Oral Presentation

A LOOK AT OTTO TOEPLITZ'S (1927) "THE PROBLEM OF UNIVERSITY INFINITESIMAL CALCULUS COURSES AND THEIR DEMARCATION FROM INFINITESIMAL CALCULUS IN HIGH SCHOOLS"[1]

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This paper discusses Otto Toeplitz's 1927 paper "The problem of university infinitesimal calculus courses and their demarcation from infinitesimal calculus in high schools." The "genetic approach" presented in Toeplitz's paper is still of interest to mathematics educators who wish to use the history of mathematics in their teaching, for it suggests a rationale for studying history that does not trivialize history of mathematics and shows how history of mathematics can supply not only content for mathematics teaching but also, as Toeplitz is at pains to emphasize, a guide for examining pedagogical problems. At the same time, as we shall discuss in our paper, an attentive reading of Toeplitz's paper brings out tensions and assumptions about mathematics, history of mathematics and historiography.

TOEPLITZ'S LIFE IN MATHEMATICS, HISTORY OF MATHEMATICS, AND MATHEMATICS TEACHING

Before starting our examination of the paper which is our focus in this paper, we ought to have some sense of who its author, Otto Toeplitz, was as an intellectual and educational figure. Toeplitz was born in Breslau, Germany (now, Wrocław, Poland) in 1881 and died in Jerusalem in 1940. His doctoral dissertation, *Über Systeme von Formen, deren Funktionaldeterminante identisch verschwindet* (*On Systems of Forms whose Functional Determinant Vanishes Identically*) was written under the direction of Jacob Rosanes and Friedrich Otto Rudolf Sturm at the University of Breslau in 1905. The following year, Toeplitz left Breslau and went to Göttingen. Heinrich Behnke (1963, p.2), describes that move as one from "a quiet provincial town to a gleaming metropolis" – an apt expression, for Göttingen at that time was blessed with the luminary presence of Klein, Hilbert, and Minkowski. Soon, Behnke tells us, Toeplitz was included among the students of Hilbert's inner circle. Toeplitz's doctoral dissertation had already touched on topics related to systems of bilinear and quadratic forms, but with Hilbert Toeplitz's interests in this direction crystalized and became the work on infinite linear, bilinear and quadratic forms and infinite matrices for which he is known. After he left Göttingen, he went to Kiel in 1913 and then to Bonn in 1928. Throughout, however, he continued the work on linear operators on infinite dimensional spaces which had been his general focus in Göttingen.

Toeplitz's mathematical accomplishments are not our main concern here though. What interests us is his involvement in the history of mathematics. Still, it is important to know that he *was* a mathematician of the first rank, for his identity as a proponent of the history of mathematics was bound together with his identity as mathematician (and, as we shall see, as teacher of mathematics). Indeed, Abraham Robinson remarks that Toeplitz "...held that only a mathematician of stature is qualified to be a historian of mathematics" [2] (Robinson, 1970, p. 428). That said, it cannot be claimed that his work in the history of mathematics as such was remarkable, at least relative to his deep mathematical work; however, he had, by all accounts, a profound interest in the history of mathematics, and he made efforts to promote its study. For example, together with Otto Neugebauer and Julius Stenzel he established *Quellen und Studien zur Geschichte der Mathematik* (*Sources and Studies in the History of Mathematics*) in 1929. It was for this journal that Toeplitz first received the papers which later became Jacob Klein's famous *Greek Mathematical Thought and Origins of Algebra*.[3] Toeplitz's enthusiasm for those papers mirrored his particular interest in the relationship between Greek thought and Greek mathematics (Robinson, 1970): "He was a classical scholar able to read Greek texts and he knew his Plato just as well as his Gauss and Weierstrass" (Born, 1940, p.617). The breadth of Toeplitz' scientific interests is also reflected in the good relations he cultivated with the philosopher and psychiatrist Karl Jaspers. Indeed, Jaspers dedicated his 1923 book "Die Idee der Universität" ("The idea of the university") to Toeplitz.[4]

