MATHEMATICS AS A TOOL-DRIVEN PRACTICE: THE USE OF MATERIAL AND CONCEPTUAL ARTEFACTS IN MATHEMATICS

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In this paper we introduce the concept of cognitive artefact and show how such artefacts are used in mathematical activities. By analysing different instances of artefact use we argue that our use of cognitive artefacts can lead to (at least) three different types of qualitative shifts in our mathematical capacity. Cognitive artefacts may allow: 1) expansions of practices in otherwise impossible ways, 2) extensions of mathematical domain, and 3) creative mediation of different mathematical areas. We argue that the use of cognitive artefacts – and in 2) and 3) – the choice in artefacts influence the development and content matter of mathematics. Our analysis of the role played by cognitive artefacts shows that mathematics is essentially a tool driven practice. We close the paper by discussing consequences of this realization for the choices we face concerning the introduction of CAS-tools in mathematics education.

INTRODUCTION

The computer has made its entry into mathematics teaching and learning – which has created heated debates with very strong opinions for and against. This paper is not a part of this debate, at least not directly. We are not addressing the advantages and disadvantages of computer assisted teaching and learning of mathematics – as a matter of fact, we are not discussing computers at all. Rather, we take this debate as an opportunity to shift the focus from the computer as such to the use of tools in mathematics in general, to move beyond the “good”/”bad” discussion of computers and instead ask: What can we learn about mathematics if we view mathematics as a tool-driven practice in research and in every-day (or practical) mathematics?

The concept of cognitive artefacts has drawn a lot of attention in contemporary cognitive science (see Heersmink 2013 for an overview and Hoyles & Noss 2009 for some educational implications), and in this paper we are using this concept to explore how tools affect the development of mathematics. In the following we introduce the concept of cognitive artefacts, and we use it to analyse four concrete pieces or episodes in ancient and modern mathematics in order to explore and pinpoint different ways in which mathematics can be viewed as driven by tools. We identify three kinds of qualitative shifts in these pieces of mathematics that are due to the use of such tools. We will close the paper by discussing what implications the perspective offered by the concept of cognitive artefacts could have on mathematics education.

WHAT ARE COGNITIVE ARTEFACTS?

A cognitive artefact is a human made object that is used to aid, improve or enhance human cognition (cf. Hutchins, 2001, p. 126). Typical examples of cognitive artefacts include shopping lists, calendars, address books and GPS navigation devices. Such tools allow us to think better, more reliably or with less effort (cf. Kirsh & Maglio, 1994). They do so not by enhancing our mental capacity, but rather by changing the cognitive landscape and offer new and cognitively less expensive ways of solving a given task.

In parts of the literature cognitive artefacts are exclusively associated with physical objects (e.g. Hutchins, 2001), while other theorists operate with a more inclusive definition where conceptual artefacts such as procedures, rules and certain concepts are also accepted as cognitive artefacts (e.g. Norman, 1993, p. 4). In this paper we will use the concept in this last more inclusive sense. This choice is motivated by several observations. Firstly, as also noticed by Norman, algorithms and rules of thump are clearly human creations, they are artefacts, and they can in some cases play the same role in human cognition as physical cognitive artefacts, i.e. they aid, improve or enhance our thinking. Secondly, in many – if not most – cases the physical device taken in isolation is not enough to accomplish the given cognitive task. You will also need to know certain algorithms or rules for operating the device. Thus it is natural to include the conceptual artefacts in the totality of resources needed in order to accomplish the task. Lastly, in some cases the physical part of the artefact can even be internalised. The alphabet for instance can be seen as a cognitive artefact that is used to reduce the cost of search operations; if the books in the library were not alphabetized it would be much harder to find the one you need. However, whether you carry a piece of paper with the alphabet written down or have memorized the alphabet is not important. In both cases you use the same artefact.

In the following we will describe how cognitive artefacts are used in mathematics and identify three different ways in which artefacts have led to qualitative shifts in our ability to perform mathematical cognition.

