Panel Discussion 1

TECHNICS AND TECHNOLOGY IN MATHEMATICS AND MATHEMATICS EDUCATION

Morten Misfeldt[a], Mario Sánchez Aguilar[b], Mikkel Willum Johansen[c], Mirko Maracci[d] & Per Jönsson[e]

[a]Aalborg University, [b]CICATA Legaria, National Polytechnic Institute of Mexico, [c]University of Copenhagen, [d] University of Pavia, [e]Malmö University

The use of computer technology for teaching and learning of mathematics has several consequences and does sometimes give rise to both controversies and misunderstandings. We address these problems by both a philosophical and a historical approach, investigating what it actually is that goes on when new technologies enter mathematics as a discipline and mathematics education as a societal practice. Our analysis suggests a focus on continuities in time and place in the sense that it is necessary to understand the history of "tool use" in mathematics and the various ways that scholastic and non-scholastic mathematical practices adopt such tools. Furthermore we point to the strong interrelation between mathematics as a body of knowledge, mathematical activity and the technologies used for mathematical work. Finally we discuss how different theoretical lenses and epistemological outsets give rise to different guidelines and conclusions regarding the use of computer technology in mathematics education.

TECHNOLOGY IN MATHEMATICS EDUCATION; BEYOND PRO AND CON

Despite 30 years of use in mathematics education and substantial research and development activities, computer technology has not brought the positive changes originally envisioned (Artigue, 2010, Hoyles, 2014). In this plenary panel discussion we have allowed ourselves to take a helicopter view on the understandings of the uses of technology and ask some of the big questions that become apparent. In a sense we wish to understand why the use of technology in mathematics education can give rise to such hopes and at the same time be considered as a major disappointment. The panel should bring us further in an understanding of how to conceptualize the use of computer technology in the teaching of mathematics, and illuminate the debate pro and con the use of such technologies for teaching mathematics. In the panel we address the following questions:

- How, and to what extent, does the use of computer technology in mathematical activities change *mathematical work processes*, what *mathematics is* and how it is *understood and learned*? More specifically:

- **1a.** How does the use of computer technology in mathematical activities change *mathematical work processes?*
- **1b.** How does the use of computer technology in mathematical activities change what *mathematics is*?
- **1c.** How does the use of computer technology in mathematical activities change how mathematics is *understood and learned*?

- Is the use of computer technology in mathematics and mathematics education best viewed as *in continuity* with *or* as a break away from the use of non-computer technology?
- How can different theories describe doing and learning mathematics with computer technology?

We have struggled to negotiate a version of the questions that can be embraced by all of us. And we do suggest that any attempt to answer these questions will at least allow a more fine-grained discussion of the reasons for bringing computer technology to the mathematics classroom as well as an increased understanding of the resulting changes to classroom practice.

UNDERSTANDING THE QUESTIONS

When we tried to answer the questions, we realized that all of the question could be answered both from the perspectives of activities in education and from the perspective of activities in mathematics (such as we have asked the second question). However this leads to another unclarity – what is meant by *in education* and *in mathematics?*

This unclarity invites us to consider mathematical practices in various settings. For simplicity we will talk about *educational settings* and *research settings.* Furthermore the educational setting refers both to students at different levels and to teachers of mathematics. Of course aspects of *vocational/work life*, *citizenship,* and *private life* also involves mathematics, but for the sake of simplicity we will address the questions from three perspectives: *researchers of mathematics*, *the mathematics student*, and *the mathematics teacher*. And hence our discussion speaks into the organization suggested by table 1.

	Mathematics Student	Mathematics Teacher	Mathematician
How the use of computer technology in mathematical activities changes *mathematical work processes*	Addressed in the section “technology and mathematical work processes”		
How the use of computer technology in mathematical activities changes what	Addressed in the section “technology and the nature of mathematics”		

mathematics is	
How the use of computer technology in mathematical activities changes how mathematicsis *understood and learned?*	Addressed in the section: "How is the use of computer technology in mathematical activities changing how mathematics is understood and learned"
Is the use of computer technology in mathematics and mathematics education best viewed as *in continuity with* or as a *break away* from the use of non-computer technology?	Addressed in the section: "computers as continuity or rupture in the development of mathematics"
How can different theories describe doing and learning mathematics with computer technology?	Addressed in the section: "how do different theories describe doing and learning mathematics with computer technology?"