Jasper's decision to dedicate his book on education to Toeplitz is pertinent to our story. For it was in his role as an educator that Toeplitz's interest in the history of mathematics was decisive. Toeplitz's father was a school teacher, and teaching was of great importance to Toeplitz. It is not by chance then that Behnke entitled his tribute to Toeplitz mentioned above, "Der Mensch und der Lehrer" ("The Man and Teacher"). In the early 1930's Toeplitz and Behnke initiated yearly "meetings for cultivating the relations between university and high school" ("Tagungen zur Pflege des Zusammenhangs von Universität und höherer Schule") (Schubring, 2008; Hartmann 2009, pp. 186-195) and they jointly founded the education oriented journal "Mathematische und physikalische Semesterberichte" ("semester reports on mathematics and physics") in 1932 (Hartmann 2009, 199-209). Toeplitz published articles on mathematics education in nearly every issue of the Semesterberichte. A study of these interesting papers is still outstanding and would be a rewarding task.[5]

THE 1927 PAPER AND THE GENETIC APPROACH

In discussing Toeplitz's intellectual life we have emphasized the three streams of mathematics, history of mathematics, and teaching. These three streams came together in what he called the "genetic method" for the teaching. It is beyond the scope of this paper to examine what records may exist of Toeplitz's actual teaching in Göttingen, Kiel or Bonn, but it is clear that he did to some degree introduce the genetic method into the classroom and that he had intended to write a textbook based on it. This Toeplitz's former student Gottfried Köthe tells us, adding that,

> He worked on [the method] for many years, pursuing intensive historical studies of the development of infinitesimal calculus. In his lectures he constantly tried out new approaches, discussing the several parts with his students and searching always for new formulations (Toeplitz, 1963, p.xii)

Köthe edited Toeplitz's lectures and in 1949, after Toeplitz's death, published the textbook that Toeplitz never finished as *Die Entwicklung der Infinitesimalrechnung: Eine Einleitung in die Infinitesimalrechnung nach der genetischen Methode, vol.1* (another volume was intended). This was later republished in English as *The Calculus: A Genetic Approach* (Toeplitz 1963) in 1963. The book, however, did not present the idea of the genetic method, its rationale and overall strategy. That was set out in the 1927 paper which is our focus here, "Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen" (Toeplitz, 1927).

In this paper, which was the published version of an address Toeplitz delivered at a meeting of the German Mathematical Society held in Düsseldorf in 1926, Toeplitz clearly wanted to be understood as solving a specifically *educational* problem, that of designing an introductory course in calculus for beginning university students. Thus the "The problem of university infinitesimal calculus courses and their demarcation from infinitesimal calculus in high schools," mentions neither history nor the genetic approach which is based on history and which is the centerpiece of the address.

The educational problems he wishes to solve are in truth not immediately related to history. He defines these problems in terms of three basic dilemmas, or "moments." The first is contending with two existing schools of thought as to what the guiding principle should be regarding such a course. One school of thought maintains that beginning university students should have an introduction to the calculus that is exact and rigorous, the other, that the course should be intuitive and approachable. Neither "path" (*Richtung*) alone is completely satisfying, but nor is a hybrid which takes a little from each; they are, Toeplitz says, unbridgeable (*unüberbrückbar*).

The second moment is closely related to the first, in some ways, it is a version of the first. It is the tension between two aims of an introductory course in calculus, one being that students acquire necessary tools and concepts in order to ground further work in mathematics and science, and the other, that students acquire a taste for the subject. The latter is particularly important to Toeplitz. He wants students to appreciate how mathematics can be exciting and beautiful. Yet, particularly in the case of calculus, it is all too easy, he believes, to destroy any pleasure of the subject by a tendency to teach mere rules and formulas. He also stresses in this connection that he wants to reach students who are capable of studying mathematics seriously, but who are not necessarily mathematical-types.

The third moment shows, in a more immediate way, Toeplitz's concern with the intellectual character of students. The difficulty here is that there are two different groups of students who are likely to take a first course in calculus. One group comes from the science-oriented Oberrealschule and the other from the humanist Gymnasia, and each is problematic in its own way. One can easily guess the advantages of the Oberrealschule students and the disadvantages of the Gymnasium students, but Toeplitz is also astute enough to recognize that the latter have something to offer and not all is well with the former. As we remarked above, Toeplitz himself had a firm humanist training, so he could well appreciate the value of an education obtained at the Gymnasia. The problem with the Oberrealschule students is precisely what others might see as an advantage, namely, their greater exposure to the technical side of calculus. Toeplitz points out that this too often and too easily prevents them from seeing that there is more to know, that knowing some techniques of calculus is not the same as knowing calculus. These are the greater challenge for Toeplitz.