EXPANDING THE GIVEN

The first claim we wish to make in this paper is that mathematics is essentially a tool-driven activity. Over the last two decades cognitive science has shown that humans and several other species of animals have an inborn ability to solve tasks we would describe as mathematics. In short, we can do basic arithmetic on sets with less than four elements, and we can judge the approximate size of larger sets (Feigenson et.al. 2004; see also Johansen 2010, pp. 49 for discussion). Our inborn abilities however do not allow us to do anything more than that. So if we want to find out what 5+6 is or judge whether a set contains 9 or 10 elements, we have to use qualitatively different cognitive abilities and strategies (Núñez 2009).

The limits of our inborn abilities were effectively demonstrated in a study on members of the Amazonian Pirahã tribe (Frank, Everett, Fedorenko, & Gibson 2008). This tribe is especially interesting in this context because their language does not contain number words, and consequently the Pirahã does not have access to the technology of counting. In the study a subject was shown a small number of objects and was asked to match the sample by placing a similar number of objects on a table. In test conditions where the sample was hidden the performance of the subjects decreased as the size of the sample increased; with a sample size of four objects most subjects were able to match the sample correctly, but with a sample size of ten objects most subjects would fail the test. In a follow-up study similar results were obtained with participants from Boston who were deprived the ability to count (Frank, Fedorenko & Gibson 2008).

Tests such as these show that normal adult humans cannot perform simple tasks such as matching a hidden sample of ten objects without cognitive support. We simply have to use some kind of tool in order to solve this task. One of the tools that can be used in this respect is counting. Counting involves a large amount of highly complex cognitive mechanisms, such as the ability to group objects in certain ways, but first and foremost it involves a counting sequence, such as the sequence of words “one”, “two”, “three” etc. In our analysis a counting sequence is a clear example of a conceptual cognitive artefact.

From a mathematical point of view the example might be banal, but there is a more general lesson to be learned from it. Our ability to think – also mathematically – is determined by the cognitive context we are positioned in, that is: by the cognitive artefacts and other cognitive support available to us. An Amazonian Indian cannot suddenly begin to count, even if she wants to and even though she has the cognitive hardware (so to speak) needed in order to do so. It is simply not within her cognitive reach. The introduction of counting thus constitutes a radical change in our cognitive landscape. With access to counting we can perform tasks that are impossible for us to do without. Counting allow us to expand our inborn ability to handle the size of sets with digital precision. Without counting (or similar techniques) we can handle sets with 1 to 4 elements, but with counting we can handle larger sets with the same degree of precision.

A similar story can be told about basic arithmetic. We seem to have an inborn ability to do addition and subtraction, but only on small sets. With the introduction of the proper cognitive artefacts these abilities can be expanded so as to be applicable to sets of arbitrary size. In this case the proper artefacts could be conceptual artefacts such as rules and algorithms or tables of basic products, but also physical artefacts such as the abacus, counting boards or representational systems that allow basic calculations to be performed (see e.g. Menninger 1992, pp. 299 and Johansen & Misfeldt 2015 for examples and analysis). It is not our ambition at this place to provide historical analysis or account for the genealogy of counting or arithmetic. The fact that we use tools is not due to historical contingencies. It is due to the cognitive conditions we face as human beings; without cognitive support our mathematical abilities are extremely limited. The kind of tools and cognitive artefacts we use is however a result of historical development and below we will provide historical case studies illustrating the importance of such developments.

CHOICE MATTERS

In this section we will expand our analysis by showing some of the roles cognitive artefacts play in academic mathematics and by illustrating why the choice of cognitive artefact matters.

We will begin by looking at Proposition 18 from Book V in Euclid's *The Elements*. The proposition is stated and explained in the following way:

> **Proposition 18**
>
> *If magnitudes be proportional* separando*, they will also be proportional* componendo.
>
> Let *AE*, *EB*, *CF*, *FD* be magnitudes proportional *separando*, so that, as *AE* is to *EB*, so is *CF* to *FD*; I say that they will also be proportional *componendo*, that is, as *AB* is to *BE*, so is *CD* to *FD* (Heath, 2006, p. 427).

Even with this explanation it might be difficult to understand the exact content of the theorem. In Heath's translation the reader is offered cognitive support in form of the following diagram (here, slightly simplified):

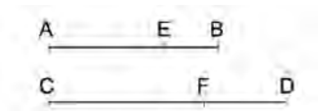

Figure 1: Diagram representing Euclid V.18

In fact, there are diagrams (or rather: figures) like this on almost every page of Heath's translation. This is puzzling in the sense that Euclid carefully describes all of the needed constructions in the text. So why has Heath included the figures in the book? They add nothing to the content of the text and thus seem completely superfluous.

