Oral Presentation

REFLECTING ON META-DISCURSIVE RULES THROUGH EPISODES FROM THE HISTORY OF MATRICES

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In this work, we present a teaching proposal about history of matrices. Our goal is to create conflictive situations in which students are encouraged to reflect upon their metadiscursive rules related to matrices, comparing them with those present in some historical writings. We have been based in the historical interpretation of Frédéric Brechenmacher and in Sfard's theory of Thinking as Communicating. The conceptual framework for using history in the teaching of mathematics was inspired by some works of Tinne Hoff Kjeldsen. We elaborated two teaching modules approaching two episodes of the history of matrices; the first has as protagonist the mathematician J. J. Sylvester and the second one has A. Cayley as protagonist. We discuss some of the results obtained in a pilot study in which the material was tested.

INTRODUCTION

Almost all Linear Algebra courses in Brazil, as well as the textbooks most often used, start with the concept of matrix as a stand-alone mathematical object. The definition is stated without reference to any problem in which the notion appears, immediately after the operations are introduced and their properties deduced in an abstract manner [1]. This sequence is thus seen as a goal in itself, no matter if it would be richer to develop further discussions about the nature or the origin of matrices and their operations.

As a consequence, when we ask students having finished their Linear Algebra courses why matrix multiplication is defined as the dot product between the rows of the first matrix and the columns of the second matrix, they generally cannot answer. The following quote shows the answer an school Mathematics teacher gave to this question (part of a questionnaire given at the beginning of our pilot study).

> Question: Imagine that a student asks you the following question during a class on matrices: "Why we have to multiply rows with columns in the matrix multiplication?" What would you answer?
>
> Answer: I would say that he should accept it as a truth. Unfortunately, this would be my answer. I wouldn't consider saying anything else.

The aim of our research is to create "conflictive situations" in which students are encouraged to reflect upon the rules that define their actions when dealing with matrices (metadiscursive rules), after comparing them with the rules that appear in some historical writings. The notion of *conflictive situation* is inspired by what Sfard calls *commognitive conflict* and the notion of *metadiscursive rule* is used here in the sense proposed by Sfard's theory of thinking as communicating (Sfard, 2008).

In the next section we explain the conceptual framework used in the research, largely inspired in the works of Tinne Hoff Kjeldsen (Kjeldsen, 2011; Kjeldsen & Blomhøj 2012; Kjeldsen & Petersen, 2014). These references made it possible to combine the historical approach we perceived as relevant to our goal with methodologies from the field of Mathematics Education.

Using historical sources about matrices, we developed teaching proposals in order to analyze the context of problems in which matrices appeared as a useful definition, so making it clear that this notion was not proposed immediately as a mathematical object. The historical work on matrices will be discussed in the third section of this paper.

Frédéric Brechenmacher (2006) is another important reference, who showed that the notion of matrix emerged and developed associated with concepts such as determinants, linear transformations and quadratic forms, to cite a few. Unlike the order in which these concepts appear in a Linear Algebra course nowadays, in history the notion of matrix was one of the last to appear. Moreover, the history of matrices shows that they come to light as a representation and their constitution as a mathematical object occurred along different mathematical practices. As Brechenmacher observed, the notion of matrix changed over time through different identities assigned to it within these mathematical practices.

The fourth section presents the pilot study carried out in the first semester of 2014 and we close the article with some initial conclusions.

CONCEPTUAL FRAMEWORK

Kjeldsen (2011) proposed a theoretical argument to integrate history in mathematics teaching based on Sfard's theory of mathematics as a discourse. According to Sfard (2008), mathematics is a well-defined form of communication or a type of discourse governed by certain rules. In this perspective, learning mathematics requires to take part in the mathematical discourse. In Sfard's words, it is necessary even for one's understanding of mathematics, since learning a mathematical discourse is "becoming able to have mathematical communication not only with others, but also with oneself" (Sfard, 2007, p. 575).