It is in this last moment that students' previous high school education comes under consideration in a concrete and pointed way. This was not by chance. Toeplitz had very much in mind a set of reforms in German high school education involving the incorporation of the infinitesimal calculus into the mathematical syllabus of the gymnasia. Bringing the infinitesimal calculus into the gymnasium was the most important project among Felix Klein's educational initiatives, and in 1925 infinitesimal analysis finally became part of the official syllabus at Prussian Gynmasia.[6] This was just one year before Toeplitz's address, which is why Toeplitz, unable to hide his objections, refers to the teaching of calculus to high school students with a discernable edge, calling it a *fait accompli*. But so it was, and, therefore, it became part of the overall difficulty in setting up a first year calculus course at the university level.

Toeplitz suggestion is that all three moments and difficulties they identify can be addressed if one gives proper attention to the history of mathematics: if one takes the development of mathematical ideas as a guide for teaching, he claims, not only will the drama of that development be revealed to students but also the logic and interconnection of mathematical ideas themselves. This is the core of Toeplitz's "genetic method." However, there are two ways the method can actually be taken up. Toeplitz calls these the "direct genetic method" and the "indirect genetic method."

The "direct method" uses the historical development of mathematical ideas to inform the presentation of ideas in the classroom – it is a way of teaching. It answers the first moment, for example, not by bridging the divide between rigor and intuition but by providing a third alternative: the student arrives at mathematical ideas by following the same slow and gentle ascent by which the ideas themselves were arrived at historically. In other words, he does want to present rigorous ideas in a softened intuitive fashion, or to use intuitive notions as a springboard for jumping to rigorous formulations, but, rather, to allow the rigorous ideas to unfold for the students according to the very same gradual process that they themselves unfolded and, by doing so, not only bring the rigorous ideas into the classroom but also to show in a *natural* way why those ideas slowly became clear and necessary.

The "indirect method" is a way to analyze problems of teaching and difficulties in learning rather than a way of teaching itself: it is, as Toeplitz puts it, "...the elucidation of didactic difficulties, ...didactic diagnosis and therapy on the basis of historical analyses, where these historical analyses serve only to turn one's attention in the right direction." The qualification at the end is crucial: the application of the "indirect method" does not necessarily mean that history itself must be brought into the classroom. In the guise of the "indirect method," history takes on a role that might be compared to the psychology of learning: teachers use it to guide their teaching strategies and decision-making, but it is not what they teach their classes. Thus, Toeplitz argues, from the historical claim that knowledge of the definite integral was possessed by the Greeks or, at any rate, predated other ideas from the calculus, the study of the calculus for modern students ought to begin with the definite integral.

Toeplitz maintained that his "genetic method" was not unprecedented.[7] Felix Klein, he recalls, adopted the biogenetic law in his teaching already in 1911. In fact, this direction in Klein's thinking was evident more than a decade and a half earlier. In 1895, Klein delivered a talk at a public session of the Göttingen Royal Association of the Sciences on the "Arithmetization of Mathematics" (Klein 1895). At the end of his talk, Klein remarked about the teaching of mathematics. He said that, in his view, a paradoxical situation exists whereby teachers at gymnasia tend to stress "Anschauung" too much while professors at universities do so too little. More to the point, university professors put "Anschauung" aside completely whenever possible. Klein argued that at least the elementary courses meant to introduce the beginner to higher mathematics should take "Anschauung" as a starting point, "*...since on a small scale the learner always passes naturally through the same development which has been passed through by science on a large scale*". ("wird doch der Lernende naturgemäß im kleinen immer denselben Entwicklungsgang durchlaufen, den die Wissenschaft im großen gegangen ist"). Although Klein did not use the term "biogenetic law" explicitly, Klein's intention was obvious to Alfred Pringsheim (1850-1941). Thus, when he spoke at the German Mathematical Society in 1897, Pringsheim asked whether it is "suitable to transfer Häckel's principle of the concurrence between phylogenesis and ontogenesis in such an unrestricted way to teaching" (Pringsheim 1897, p. 74) as Klein had done in his talk of 1895. After a lengthy discussion involving mostly non-mathematical examples, Pringsheim concluded that "the principle referred to by Herr Klein as a principle of teaching [i.e. the biogenetic law] appears to be anything but conclusive and at least needs an examination from case to case" (Pringsheim 1897, p.75). With that, Pringsheim stated his own principle, namely "Every individual passes essentially through the same development as science itself as long as he is not shown a better way" (loc. cit.). It was to continue these discussions that Klein and Pringsheim were invited to give lectures on the issue of the university courses for beginners at the meeting of the following year 1898. Toeplitz refers precisely to these lectures right at the start of 1927 paper, forming a rhetorical if not real context for his own ideas.