In order to answer this we must turn to the cognitive role such visual representations play. Of course we could read the text and imagine the appropriate figure in our mind's eye. It would however take a considerable effort – even in simple cases such as the above. Our short-term memory is very limited and not completely reliable, so from a cognitive point of view it makes sense to off-load some of the cognitive work to a material object, in this case: a figure drawn on paper. The figure is in other words a highly specialised cognitive artefact. To introduce a more precise concept, we can say that in this case the artefact has an *anchoring* role for our cognition (Hutchins 2005).

The conceptual structure we need to build in order to understand the content of Euclid's theorem is anchored in the physical drawing. The anchor keeps the general structure stable and allow us to focus on and manipulate local parts of the structure; we can for instance imagine what would happen if we moved the point *E* or we could add new elements to the drawing (as Euclid actually does in the proof of the theorem). In this case the anchor seems to be a fairly natural depiction of the content it anchors; it simply represents magnitudes as line segments.

We will not go through the details of the proof here and the reader does not need to understand it in details, but we will nevertheless include the proof in full in order for the reader to form an impression of the cognitive workload it would take to actually understand and read the proof. In other words, we want to prove a point, not a theorem. This being said, the proof goes like this:

> For, if *CD* be not to *DF* as *AB* to *BE*, then, as *AB* is to *BE*, so will *CD* be either to some magnitude less than *DF* or to a greater. First, let it be in that ratio to a less magnitude *DG*. Then, since, as *AB* is to *BE*, so is *CD* to *DG*, they are magnitudes proportional *componendo*, so that they will also be proportional *separando*. Therefore, as *AE* is to *EB*, so is *CF* to *FD*. But also, by hypothesis, as *AE* is to *EB*, so is *CF* to *FD*. Therefore also, as *CG* is to *GD, so is CF* to *FD*. But the first *CG* is greater than the third *CF*; therefore the second *GD* is also greater than the fourth *FD*. But it is also less: which is impossible. Therefore as *AB* is to *BE* so is not *CD* to a less magnitude than *FD*. Similarly we can prove that neither is it in that ratio to a greater: it is therefore in that ratio to *FD* itself. Therefore, etc. (Heath 2006, p. 427).

As we can see, even with the cognitive support offered by the diagram in figure 1, it would take a considerably effort to follow the proof. As it is, Heath gives us a hint to another way to attack the problem. He translates the problem to algebraic symbols. In this representation the theorem states that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{b}=\frac{c+d}{d}$. Once the theorem is stated in this way, its proof is no more than a simple calculation:

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}+1=\frac{c}{d}+1$$

$$\frac{a}{b}+\frac{b}{b}=\frac{c}{d}+\frac{d}{d}$$

$$\frac{a+b}{b}=\frac{c+d}{d}$$

Here, we use another cognitive artefact; abstract symbols. Contrary to the Euclidian proof we do not need to consider the content of the operations we perform. We just need to know a few fully formal rules that tell us how we are – and how we are not –

allowed to operate on the symbols. In other words, the artefact allows us to externalise the problem and solve it as a series of physical actions.

This example shows that different artefacts have different affordances. A diagram such as figure 1 offers a qualitatively different type of cognitive support than algebraic symbols, and tasks that might be difficult to perform using only the figure might be relatively easy to perform when using algebraic symbols (and vice versa). Thus, cognitive artefacts are not just cognitive artefacts. Different artefacts shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts one have access to. What one can do – and maybe even what one can think – is determined by the cognitive artefacts one has access to.

ARTEFACTS AND THE DEVELOPMENT OF MATHEMATICS

We should keep in mind that cognitive artefacts are artefacts; they were not always around, but were developed by humans. Furthermore, as we argued in the second section, cognitive artefacts are necessary in order to do more than rudimentary mathematics. However, with the example analysed in the previous section it can also be asked whether the introduction of new cognitive artefacts into the mathematical practice can change the cognitive landscape in such a way that it not only allows us to expand our given abilities or to do something well-known more easily, but also allows us to perform qualitatively new tasks. In other words: Can the introduction of new cognitive artefacts lead to qualitative changes in the content matter of mathematics?