The rules that control the discourse are divided into two types: object-level rules and metadiscursive rules. The first concerns "narratives about regularities in the behavior of objects of the discourse" and the second one concerns "patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201).

In the mathematical discourse, object-level rules relate to the properties of mathematical objects. Examples include: (1) in Euclidean geometry, *the interior* *angles of a triangle always add up to 180°*, and (2) in algebra, *ab = ba, where a and b are real numbers*.

The metadiscursive rules (or metarules) concern the actions of the discussants. They are usually implicit in the discourse and manifest themselves when one judges, for example, *if a particular description can be regarded as a definition or if a proof can be accepted as correct*.

Metarules govern "when to do what and how to do it" (Sfard, 2008, p. 208). So, they affect the way in which participants of a discourse interpret its content. Learning of mathematics is thus the developing of appropriate metarules. On the other hand, as these rules are contingent and tacit (Sfard, 2008, p. 203, 206), participants do not observe them in a conscious and natural way. For this reason, it is unlikely that participants can learn metarules by themselves.

The term *metarule* in Sfard's approach is quite broad, including, for example, norms, values, and goals. It can also be used to designate repetitive patterns in different activities.

> (…) it is possible to talk about the metarules regulating participation (e.g., raising hands before speaking, working in groups), or metarules characterizing participants' intentions (e.g., genuinely engaging in mathematical activity versus acting to please the teacher), or the metarules regulating the object-level rules of mathematics (e.g., using the metaphor of motion to compute limits, using graphs to realize functions). (Güçler, 2013, p. 441)

In what concerns our particular subject of research, Kjeldsen argued that history of mathematics plays a fundamental role in order to "illuminate metadiscursive rules". These kind of rules are historically established and they may thus be treated as the object-level of a historical discourse. In this way, metadiscursive rules stop being tacit and can be made explicit objects of reflection (Kjeldsen, 2011).

The idea is then to promote situations in which students are encouraged to investigate the development of mathematical practices through historical sources and to understand the vision mathematicians had about their own practices. An approach of this kind can help the students to grasp how mathematicians conceived their objects of study and how they formulated their mathematical statements. Doing so, students can have contact with discourses governed by metarules that are different from the modern ones and different from their own metarules:

> (…) the historical texts can play the role as "interlocutors", as discussants acting according to metarules that are different than the ones that govern the discourse of our days mathematics and (maybe) of the students. (Kjeldsen, 2011, p. 52)

In the present research, we developed teaching and learning situations with the aim to clarify the metarules found in mathematical texts from the past, so the participants can compare them with their own metarules. The use of historical sources can thus lead to the situation that Sfard calls *commognitive conflict*, defined as "a situation in which communication is hindered by the fact that different discursants are acting according to different metarules" (Sfard, 2007, p. 576).

Guided by such a theoretical argument, Kjeldsen and Petersen (2014) implemented, in a Danish high school, an experimental course on the history of the function concept. In addition to using Sfard's theory of thinking as communicating, the course was also designed by using a multiple perspective approach to history (Kjeldsen, 2011) and the theories related to concept image, concept definition (Tall & Vinner, 1981) and concept formation (in the sense of Sfard, as cited in Kjeldsen & Petersen, 2014, p. 32). The researchers used extracts from primary sources written by Euler (1748) and Dirichlet (1837) in order to explore two metarules:

- *General validity of analysis*. This rule assumes that results, rules, techniques, and statements of analysis are generally valid.
- *Generality of the variable*. This rule states that a variable in a function can take on all values.

These two metarules were dominant in the analysis of 18th century and Euler assumed both of them in the definition of a function he presented in 1748. His definition considered a function of a variable quantity as an analytical expression composed in any manner from that variable quantity and numbers or constant quantities (Kjeldsen & Petersen, 2014, p. 37).

Afterwards, the students get in touch with Dirichlet's definition, which departs from metarules that are different from the ones Euler assumed. In this last case, a variable quantity was used to propose a definition of a function as a relation of dependence between variables, which is not necessarily given by one same law in the whole interval; and not thought of as relations that can be expressed by mathematical operations (Kjeldsen & Petersen, 2014, p. 37).