THE BIOGENTIC LAW AND THE SCIENTIFIC APPROACH TO TEACHING WITH HISTORY

Even though Klein did not mention the biogenetic law by name in 1895, he was clearly an adherent to the doctrine and referred to it explicitly in the first decade of the century, when Toeplitz would have known him at Göttingen. In the appendix to his *Elementary Mathematics from an Advanced Standpoint* (Klein, 1908/1939), a book meant for mathematics teachers, Klein states the law and its implications in a way that shows why Toeplitz should find it so enticing as a principle for his own pedagogical strategy:

> From the standpoint of mathematical pedagogy, we must of course protest against putting such abstract and difficult things before the pupils too early [he is referring to the theory of sets]. In order to give precise expression to my own view on this point, I should like to bring forward the biogenetic fundamental law (*das biogenetische Grundgesetz*), according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. Such thoughts have become today part and parcel of the general culture of everybody. Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. Taking into account the native ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge. It is necessary to formulate this principle frequently, for there are always people who, after the fashion of the mediaeval scholastics, begin their instruction with the most general ideas, defending this method as the "only scientific one." And yet this justification is based on anything but truth. To instruct scientifically can only mean to induce the person to think scientifically, but by no means to confront him, from the beginning, with cold, scientifically polished systematics.
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> An essential obstacle to the spreading of such a natural and truly scientific method of instruction is the lack of historical knowledge which so often makes itself felt. In order to combat this, I have made a point of introducing historical remarks into my presentation. (p.268).

Klein's favorable view of the biogenetic law should not be viewed as eccentricity on his part, nor for that matter on the part of Toeplitz. The biogenetic law had a strong presence in their time and long history preceding it.[8] In biology itself, it was popularized by Ernst von Haeckel (1834-1919) and was generally identified with him, as in the quotation above from Pringsheim's 1897 talk. It is fairly clear, moreover, that it is Haeckel's formulation of the rule that "ontogeny recapitulates phylogeny," that the development of the individual organism follows the development of the species, which Klein and Toeplitz refer to as the biogenetic law. Haeckel of course did not invent the law; complete and consistent statements of it can be found already in the early part of the 19th century (see Gould, 1977 and Mayr, 1994 for critical accounts). And there were variations of the law, ranging from more respectable forms in which the individual development merely paralleled species development to less respectable forms in which the latter actually caused the former. But what is centrally important is the biogenetic law was, even when rejected, viewed as a scientific matter and a serious scientific hypothesis.

Alongside the "scientific" biogenetic law was a cultural version of the same idea, namely, that the *intellectual* development an individual person follows that of civilization. This of course had educational implications and was often taken up of educational theorists and practitioners. Thus, for example, in Froebel's 1826 *The Education of Man* we read, "Every human being who is attentive to his own development may thus recognize and study in himself the history of the development of the race to the point it may have reached, or to any fixed point" (Froebel, 2005, p.40). This view was held also by the followers of the influential Friedrich Herbart (1776-1841) such as Tuiskon Ziller (1817-1883) who called it the *Kulturstufentheorie*, or the *cultural epoch theory* (see Gould, 1977, pp.149ff). More importantly for us, Florian Cajori (1859-1930), the Swiss-American educator and historian of mathematics, refers again to the law as it comes down through educational thinkers in his *History of Elementary Mathematics with Hints on Methods of Teaching* (Cajori, 1896).Cajori opens the preface of that work with a quotation from Herbert Spencer in which the genetic principle is stated:

> "The education of the child must accord both in mode and arrangement with the education of mankind as considered historically; or, in other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race" [Cajori quoting Spencer]

Cajori then uses this to justify his own use of history of mathematics for mathematics teaching:

> If this principle, held also by Pestalozzi and Froebel, be correct, then it would seem as if the knowledge of the history of a science must be an effectual aid in teaching that science. Be this doctrine true or false, certainly the experience of many instructors establishes the importance of mathematical history in teaching (p.v).

Toeplitz, curiously enough, never refers to the "cultural epoch theory" or any other of these educational versions of the "biogenetic law." One might speculate that, unlike the "biogenetic law," Toeplitz might have found these other theories were somehow unscientific. Whether or not Toeplitz accepted the "biogenetic law" in the literal way Haeckel framed it, the scientific status of the "biogenetic law" would certainly provide his own genetic approach with a firm basis as Felix Klein seemed to think regarding his own pedagogic method – for Klein, recall, referred to his approach, in the passage quoted above, "a natural and truly scientific method of instruction." This "scientific" or “natural” rationale for the genetic approach, this historical oriented method of teaching mathematics, would also imply a view of history itself as something natural, like biology. Seeing history in this way would, implicitly, allow him to approach turns in history as developments that could be rationally reconstructed on solid ground without resorting to a kind of logical axiomatic structure.

It is surprising then that Toeplitz does not embrace this view of history with conviction and, rather, denies that what he is doing has anything at all to do with history. He says (p.94) that a historian must write down everything that happens, good or bad, while he is interested in only what has successfully entered into mathematics. He tells us that his is *not* a course in history: “Nothing could be further from me than to lecture about the history of infinitesimal calculus: I myself ran away from such a course when I was a student. It is not about *history*, but about the *genesis* of problems, facts, and proofs, about the decisive turning points within that genesis” [emphasis in the original].

But can Toeplitz really separate history from this genesis of problems, fact, and proofs? The very reason why taking the genesis of problems, facts, and proofs into account should help students is that it is natural, fitting to the students’ own ways of learning. Is it not for this reason that Toeplitz uses the medical language of “diagnosis and therapy” in describing how historical analysis is supposed to benefit teaching? But this is only possible if historical ideas are themselves somehow natural, just as diagnosis and therapy presuppose certain biological facts. The distinction Toeplitz makes between history of mathematics and the genesis of mathematical ideas and techniques is ultimately therefore an artificial one. Indeed, Toeplitz does not hesitate to make historical pronouncements as if they were indisputable facts – that “The Greeks discovered the definite integral” (p.96), that “The Dedekind cut is essentially in Euclid’s fifth book” (p.97), that “Barrow possessed differential and integral calculus in its entirety” (p.98).

The problematic separation between history and genesis presents itself with even more force when once realizes that Toeplitz makes what are really historical claims *on the basis of his genetic approach,* even while he denies it. Thus, having argued that the *relationship* between the definite integral on the one side and the differential calculus and indefinite integral on the other is what needs to be highlighted in the calculus course, he says, “You see clearly here the difference between genesis and history. Historians place the bitter priority quarrel between Newton and Leibniz in the foreground of the historical development of the differential calculus; from the genetic perspective, completely different moments form the central focus.” But even if these judgments of what should or should not be in the foreground are based on history via the “indirect genetic approach,” they must, nevertheless, have some *status* as history. One cannot just dismiss the issue by saying one judgment is for history and the other for education: parallelism is a symmetric relation.