In this section we will discuss the possible connection between the development of new cognitive artefacts and developments of mathematics by analysing two cases: Cardano's introduction of complex numbers [1] and Minkowski's use of n-dimensional lattices. The first case involves relatively simple mathematics and is relatively distant in time, whereas the second case involves advanced mathematics and describes a relatively recent development.

Cardano and the complex numbers

In *Ars Magna* (1545) Cardano considered several problems of the type: Divide a given number into two parts such that the product of the parts is equal to another given number. In one of the cases he considered how to divide ten into two parts such that their product is 40 (Cardano 2007, p. 219). This type of problems has been known since antiquity and in Euclid's *The Elements* we have an algorithm that makes it possible to construct solutions geometrically in special cases (Proposition VI.28). The Euclidian algorithm however can only be applied if the square of half of the given number is greater than or equal to the given product (this is explicitly stated as a condition to the theorem (Heath, 2006, p. 518)). In this case the square of half the number is 25 and the given product is 40, so the condition is not fulfilled, and Cardano began his treatment by stating that "it is clear that this case is impossible" (Cardano 2007, p. 219). Nevertheless, Cardano pressed on and applies the Euclidian algorithm (or a version hereof). He constructs the square of half of the given line and represented the result geometrically, as seen in figure 2.

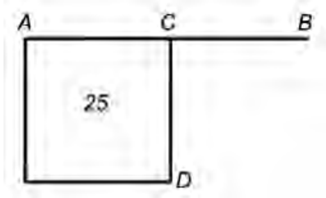

Figure 2: Drawing from Cardano (redrawn). The given line is represented as the line segment AB

As the next step the algorithm requires us to subtract the given area from the square of the given line and to find the square root of the result. In this case we will have to subtract 40 from 25 and construct the square root of the resultant area. This cannot be done geometrically – hence the condition in Euclid's proposition. Cardano responded to this problem by abandoning the geometric interpretation and representation of the situation. He simply replaced the geometric representation with abstract algebraic symbols, and then carried through with the rest of the steps in the algorithm interpreted not as geometric constructions, but as algebraic operations. This led him to the conclusion that the problem has the solutions $5+\sqrt{-15}$ and $5-\sqrt{-15}$, as the sum of these numbers are 10 while their product is 40 [2].

Solutions such as those found by Cardano cannot be found or even seen as long as one is using an algorithm based on a geometric interpretation of the situation. One cannot represent negative areas geometrically and hence from a geometrical point of view it does not make sense to subtract a larger area from a smaller one or to construct the square root of the resultant (negative) area. From an algebraic point of view the situation is different. With the proper representational system in place one can represent the square root of -15 just as well as one can represent the square root of 15 (although we might not be able to evaluate the former or understand it as a constructable geometric object, as Cardano was well aware. It was merely ink on paper, so to speak). In other words, the algebraic symbols used by Cardano allowed him to anchor and thus introduce and operate on a class of objects (square roots of negative numbers) that could not be anchored in the traditional geometrical representations. So in this case the development of a particular cognitive artefact (algebraic symbols) allowed a qualitative shift in the content of the mathematics Cardano was able to develop and work with (c.f. De Cruz & De Smedt 2013).

Minkowski lattice – An artefact in geometry of numbers?

Our final example is an episode in the history of modern mathematics regarding the German mathematician Hermann Minkowski's development of geometry of numbers and the concept of a general convex body. Before we enter into the mathematical details, we introduce a methodological triangle (figure 3) [3] that displays the relation between the historian, the materials/historical artefacts and the historical actors.