The goal is to make the conflict to emerge between the different metarules found in the historical texts, and also between these metarules and their own. Although our research is much inspired by Kjeldsen's theoretical argument it concerns a different mathematical subject. We prepared two teaching modules focusing on episodes in the history of matrices and selected three metarules we found appropriate to provoke a conflict about the way matrices were and are conceived. In the next section, we explain the historical content and the metarules that have been selected.

HISTORICAL PRACTICES ON MATRICES AND SOME OF ITS METARULES

Two research episodes about matrices were analyzed in order to investigate the different roles that the notion of matrix acquired within two practices developed in the 1850s by the mathematicians James Joseph Sylvester and Arthur Cayley (Bernardes, 2012). The historical discussion of these works is based on the interpretation suggested by Frédéric Brechenmacher (Bechenmacher, 2006).

Sylvester introduced the word “matrix” in his research about the classification of the types of contacts between two conics. In this context, matrices were conceived as a means of representation. This role changes in Cayley’s research. In the memoir published in 1858 (Cayley, 1858), matrices offered a new language in which known problems could be treated differently and new problems could be proposed. Moreover, Cayley established the rules for operations with matrices.

In 1850 the British mathematician James Joseph Sylvester published a memoir in *The Cambridge and Dublin Mathematical Journal*, (Sylvester, 1850a) addressing one problem of a geometric nature: the classification of the types of contact between two conics. The term “contact” was used to denote an intersection point in which the two conics are tangent to each other.

The main mathematical tool used by Sylvester in order to solve the contact problem was the notion of determinant. However, he did not compute determinants of matrices, this last notion was introduced later.

In order to classify the type of contact between two conics, Sylvester analyzed the multiplicity of the roots of the equation $\det(U + \mu V) = 0$, U and V being homogeneous quadratic equations in three variables that represent the conics. To let it clear:

$$U : ax^2 + by^2 + cz^2 + 2a'xy + 2b'xz + 2c'yz = 0$$

$$V : \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' xy + 2\beta' xz + 2\gamma' yz = 0,$$

and the coefficients are real numbers. The equality $\det(U + \mu V) = 0$ yields a cubic polynomial equation [2].

In the articles concerning the contact problem [3], Sylvester computed determinants of (homogeneous) polynomials functions. This was a recurrent procedure in his practice and sometimes he also used auxiliary tables, in which the entries were functions of the coefficients of the conic-defining equations.

The analysis of these works motivated us to identify a metarule underlying Sylvester's practice: *determinants were tools computed from functions (homogeneous polynomials) and were useful in the investigation of properties of curves and surfaces*.

We can note immediately a huge difference between this metarule and ours, since nowadays in linear algebra determinants are defined by means of (square) matrices and seen as a property depending on these mathematical objects.

Returning to Sylvester’s practice, analyzing the multiplicity of the roots of the equation $\det(U + \mu V) = 0$ was not sufficient to classify all four kinds of contact. In the case of multiplicity two or three, there are two kinds of contact, as illustrated by the examples in Figures 1 and 2.

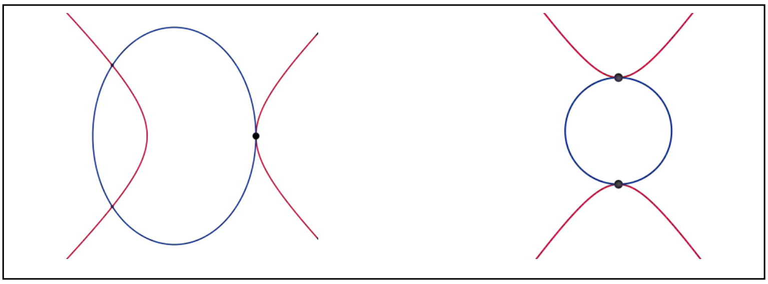

Figure 1: Simple (left) and diploidal (right) contact.