Table 1: Matrix showing different approaches to the question of technology and mathematics learning.

Finally the perspective that we take also affects our possible answers. The questions mean different things if addressed from specific theoretical perspectives, and they hence have different answers. In our panel debate we have addressed the questions from cognitive, didactical and disciplinary perspectives. We will not fill out the entire matrix from each perspective. Rather we will use the matrix as a guide to navigate when several approaches address the same question. In the following we shall address the questions one by one.

TECHNOLOGY AND MATHEMATICAL WORK PROCESSES

We address the question of technology for mathematical work process from the perspective of students, teachers and researchers work processes.

Students' mathematical work processes

In general, the use of computer technology promotes the emergence of new solving techniques, which can facilitate many calculations to the students (Lagrange, 2005).

These instrumented techniques allow students to try many individual cases eventually reaching generalizations; Trouche et al. (1998) (cited in Lagrange, 2005) showed for example how some students obtained an expression of the n^{th} order derivative of $(x^2 + x + 1)e^x$, by reflecting on several particular cases using a calculator with CAS (Computer Algebra System) capabilities.

The distribution of algebraic and arithmetic work to computer technologies does give rise to some problems, certain tasks and topics (for instance trigonometric triangle calculations) cannot be worked with by students in meaningful ways, and do not train the algebraic skills they did in previous technological situations (Misfeldt, 2014).

Computer technologies provide users with different representational resources and new possibilities for using familiar forms of representation (Morgan, Mariotti & Maffei, 2009). For example, students can quickly draw graphical representations of mathematical objects, but they can also manipulate and explore these representations dynamically. While the benefits of such resources seem intuitively clear, it should be pointed out that there still lacks a proper understanding of external cognition and how graphical representations work (Scafie & Rogers, 1996).

Finally computer technology can also modify mathematics students' study processes. Due to the omnipresence of the Internet and mobile devices, students can have immediate and unlimited access to various sources of mathematical information. Contemporary mathematics students rely on non-traditional sources of mathematics. For instance, the study of van de Sande (2011) shows how mathematics students from different regions of the world are turning to Internet-based open forums looking for advice that could help them to solve their doubts related to their mathematical tasks.

Mathematics teachers' work processes

The availability of computer technology affects the teachers' work in many respects. For instance, mathematical tasks that the teacher can offer to her/his students could become obsolete when the use of computer technology is allowed in the classroom (Lagrange, 2005). A task that could be considered a challenging problem in a setting where computer technology is not available can become a trivial exercise in a technological-aided setting, in the sense that only applying a command or pressing a button on the calculator could solve it. Thus the need for redesign of mathematical tasks arises. It is necessary to rethink the mathematical activities in order to make them more meaningful and challenging in a technology-aided environment.

Computer technology offers the possibility to enrich teachers' instructional techniques. The use of technology may promote the emergence of new teaching techniques; for example, the work of Drijvers, Doorman, Boon, Reed & Gravemeijer (2010) provides a taxonomy of various forms of work that can arise when teachers teach mathematics with the aid of computational tools. Internet resources such as YouTube can make the mathematical problems given to students more interesting by providing them with realistic contexts in which these problems could be embedded (Stohlmann, 2012).

Technology can also help to expand teachers' instructional spaces, i.e., teachers can provide their students with mathematics instruction beyond the walls of the classroom. There are for example mathematics teachers who video record their mathematics lessons and make them available to their students so they can review the lesson in the privacy of their home (see for example the concept of *flipped classroom*, Talbert, 2014; Tucker, 2012). On the other hand, the use of mobile technology can help teachers to organize mathematical activities outside the classroom where students can use elements of the real world to study mathematical objects and their properties, for instance (Wijers, Jonker & Drijvers, 2010) report the use of a mathematical game based on the use of mobile devices and GPS technology, in which students draw geometric shapes and explore their properties.