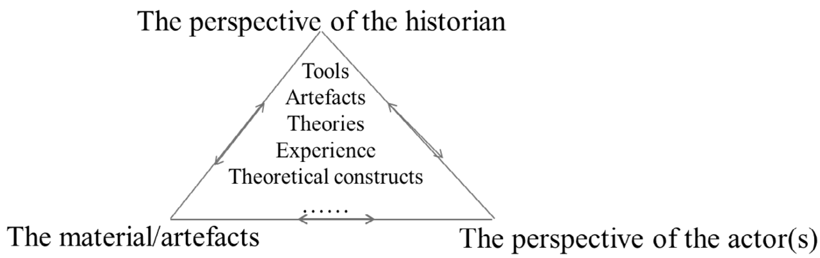

Figure 3: Methodological triangle

Reliability and validity of historical analyses depend on the relation between these three i.e. the relation between the perspective of the historian (from which perspective(s) is the historian writing his/her history?), the perspective of the historical actors (what were/was their intentions at the time?) and what material/artefacts does the historian have access to. In the following we will use Kjeldsen's (2008, 2009) historical analyses of Minkowski's development of the concept of a general convex body to pinpoint yet another way in which mathematics can be considered to be tool-driven. The relations in the methodological triangle and the validation of the historical analyses with respect to our agenda in this paper will be unfolded and discussed as we move along.

The idea of a general convex body was crystalized and constructed in the period 1887-1897. Two instances have been found: 1) Hermann Brunn's theses at Munich University from 1887 in which he introduced and investigated what we today will think of as general convex bodies in two and three dimensions. 2) Hermann Minkowski's work on positive definite quadratic forms that led to his development of geometry of numbers and the beginning of a theory for general convex bodies in the period 1887-1897. The short introductions to the history of convex analysis and geometry that can be found in textbooks and some historical accounts (see e.g. (Bonnesen and Fenchel 1934; Klee 1963; Gruber 1993)), are mostly written from the perspective of the present status and practice of the theory of convexity that is, from the conceptualization of modern mathematicians. In Kjeldsen's analysis there is a change of perspective from what we can call timeless "sameness" or from the universality of mathematics which the above mentioned historical accounts are written from, to the situatedness, to the local development of mathematics, to the practices of Brunn and Minkowski, a perspective where attention is paid to the tools and techniques they used, to their intensions and to unintended consequences of their work. She moves into Brunn's and Minkowski's "workshops" (with their tools, techniques, objects and their theories) through their manuscripts, their institutional affiliations and their mathematical cultures. She uses the historiographical tool of epistemic configuration (Rheinberger 1997; Epple 2004) in her historical analyses. Among other things, she argues that Minkowski's construction of the concept of a general convex body appeared as an unintended consequence of his work on positive definite quadratic forms.

Hermann Brunn introduced what he named "ovals" and "egg forms" in his thesis written in 1887 at the University of Munich. He defined an oval as a closed plane curve that has two and only two points in common with every intersecting straight line in the plane, and a full oval as an oval together with its inner points. Egg surfaces and egg bodies were defined as the corresponding objects in space. A mathematician of today will recognize these objects as convex sets in two and three dimensions. For Brunn they were what we could coin quasi-empirical objects whose mathematical properties such as curvature, area, volume and cross sections were unknown. The visual and intuitive, the quasi empirical status of Brunn's objects, were essential for his mathematical practice. He had very strong opinions about the methodology of geometry as he wrote in his thesis:

> I was not entirely satisfied with the geometry of that time which strongly stuck to laws that could be presented as equations quickly leading from simple to frizzy figures that have no connection to common human interests. I tried to treat plain geometrical forms in general definitions. In doing so I leaned primarily on the elementary geometry that Hermann Müller, an impressive character with outstanding teaching talent, had taught me in the Gymnasium, and I drew on Jakob Steiner for stimulation. (Brunn, 1887)

Brunn's mathematical objects can be seen as developing from artefacts from our material world – artefacts which Brunn turned into quasi empirical mathematical objects. In the discussion we will compare Brunn's objects with Minkowski's and discuss the role of their objects in the development of mathematics.

David Hilbert wrote in memory of his friend and colleague Hermann Minkowski (1864-1909) that Minkowski's geometrical proof of the so-called minimum theorem for positive definite quadratic forms was "a pearl of the Minkowskian art of invention" (Hilbert 1909). Besides being a very intuitive proof and providing a better upper bound for the minimum, Minkowski's work with the geometrical proof of the minimum theorem led to a new discipline in mathematics, geometry of numbers, it led to the idea of a general convex body hereby launching the beginning of the modern theory of convexity, and it led to the generalization of the concept of a straight line through Minkowski's introduction of what he called radial distance (which we today would call an abstract notion of a metric). A key object in these developments is the concept of a lattice which Minkowski used in his investigations of the minimum problem for positive definite quadratic forms in n variables. In the following we will explain the role of the lattice in Minkowski's work in order to discuss if and if so in what sense the lattice can be seen as a cognitive artefact and how these developments of Minkowski's in modern mathematics can be said to be driven by this tool.