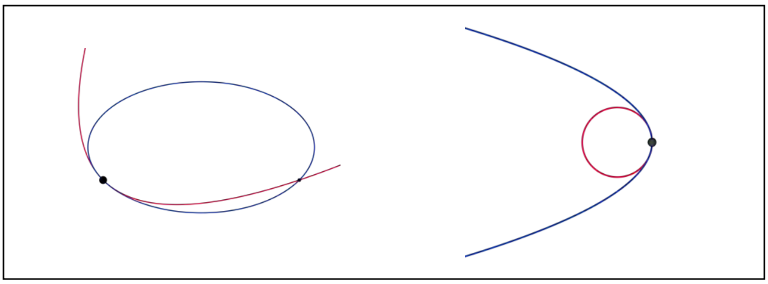

Figure 2: Proximal (left) and confluent (right) contact.

The types of contacts may be distinguished by studying the multiplicity of the intersection points in which the conics are tangent (the black dots in Figures 1 and 2). In the situation of a simple contact, there is one double intersection point (Figure 1, left); in a diploidal contact, there are two double intersection points (Figure 1, right), in a proximal contact, there is one triple intersection point (Figure 2, left); and in a confluent contact, there is one quadruple intersection point (Figure 2, right).

So, in order to solve the contact problem, Sylvester introduced the notion of minor determinants and developed a technique consisting of extracting systems of minor determinants from the complete determinant.

The term "matrix" was introduced in this context and with the goal of generalizing a property of minor determinants.

> (…) we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p and selecting at will p lines and p columns, the square corresponding to which we may be termed determinants of the pth order. (Sylvester, 1850b, p. 369)

In this quote Sylvester makes explicit his understanding of a matrix as a source of minor determinants, concisely called by Brechenmacher as "*mère de mineurs*" (2006, p. 15). This understanding was reinforced in another article:

> I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent (…). (Sylvester, 1851b, p. 302)

We thus propose there is a second metarule concerning matrices that underlies these works of Sylvester: *matrix is a mother of minors*. This metadiscursive rule is expressed in Sylvester's idea of a matrix as a representation from which systems of minor determinants can be generated. Before stating the next metarule that guided our work, we need to describe briefly another actor who was important in this research.

Eight years after the introduction of a matrix by Sylvester, his friend Arthur Cayley published a memoir in which he defined the matrix operations and stated their properties (Cayley, 1858). According to Cayley, matrices arise naturally from "an abbreviated notation" for linear systems. Consequently, he defined matrix operations from similar operations possible to be accomplished with linear systems.

In the first page of the article, Cayley makes an analogy of matrices with simple quantities (numbers):

> (…) It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities; they may added, multiplied or compounded together (…). (Cayley, 1858, p.17)

This analogy pushed him to consider a certain type of matrix as a simple quantity:

$$\begin{array}{l} \quad(m, \quad 0, \quad 0) \\ m = |\,0, \quad m, \quad 0\,|\,, \\ \quad |\,0, \quad 0, \quad m\,| \end{array}$$

> The matrix on the right-hand side is said to be the single quantity m considered as *involving the matrix unity*. (Cayley, 1858, p. 20, italics in the original)

Cayley developed a practice of computation with matrices based on a dual interpretation of a matrix: either as a system of numbers and as a number (Brechenmacher, 2006, p. 20). This duality is expressed in the statement of his "remarkable theorem", announced in the first page of the memoir:

> 21. The general theorem before referred to will be best understood by a complete development of a particular case. Imagine a matrix
>
> $$\begin{array}{l} M = (a, \quad b), \\ \qquad |c, \quad d| \end{array}$$
>
> and form the determinant
>
> $$\begin{vmatrix} a-M, & b \\ c, & d-M \end{vmatrix},$$
>
> the developed expression of this determinant is
>
> $$M^2-(a+d)M^1+(ad-bc)M^0;$$
>
> (…) and substituting these values the determinant becomes equal to the matrix zero, (…). (Cayley, 1858, p. 23)

Cayley explains afterwards that:

$$\begin{array}{ll}(a-M, & b\ \ ) \\ |c, & d-M| \end{array} = \begin{array}{ll}(a & b) \\ |c, & d|\end{array} - M\begin{array}{ll}(1, & 0) \\ |0, & 1|\end{array}$$

is the “original matrix, decreased by the same matrix considered as a simple quantity involving the matrix unity” (Cayley, 1858, p. 24).