Teachers' work has been described within the documental approach focusing on the interplay between various resources including computer technologies, that the teachers use in preparation, conduction, and documentation of their teaching, and their actual practice (Gueudet, Buteau, Mesa, & Misfeldt, 2014; Gueudet, Pepin, & Trouche, 2012). Despite the undeniable potential, integrating technology in the mathematics classroom also raises several difficulties, and increased the complexity of teaching mathematics (Tabach, 2013).

Mathematicians' work processes

It is undeniable that the work processes of professional mathematicians benefit from the calculation capabilities of computational tools to the point that it can be argued that the introduction of computer technology in mathematics has changed mathematics in several different ways. Four main points can be mentioned:

- Computers have made it easier to search, store and share information.
- Computers have opened the possibility of more powerful explorative experimentation.
- Computers have made certain types of computationally heavy proofs possible.
- Computers, and associated complex and large data sets from various fields, have changed what problems are considered interesting.

Hence computational tools support already existing work processes (such as communicating, searching information etc.), allow mathematicians to conduct experiments that could lead to the formulation of conjectures and new theorems that can subsequently be demonstrated in a more formal way.

TECHNOLOGY AND THE NATURE OF MATHEMATICS

The use of tools has accompanied mathematical work processes throughout the history of mathematics: ruler and compass, abacus, curve-drawers, perspectographs, planimeters are examples of historical mathematical tools.

Such tools were used to support mathematical activities and at the same time they contributed to and influenced the progress of mathematical knowledge.

As one example of this we can consider the abacus (this example is thoroughly discussed in Bartolini-Bussi & Mariotti, 2008). The abacus can "easily" evoke to experts the place-value notation of integer numbers, and indeed it is often used in primary schools as a didactical aid, and it is still used in some countries in everyday life.

The first appearance of the Sumerian abacus dates back to the period 2700–2300 BC (Selwyn, 2001). Anyway it took centuries to pass from the computation practice based on the use of the abacus to the development of a "new" way to represent written numbers (the place-value notation was originally developed by Indians and introduced in Europe in the XIII century by Fibonacci; and it took centuries before it was widely accepted).

> "From an historical perspective, the positional system is not "embedded" but rather an important yet unexpected "by-product" (and even a late one) of the century use of abaci in computation". (Bartolini-Bussi & Mariotti, 2008, p. 761).

This example illuminates the role that tools played and still play in the historical development of mathematics. Tools help represent mathematical actions and objects, create new representations, develop new forms of treatment of representations, and give birth to new mathematical objects and new ways of thinking of mathematical objects. The example also shows how complex this process can be and how unexpected the results may be. The potential of representational, communicative, data-storing and data-processing affordances of todays computer technology are strong and hence we will describe below how computer technology is destined to impact the development of mathematics in unforeseeable ways. Drawing on evolutionary approaches to cognition, Kaput and Shaffer argue that "computational media are in the process of creating a new, virtual culture based on the externalization of highly general algorithmic processing that will in turn lead to profoundly new means of embodying, enriching and organizing all aspects of human experience" (2002, p. 288), that is a new stage of human cognitive development. In the next sections we will zoom in on the effects that tools has on mathematics as a discipline, and see how it changes for *researchers* and for *teachers* and *students*.

The researcher perspective

None of the changes in work processes of mathematicians described above are philosophically innocent, since such changes in the work practice might lead to more fundamental changes in the field of mathematics.

The fact that computers have made it easier to search, store and share information has not only made the day-to-day work of mathematicians easier, but has also introduced qualitatively new ways of conducting mathematical research. An illustrative example is the On-Line Encyclopedia of Integer Sequences (OEIS.org) that by June 1, 2013, had been cited in 2399 papers (according to https://oeis.org/wiki/Works_Citing_OEIS). Thus, computer based tools for sharing and searching information has provided a new ways for finding and exploring mathematical theorems.