A positive definite quadratic form f in n variables has the following form:

$$f(x) = \sum a_{hk}\, x_h\, x_k\ , \quad x = (x_1, x_2, \ldots, x_n)\ , \quad a_{hk} = a_{kh}$$

where a_{hk} are real numbers.

The minimum problem for such forms is to: *Find the minimum value of the quadratic form for integer values of the variables – not all zero.*

Minkowski was inspired by Gauss and Dirichlet who had outlined and shown how positive definite quadratic forms in two and three variables, respectively, could be represented geometrically.

Following Gauss, we let

$$f: \quad axx + 2bxy + cyy$$

be a positive definite quadratic form in two variables. In a rectangular coordinate system, the level curves of such a form will form ellipses. Gauss (1863, p. 188–196) outlined how such a form can be associated with a lattice that is built up of congruent parallelograms through a coordinate transformation (see figure 4).

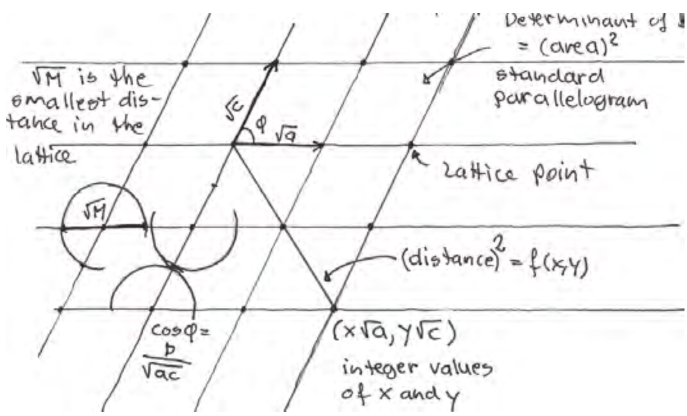

Figure 4: The lattice

The angle φ between the coordinate axes in the lattice is determined by $\cos \varphi = b/\sqrt{ac}$. The points $(x\sqrt{a}\ , y\sqrt{c})$ for integral values of x and y are called lattice points. They form the vertices of the parallelograms. In this coordinate system the quadratic form measures the distance from lattice points to the origin for integral values of the variables:

$$f(x,y) = (\text{distance from the lattice point } (x\sqrt{a}\ , y\sqrt{c}) \text{ to the origin})^2$$

In this geometrical representation the minimum problem becomes the problem of finding the smallest distance between two points in the lattice. Minkowski reached an upper bound for the minimum for forms of three variables through geometrical reasoning in his probationary lecture for the habilitation in 1887. The technique he used was to place spheres with the smallest distance in the lattice as diameter around lattice points. Since the spheres will not overlap and they do not fill out the volume of the standard parallelotopes, he could deduce the following inequality:

$$V_{sp} < V_{par}$$

$$\left(\frac{4}{3}\right)\pi\left(\frac{\sqrt{M}}{2}\right)^3 < V_{par}$$

$$M < kD^{1/3}$$

Hereby he reached an upper bound for the minimum M of the quadratic form that depends solely on the determinant D of the form and the dimension. In 1891 he published a proof for the n-dimensional case.

Minkowski developed what he called Geometry of Numbers as a general theory of which positive definite quadratic forms could be treated geometrically. He realized that the essential property was not the ellipsoid shape of the level curves for positive definite quadratic forms but what we today will call the convexity property of these bodies. In a talk from 1891 Minkowski introduced the 3-dimensional lattice, not as a representation of a positive definite quadratic form in three variables, but as a collection of points with integer coordinates in space with orthogonal coordinates. In the lattice, he considered what he called a very general category of bodies that consists: “of all those bodies that have the origin as middle point, and whose boundary towards the outside is nowhere concave.” (Minkowski 1891). By then he had realized that it does not have to be a positive definite quadratic form that measures the distance in the lattice. It can be any body belonging to this category of bodies. The lattice had changed function from being a geometrical representation of a positive definite quadratic form to function as scaffolding for investigating the general categories of bodies mentioned above. A scaffolding which Minkowski began to investigate within the context of geometry of numbers that he was developing.