Relying on his dual interpretation about what a matrix is in Cayley’s works, we state a third metarule: *dual interpretation of a matrix.* A matrix was interpreted either as a system of numbers or as a number.

The three metarules that we have defined were explored in two teaching modules in which we presented, in an abbreviated manner, the works of Sylvester and Cayley. Original excerpts are used as much as possible, but sometimes we inserted text to make the links between parts of the text that we chose as the most relevant to our goal. We describe in the next section how these teaching modules were tested in a pilot study.

THE PILOT STUDY

We carried out an experiment in a pilot study with the goal of testing the teaching modules. We offered a mini-course for six volunteers called “Different roles of the notion of matrix in two episodes of the history of matrices”. The mini-course was taught by the first author of this paper and the meetings took place on two Saturdays, lasting about five hours each.

The mini-course students were school Mathematics teachers, ranging from 6^{th} to 12^{th} grades (corresponding to students aged 11 through 17). He time of experience of the teachers varied from 3 to 12 years, and they were all taking a Linear Algebra graduate course as part of the requirements of a professional master’s degree in Mathematics, offered for teachers currently teaching in the public system. In the quotes bellow, the participants will be identified by the letters M, T, Fa, Fe and J. During the meetings, they worked in groups in order to answer the historical activities proposed in the two teaching modules. Our data sources were: 1) audio recordings of the groups’ discussions; 2) written answers to the activities; 3) a summary in written form explaining what they learned in each module; 4) two questionnaires, one filled before the first meeting and the other after the final meeting.

The goal of the first questionnaire (Figure 3) was to understand the profile of the participants and to get a glimpse of how they were learning matrices in their Linear Algebra course.

1. From what institution did you earn your Bachelor´s degree? When did you finish it?
2. How long have you worked as a Mathematics teacher?

3. Tell us about the Linear Algebra courses you took, at both undergraduate and graduate levels: How was the subject taught? Did you enjoy it? How hard was it?
4. Tell us about the teaching of matrices in the Linear Algebra courses mentioned above: Were matrices the first topic taught? Did it make sense to you to learn about matrices and their operations and properties?
5. Imagine that a student asks you the following question during a class on matrices: "Why do we have to multiply rows with columns in the matrix multiplication?" What would you answer?
6. Have you ever had a course in History of Mathematics?
7. Do you think it is important to learn about the history of Mathematics?
8. Do you think that mathematical notions change over time? Explain your position.

Figure 3: Questionnaire answered by the participants before the first meeting.

Based on the answers, we conclude that matrices were taught using the approach we mentioned in the introduction. Nobody answered properly the Question 5, about the definition of matrix multiplication. It seems that most of the teachers themselves did not know the reason for the rule:

> I would say that matrix multiplication is defined in that way. Each element of the matrix is determined through the inner product of a line by a column […] I would try to convince them that this theory is grounded in a higher Mathematics […] (Participant Fa, first questionnaire)

Two participants had not studied history of mathematics in the university, but this was neither a pre-requisite for the mini-course, nor did it prove to be a problem. In the last question (Question 8), two participants expressed their opinion saying that mathematical notions do not change over time but they admitted that something can change as, for instance, the way we teach the concepts, our views, etc.

> The notions did not change much over time, but they are no longer addressed in a mechanical way. Context plays an increasingly important role and the topics become closer to everyday life. (Participant J, first questionnaire)

In the questionnaire given at the end of the mini-course we asked them to write a short essay expressing their views and opinions about the study.