Explorative experiments are certainly not something new to mathematics. Gauss' discovery of the prime number theorem which gives an estimate for the total number of primes less than a given number[1], could serve as a historical example (Goldstein 1973). However, the introduction of computers has given us new, powerful tools for explorative experimentation (see e.g. Sørensen 2010) and has led to a new recognition of the experimental aspects of mathematical research, most notably with the birth of the journal *Experimental Mathematics*, which is specifically devoted to increase the awareness of the role played by experiments in mathematical discoveries (Epstein et. al., 1992; current statement of the journal's philosophy: http://www.emis.de/journals/EM/expmath/philosophy.html). Consequently, it is fair to say that the introduction of computers has led to an increase in both the awareness and power of explorative experiments as a method for mathematical discovery.

The advent of computer assisted proofs such as the Appel and Haken's 1976 proof of the four colour theorem (Appel & Haken 1977a & b) has not only opened the possibility of using computation heavy proofs, but has also led to the recognition that mathematics can no longer be viewed as a priori knowledge (for discussion, see Johansen & Misfeldt, n.d.). Other mathematicians have suggested more radical reforms. Most notably, Doron Zeilberger has argued that mathematicians should not invest energy in actually proving mathematical theorems. Instead they should focus their work on transforming mathematical problems into a form, where computers can attack them (e.g. Zeilberger 1999a, 1999b). Zeilberger furthermore has argued that the introduction of computers should lead to a fundamental change in the mathematical epistemology, where we accept a class of 'almost-true' theorems (Zeilberger, 1993). These observations suggest that the introduction of computers in the mathematical practice has led to pragmatic changes in the day-to-day work of the mathematicians, as well as in the methodology and epistemology of mathematics.

The student and teacher perspective

The teaching of mathematics requires a shared conceptualisation of what is being taught. Hence discussing what mathematics "is" in a technological society becomes important in order to develop learning goals and curriculum. As we saw above, the change in researchers' practice caused by the use of computer technology has affected mathematics as a discipline, and in the same way students' and teachers' use of technology in the classroom affects what mathematics is for them.

Hence two types of change can be observed; development that results from the practice of teaching and learning of mathematics in classroom settings and development from the way mathematics is done in research and professional life, affecting the target knowledge for teaching mathematics. As described in the previous

[1] If we let $P(x)$ designate the total number of primes less than or equal to a given positive real number x, the theorem more precisely states

$$\lim_{x\to\infty} \frac{P(x)}{x/\ln(x)} = 1$$

chapter digital technology allows students and teachers to distribute calculations to computational technology and to communicate more, and in different modalities. Different educational systems address these changes and possibilities differently, but potentially these technologies tone down the value of computational skills, and tone up the ability to communicate and make meaning from diverse digital representations.

The effects from outside of the mathematics classroom come from many sources. We have here discussed mathematical research, which is one obvious source for conceptualising what mathematics is. However more such sources exist. The way mathematics is used in professional life is affected by technology, and so is the relevance of studying mathematics both in order to cope with various aspects of life and in order to understand and participate in the democratic debate. Hence, all reasons for studying mathematics (Niss, 1996) are somehow affected by technology. The emerging goals for mathematics teaching as a result of technology is described in the next section.

HOW IS THE USE OF COMPUTER TECHNOLOGY IN MATHEMATICAL ACTIVITIES CHANGING HOW MATHEMATICS IS UNDERSTOOD AND LEARNED

In the previous sections we have described how educational research around computer technology, for example CAS and DGS (Dynamic Geometry Systems), have studied the micro processes of learning mathematics with computers. These tools change students' mathematical work processes and hence affect their learning. We suggest that the resulting changes can be described as questions of new goals, new didactical problems, and new didactical potentials.