In a talk from 1893 he presented his ideas in more details. He introduced what he called the radial distance $S(ab)$ between two points, where S is positive if a and b are not equal to one another, otherwise S is zero. He also defined what he called the corresponding “Eichkörper” which consists of all the points u which radial distance to the origin is less than or equal to one: $S(ou) \leq 1$ (we would call this the unit ball). He emphasized that:

> If moreover $S(ac) \leq S(ab) + S(bc)$ for arbitrary points a, b, c the radial distance is called “einhellig”. Its “Eichkörper” then has the property that whenever two points u and v belong to the “Eichkörper” then the whole line segment uv will also belong to the

> "Eichkörper". On the other hand every nowhere concave body, which has the origin as an inner point, is the "Eichkörper" of a certain "einhellig" radial distance. (Minkowski 1911, vol I, p. 272-273)

Today we would recognize a radial distance that fulfills the triangular inequality and is reciprocal as a metric that also induces a norm.

Minkowski formulated his famous lattice point theorem in the talk: If $J \geq 2^3$, where J is the volume of the Eichkörper, then the Eichkörper contains additional lattice points. Minkowski's lattice point theorem connects the volume of a body with certain geometrical properties with points with integer coordinates. In his book *Geometry of Numbers*, he developed his theory for bodies in *n*-dimensional space.

In the course of Minkowski's research the lattice changed epistemic function from being a representation of positive definite quadratic forms, to become of interests in itself when Minkowski began to investigate the lattice and its corresponding bodies, to function as a tool – a scaffolding. Viewing the mathematical practice of Minkowski in this research episode from this particular perspective of mathematics as a tool driven enterprise, we can see that the lattice played a major role as a cognitive artefact, a tool that caused a qualitative shift in the research on the minimum problem for positive definite quadratic forms, in at least two ways:

- It provided the structure in which the "very general category of bodies" could be considered (Minkowski's talk from 1891).
- It functioned as a link between integer coordinates and the seize (volume) of the convex body.

We will finish this example by further exploring how the cognitive artefact of the lattice in this concrete episode of mathematical research enhanced our mathematical thinking, in what sense it led to a qualitative shift in our ability to perform mathematical cognition.

Brunn's egg-forms and ovals are quasi empirical mathematical objects which he investigated and proved theorems about by using the method and technique from synthetic geometry. In the preface or introduction to text books about convexity we can often read that general convex bodies were first investigated by Brunn and then further explored and extended by Minkowski (see e.g. (Bonnesen and Fenchel 1934; Klee 1963; Gruber 1993)). These short accounts of the development of the theory of convexity are written from the perspective of the modern theory, from the conceptualization of the writer, who focuses on the similarity of the bodies investigated by Brunn and Minkowski respectively. This is in the tradition of modern writings in mathematics where mathematical objects are presented as timeless entities (cf. Epple 2011).

If we change perspective from considering mathematical objects as timeless entities and instead focus on the situatedness in the actual production of mathematics, we have two trajectories of research emanating from each local context with a concrete mathematical practice. This is illustrated in figure 5.

Figure 5: The figure is adapted from Kjeldsen (2014)

There were two local contexts, Brunn's and Minkowski's, having each a concrete mathematical practice that was very different from one another. Minkowski and Brunn worked independently of each other and only became aware of each other's work after they both had developed and formulated their ideas. They met around 1893, and realized that they were both working on bodies with nowhere concave boundaries (see Kjeldsen 2009).

In order to explore this "sameness" or timelessness of mathematical entities from a historical perspective of mathematics we can play with the question whether Brunn, working with ovals and egg-bodies, within his mathematical workshop or "lab", could have reached the results of Minkowski, as illustrated by the stipulated trajectory in figure 6.