So although he purports to be taking up purely educational questions, Toeplitz finds himself unavoidably, though perhaps unwittingly, adopting historical and historiographical positions – positions that, reflexively, are also perspectives on the historical character of mathematics itself. The jury may still be out on whether this is a result of Toeplitz's own idiosyncratic way of thinking or built into the genetic method itself; however, for anyone today who wishes to use the genetic method in teaching, the question whether what we see in Toeplitz is in fact an ineluctable tendency of the genetic approach must be confronted. For even if one intends only to adopt the genetic method in the form of "history as a tool," one may be forced to adopt the method in the form of "history as a goal" (Jankvist, 2009), but not the history that one intended.

In general, the ways historiography of mathematics and teaching of mathematics, even without an immediate concern for history, may be deeply entangled should, in our view, be given much greater attention both in historical and educational research. For the latter, the issue is particularly important since the introduction of history of mathematics into mathematics teaching is taken up all too often in a purely instrumental fashion with little cognizance given to what it means to look at mathematics historically in the first place.[9] Typically, it is not asked, for example, whether the ends mathematics education aims towards are necessarily in harmony with those pursued by the history of mathematics. Of course if the genetic principle, as Toeplitz understood it, were unproblematic then such questions would lose their force; but, if not – and if one is guided by the needs of teaching modern mathematics – then one would have to confront the difficulties of anachronism and its inevitable distortions of history.

NOTES

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[1] This paper has been adapted from the introduction to our translation of Toeplitz's 1927 paper soon to appear in *Science in Context* (Volume 28) under the title, "Otto Toeplitz's 'The problem of university courses on infinitesimal calculus and their demarcation from infinitesimal calculus in high schools' (1927)." Permission to use the latter was kindly given by Cambridge University Press who owns its copyright.

[2] In this regard, his relationship to the history of mathematics was similar to what one of us has called the relationship of a "privileged observer," that is, where modern mathematical knowledge is thought to provide special power in interpreting the past (see Fried, 2013)

[3] Klein (1968) mentions Toeplitz's historical work in two separate footnotes (notes 68 and 99).

[4] As for this, Uri Toeplitz, Otto Toeplitz's son, wrote in his autobiography that "In 1923, Karl Jaspers wrote a book, Die Idee der Universität, and dedicated it to my father. This demonstrates that even in Kiel father was no one-sided mathematician" (quoted in Purkert, 2012, p.111). Both Toeplitz and Behnke exchanged letters with Jaspers. These letters are analyzed in (Hartmann 2009, pp. 36-40).

[5] In 1938, on account of pressures of the Nazi government Toeplitz' name was removed from the title page of the "Semesterberichte" and in 1939 its last issue appeared. After the war, however, Behnke revived the Journal, and it still appears and is popular under the name of "Mathematische Semesterrichte".

[6] This itself has a background, for there was a long period of coexistence of algebraic analysis, based on Euler's *Introductio in analysin infinitorum*, and modern infinitesimal analysis in Prussian school mathematics, as discussed in Biermann & Jahnke (2013).

[7] Toepltiz does not have in mind here the "genetic method" to which Hilbert refers in, for example, his 1900 Über den Zahlbegriff (English translation, On the Concept of Number, in Ewald, 1996, pp.1089-1095). For Hilbert, the "genetic method" is an approach to defining numbers and other mathematical ideas in terms of more primitive concepts born in basic intuitions, for example, defining the real numbers in terms of a nexus moving through the natural numbers, integers, and rational numbers (see Corry, 1997, pp.125-130 and Ferreirós, 2007, pp. 218-222 for more about the "genetic method" in Hilbert's discussions about the foundations of arithmetic). That said, like the "genetic method" in Toeplitz's sense, the "genetic method" in Hilbert's sense stood in opposition to the axiomatic method, and, at the very end of Toeplitz's paper, Toeplitz emphasizes the difference between his "genetic method" and "Hilbert's foundational studies." If Toeplitz had in mind here works such as Hilbert's Über den Zahlbegriff, then he may have been contrasting his approach not only to Hilbert's formal ideas but also to Hilbert's notion of "genetic method." However, there is not enough evidence to make any firm claim in this connection.).

[8] A deep and thorough account of the genetic idea in mathematics education is Schubring (1978). As for the biogenetic law in biology itself, with its further ramifications, see Gould (1977).

[9] Both Jahnke and Fried have independently considered these questions in the context of mathematics education. See, for example, Jahnke (2000) and Fried (2001)

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