The teaching modules

Two teaching modules were elaborated with the following learning objectives:

i. Making participants reflect on their own metadiscursive rules when the matrix notion is at stake, by comparing them with the ones we observed in the historical writings, and

ii. Developing historical awareness about the meanings attributed to the matrix by Sylvester and Cayley.

It was not our goal to use the history of mathematics to introduce the concept of matrix or to teach linear algebra. We selected students who had already taken a first course in linear algebra and had learned about matrices.

The first teaching module was entitled "How matrices appeared in the study of conics by Sylvester". We introduced the geometric context in which the term matrix was proposed by Sylvester and explained how he solved the problem of the classification of types of contacts between two conics using determinants.

Some concepts from projective geometry were necessary, like homogeneous coordinates, projective points, projective lines, and projective conics. After introducing these notions, we presented a summary of the practice developed by Sylvester in order to solve the problem of contacts.

In the end, the students had to discuss historical questions in groups. The goal of this first block of activities was to raise a discussion among the students concerning the metarules we defined and, hence, to promote a reflection about their own metarules related to matrices. We list the activities proposed in this first module in Figure 4.

1. What is the main concept used in Sylvester's practice? Summarize how Sylvester classified the types of contacts between two conics U and V.

2. Describe the difference between how Sylvester used determinants and how we use it today.

3. Explain what a first minor determinant is according to the definition presented by Sylvester in Extract I. What is a second minor determinant? Finally, what is a minor determinant of order r?

4. Why Sylvester had to introduce the minor determinants?

5. Based on Extracts II and III, explain what a matrix was and what the role of this notion to Sylvester was.

6. Compare the definition of matrix presented by Sylvester in Extract II to the definition that is used nowadays. Write at least one similarity and at least one difference.

7. According to the text and Extract II, answer why or for what purpose Sylvester introduced the term matrix.

Figure 4: Activities proposed in the first teaching module.

The second teaching module was entitled "Cayley and the symbolic calculus with matrices". We started by giving a translation of one part of the 1858 memoir. In a second session, historical activities were proposed in order to give the opportunity for the students to reflect about the metarules.

Partial results: Discussion about metarules

The metarules selected in the historical works were explored in the teaching modules through specific questions. An explanation about the way Sylvester solved the problem of contacts, as well extracts of his articles and selected parts of Cayley's memoir were essential to support the discussion. As Kjeldsen (2011) affirms, concerning historical texts, the primary sources played the role of "interlocutors" or "discussants" acting according to certain metarules – different from our own. In the next paragraphs, we present excerpts of discussions that emerged from the metarules extracted from the works of Sylvester and Cayley.

Sylvester's conception of a matrix as the mother of minor determinants caused a bit strangeness in the participants. The transcript below is part of a dialogue that a group had when discussing the role of the matrices in Sylvester's work:

> M: From the womb of a common parent (reading Extract III) (astonishment) Jesus! (laughs) […] I think he sees it, then. In fact, the matrix is a way to organize determinants. So [...] the main thing is not the matrix, it is the determinant.
>
> M: Sylvester, he just thought in squares before. Only after he saw it was not exactly like this, right?
>
> T: I think he saw that there (matrix) should [...] solve a system.
>
> M: Yeah, after he formed the matrix. Then he did the opposite. Indeed, the matrix for him was a way to keep information. The main information: determinant. (Group discussion, first meeting)

The speech of participant M shows a conflict with the conception of a matrix as a representation, from which the minor determinants could be generated, or in other words, as a source to keep information about determinants. This idea places the determinant as the main object and emphasizes the order of development of these concepts. This contrasts with the understanding of the participant M. For him, the notion of matrices come first and then the notion of determinants (defined and computed by means of matrices).

All participants read and discussed the initial pages of Cayley's memoir (translated from English to Portuguese) together. From this activity, they became acquainted with Cayley's motivation to introduce matrix operations. In particular, they realized the origin of matrix multiplication as a composition of linear transformations.

The quote below, taken from one report, shows that some participants noticed the association of matrices with linear systems made by Cayley. This was important in the way the operations (matrix addition, matrix multiplication by a number and matrix multiplication) were defined.