Figure 6: The figure is adapted from Kjeldsen (2014)

However, as our historical analysis of the concrete episodes of Brunn's and Minkowski's work with ovals and egg-bodies and positive definite quadratic forms, radial distance and "eichkörper", respectively, from the perspective of Brunn's and Minkowski's mathematical practices has shown, Minkowski's lattice point theorem could not have been developed within Brunn's mathematical workshop. There is nothing in Brunn's practice in this episode that connected the volume of his egg-bodies with points in space with integer coordinates. Brunn could not have asked the question of the lattice point theorem, as illustrated in figure 7.

Figure 7: The figure is adapted from Kjeldsen (2014)

The content of concrete episodes of mathematical research, and the questions asked in such episodes depend on the objects and techniques (lattice, geometrical representation of quadratic forms vs quasi empirical egg forms, synthetic geometry) that are available and present for the mathematician in the particular research situation. Mathematicians' ability to think mathematically is determined by the cognitive context they are positioned in, that is: by the cognitive artefacts and other cognitive support available to them.

The lattice played a significant role in Minkowski's work. If we look at the dynamics of the knowledge production, we can see that in the beginning of the research episode, the lattice functioned as a representation for positive definite quadratic forms that made it possible for Minkowski to use the method of analytic geometry to work on the minimum problem. The lattice then became the object of investigation which led to Minkowski's introduction of the radial distance and the "Eichkörper". The lattice was the connecting link between the geometry of the nowhere concave bodies and arithmetic through the points with integer coordinates in the Euclidean coordinate system in *n*-dimensional space. In this sense, the lattice functioned as a cognitive artefact, a tool that drove the development of geometry of numbers. It caused a qualitative shift in the development of Minkowski's work on positive definite quadratic forms.

DISCUSSION

Through our four examples we have explored how cognitive artefacts are used in mathematics and we have identified three different ways in which artefacts have led to qualitative shifts in our ability to perform mathematical cognition: 1) as an expansion of the given (counting), 2) as an extension of what one can work and manipulate with (the square roots of negative numbers), and 3) as a scaffolding mediator between different mathematical areas (the lattice). The analyses show that cognitive artefacts are not just cognitive artefacts. Different artefacts shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts we have access to. Our ability to think – also mathematically – is determined by the cognitive context we are positioned in, that is: by the cognitive artefacts and other cognitive support available to us.

In the introduction we alluded to the debate about the use of computers in the teaching of mathematics. We took this debate as an opportunity to shift the focus from the computer as such to the use of tools in mathematics in general, to move beyond the "good"/"bad" discussion of computers and instead ask: What can we learn about mathematics if we view mathematics as a tool-driven practice in research and in every-day (or practical) mathematics? We complete the loop by returning to the educational perspective. Today almost everybody in the Western world is intimately connected with smartphones, laptops, tablets and other devices that offer powerful computational support. This has radically changed the cognitive landscape we are situated within. We have to recognise this change and take informed decisions about what consequences it should have for our mathematical practice. At the outset, doing long division with smartphone is no less mathematical than doing it using an abacus or Hindu-Arabic numerals. In all the cases, students will be using cognitive artefacts. As we have shown in our analyses of our historical cases, the different artefacts have different affordances. They shape the cognitive landscape in different ways, and for that reason it matters what type of artefacts our students have access to. What they can do is determined by the cognitive artefacts they have access to. Our decisions concerning which artefacts to use and (more importantly) which to teach our students to use, should depend on an analysis of these affordances as compared to our need.

NOTES

1. We are indebted to Professor Jesper Lützen, University of Copenhagen, for bringing this case to our attention. Lützen presented his own treatment in a talk given at the Second Joint International Meeting of the Israel Mathematical Union and the American Mathematical Society, IMU-AMS in Tel Aviv, Israel, June 16-19, 2014.

2. It should be noted that Cardano used a slightly different representation. In his original manuscript the modern symbols + and – are represented as "p" and "m" respectively, and $\sqrt{\ }$ is represented as "R2". Thus in total his two solutions are stated as: 5 p:R2:m:15 and 5 m:R2:m:15 (Struik 1969, p.68).

3. Presented by Kjeldsen in the talk "Whose History? Minkowski's development of geometry of numbers and the concept of convex sets", held at at Second Joint International Meeting of the Israel Mathematical Union and the American Mathematical Society, IMU-AMS in Tel Aviv, Israel, June 16-19, 2014.

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