquires a double root corresponding to P.

Idea 2 (Fermat): infinitesimal method, depending on adequation, or the first term of a local power-series expansion.

Weil did the following "historical observations":

Observation 1: the debate is between algebraic and mechanical curves.

Observation 2: to the "défi" of Fermat about the cycloid, Descartes do replies by the invention of the instantaneous centre of rotation.

Observation 3: At this period (XVIIth century), the distinction between differential and algebraic geometry has not been clarified. But now we can understand that Descartes'method belongs to algebraic geometry, and Fermat's method belongs to differential geometry. The first one is available with some general ground field, the second one works for more general (non-algebraic) curves.

Today, on this subject of tangents, the reader is invited to examine several analysis by Evelyne Barbin, from an historical point of view (Barbin, 2006, 2015b) and from a didactical point of view (Barbin, these Proceedings).

12. CONCLUSION: ON THE USE OF HISTORY IN THE TEACHING OF MATHEMATICS

From Weil's works and positions on history or with history, we can isolate some observations about the link between mathematics and history of mathematics, in the perspective of pursuing, transmitting and teaching of mathematics.

For various aspects of the use of a historical perspective into mathematics teaching and learning, we refer the reader to some recent published papers (Barbin, 1997, 2012, 2015c). One main idea of Evelyne Barbin (Barbin, 1997), studied again in (Guillemette, 2014), is the notion of « dépaysement épistémologique ». A decisive point is that « dépaysement » arrives if we read a mathematical text in the same way as a contemporary of the author was able to read it. Clearly André Weil contravene this attitude of mind with history, when he reads masters in order to clarify the future of theories, in order to justify the Bourbaki venture, or its own line of development in mathematics. The same observation works for the genetic approach of Toeplitz, and both, Toeplitz and Weil, admits a fictional history, eventually far from the real history, as a tool for mathematical motivation and formation. Of course any historical reconstruction is helpful for mathematical teaching, because this provides an exciting imaginary world of thinking, and in this world a personal motivation, a possibility of identification with some heroes. But a deeper insight is obtained if we work with interpretations respectful of contemporary understanding of a text; in this case we could observe the finest gestures and interpretative pulsations, and try to reconstruct the very moment of invention.

In the primary formation of André Weil, we noticed, on the one hand the influence of two great creative professionals, Hadamard and Cartan, both heirs of practize of natural care of history in science, and on the other hand the passage through preparatory classes for great schools, with Grévy. The preparatory classes (*taupe*) is an heritage from the beginnings of the École polytechnique (Barbin, 2015a), and certainly they instil the habit of taking good care of the link between mathematics and its history.

Of course for Weil, as for any mathematician, the elementary technical training inside any given closed system of calculus have to be executed as grammatical exercises, musical games, remedial gymnastics: quickly, unscrupulously and without qualms. But also the signification of such a training has to be seek out, and the natural way for that is through the reading of original mathematical texts, in the stream of an history of mathematics. And the question is the underlying conception of this history.

A point is that the core of mathematics is just its own history, and only after that point comes the questions of matters, subjects, methods which are to be considered, according to our feeling of the history. Mathematics is a culture, the history of elucidation of the necessity of mathematical ideas, rather than the history of contingent mathematical themas in which these ideas are implicated or even incarnated, or a fortiori rather than the history of its philosophical, epistemological or technical motivations. Whatever we choose to be taught from historical situations, in teaching mathematics we have to be very careful with this distinction between ideas and themes, and this is feasable especially through the reading of masters.

Another point is that mathematical works and teachings are elaborations in two directions: from problems to solutions, which are structural explanations, and conversely from structures to new problematics. So in the history appear problems, and structures, in a kind of dialectic; this dialectic is the motor of rational thinking.

From the history of mathematics we learn that the mathematician is the one which find problems where nobody could see difficulties ; and from this point derives its ability to solve problems on which everybody stumbles against. The true rigor relative to signification is there, in discovering problems; not in the process of solving, in which logical rigor is only a necessary hygiene. The point is to discover how to become subtle as far as to discover new problems. And certainly the history is the best school for that.

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