> Motivated by a simpler representation of sets of linear equations, it comes to light naturally the notion of matrices. The difference [between Cayley's matrix description and the modern definition of matrices] is in Cayley's double interpretation of the matrix,

> sometimes he sees matrices like numbers. (Requested report from participants Fe, M, and J)

The issue that gave rise to the commentary above was the difference between the description of matrix presented in Cayley's memoir and the modern definition. The trio of participants F, M, and J realized that the dual interpretation of matrices determines the difference between Cayley's conception of a matrix and their own.

When requested to judge if Cayley's proof furnished for the "remarkable theorem" would be accepted as correct today, the participants F, M, and J expressed their metarule, which they saw as being in accordance with the mathematical community.

The "remarkable theorem" states that any matrix satisfies an algebraic equation of its own order. In the proof, Cayley wrote the following determinant:

$$\begin{vmatrix} a-M, & b \\ c, & d-M \end{vmatrix},$$

where M is the following matrix:

$$M = (a, \quad b), \\ \quad |c, \quad d|$$

The quote below shows the response of F, M, and J to the question about the validity of the proof:

> As this proof is constructed to the particular case of matrix order 2, it would not be accepted today since that, in order to prove a theorem, you should use order "n". (Requested report from participants Fe, M, and J).

The participants argued that the proof should be made for matrices of order *n*. They expressed a metarule that is in accordance with the mathematical community. On the other hand, it seemed to not bother them that Cayley considered a symbolic computation involving a matrix *M* and numbers (the elements on the diagonal). Cayley justified his argument using the dual interpretation of the matrix either as a number or as a system of numbers, but his proof would not be accepted in the mathematical community today.

INITIAL CONCLUSIONS

Our purpose in this work was to use primary and secondary sources about the history of matrices in order to encourage participants in a pilot study to reflect upon their own metarules related to matrices, comparing these rules with those found in the historical writings. In this sense, we intended to create *conflictive situations*, in a sense similar to that of *commognitive conflicts* that Sfard proposes. Two teaching modules were developed based on two episodes in the history of matrices. They were implemented in a pilot study with six school Mathematics teachers.

The analysis of the results shows that during the discussions about the metarules appearing in the sources, the participants problematized their own metarules. The historical sources, treated through specific activities, made the participants elucidate the metarules they had in mind, thus confirming Kjeldsen's theoretical argument emphasizing the role of history as a strategy to make metarules become explicit and to convert them in objects of reflection (Kjeldsen, 2011).

The goal to develop historical awareness was reached in particular cases. One example is the observation that Sylvester used determinants before matrices were introduced, which made the participants notice the difference between the order in which matrices and determinants are presented today and the historical order in which these notions were developed. In addition, the study of Cayley's memoir of 1858 showed some motivations for defining matrix operations, in particular, the special way to define matrix multiplication.

The reflections on metarules also provided a perspective for the participants to reflect on the basic curriculum, regarding the topics of matrices, determinants and linear systems. They even discussed the ways in which matrices are treated at a basic level. One participant observed that:

> It was very interesting to know that the concept of matrix came from very different ideas of what is taught in schools today. What, moreover, allows us to take a more critical look at the math curriculum in high schools. (Participant Fa, final questionnaire)

We will continue this research by implementing additional activities and analyzing the discourses of participants while reflecting about their own metarules. The history of mathematics has proven to be an interesting way to create an environment for the participants to perceive the metarules they use and that they consider as being the right way to do mathematics.

NOTES

1. There are some different approaches. Stormowski (2008) proposed the teaching of matrices from the linear transformations in basic education. Cabral and Goldfeld (2012) presented matrices together with the topics systems of linear equations and linear transformations in their textbook for linear algebra courses.

2. For details, see Brechenmacher (2006).

3. Sylvester's research episode about the problem of the types of contacts between two conics was based on four articles (Sylvester 1850a, 1850b, 1851a, 1851b).

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