Since a number of mathematical work processes outside school is affected by computational technology, it is natural to reconsider goals for schooling**.** Currently the role of programming in the mathematics and science curriculum is discussed (Caspersen & Nowack, 2011; Rushkoff, 2011; Wolfram, 2010), because of the increased importance of programming in society. As a contrast long division is often described as a mathematical process that, due to the widespread use of calculators, is not necessary for lower secondary school pupils to master anymore. The discussion however is more difficult than this. Although we might all agree that it is not really important to be able to calculate the quotient of two 7-10 digit numbers fast and efficient, it does not mean that it is not important to know how it is done. And if students never do the actual process, then there is a risk that they might never learn how to do it right (the same goes for the solution of equations, algebraic simplifications and several other mathematical processes). Learning problems as a result of blackboxing is well documented (Guin, Ruthven, & Trouche, 2005; Nabb, 2010)**.** If students are consistently using a CAS to perform algebraic reductions and solutions of equations, then it is less likely that they are able to perform such calculations without the tool. This can affect learning because the student lose track of the processes that is hidden by the tool (Jankvist & Misfeldt, 2015).

The tools also offer a number of new potentials in terms of construction, inductive reasoning and experimentation. The increased potential in diagrammatic reasoning has for example been investigated in relation to DGS (Laborde, 2005; Mariotti, 2000)**.**

COMPUTERS AS CONTINUITY OR RUPTURE IN THE PRACTICE OF MATHEMATICS, AND MATHEMATICS EDUCATION

We will address the question of continuity vs. rupture through two different frameworks: distributed cognition and the theory of semiotic mediation. The two frameworks stress quite strongly the aspects of continuity between the use of computer and non-computer technology in mathematics and mathematics education rather than the aspects of rupture. Both perspectives consider computer and non-computer technology as particular 'artefacts' designed by humans in order to produce intended effects (Rabardel 1995, p. 49).

From the point of view of distributed cognition computers can be seen as epistemic artefacts that allow cognitive tasks to be distributed and completed by epistemic actions. Although the introduction of computers in mathematics has led to qualitative changes in mathematical research, the use of epistemic artefacts is not at all new to mathematics. On the contrary, throughout its history mathematics has been intimately connected with the use of cognitive artefacts; we have always strived to create tools, algorithms and representational systems that allow us to reduce the demands mathematics poses on human cognition. The use of such artefacts can be traced back to at least the Upper Paleolithic period where carved bones were allegedly used as tallying sticks. Furthermore, studies of animals, human infants and isolated tribes have shown that our ability to do mathematics without the aid of cognitive tools is very limited. To put it roughly, we have the ability to do basic arithmetic with sets containing less than five elements, and we are able to judge the size of large sets with approximation (Feigenson, Dehaene & Spelke, 2004; see also Johansen 2010, p. 49 for discussion). We are however not able to judge, say, whether there is 10 or 11 elements in a set without the aid of a cognitive tool, such as a sequence of counting words.

The theory of semiotic mediation considers the role of computer technology in fostering mathematics learning process focusing on the commonalities between computer and non-computer technology, stressing how they contribute not only to the accomplishment of mathematical tasks, but also to the individuals' construction of mathematical knowledge. Computer, ruler and compass, abacus, and curve-drawers (just to mention some materials often used in schools) are artefacts conceived and designed to be used according to certain modalities in order to solve tasks. In this sense, artefacts embody people's collective experiences, and modes of acting, thinking, and communicating; i.e. they embody collective social knowledge and experience (Stetsenko, 2004) which "assures" the correct functioning of the artefact. And for this very reason artefacts can be viewed as "bearers of historically deposited knowledge from the cognitive activity of previous generations" (Radford, 2008, p. 224).

Through the use of an artefact for accomplishing a task, the individual has in a sense access to the historically and culturally established knowledge embodied in it. In fact the process of using an artefact for accomplishing a task involves two components having opposite orientations. On the one hand, the process is oriented towards the objects of the action: the artefact is a means to transform the object; on the other hand it is oriented towards the individual, it permits the individual's consciousness-raising of the object itself of the artefact-mediated action (Rabardel, 1995). The use of an artefact even structures the individual's action and thinking, drives his attention and perception. This means that artefacts not only serve to facilitate already existing mental processes, they also transform them (Cole & Wertsch, http://www.massey.ac.nz/~alock/virtual/colevyg.htm).

The didactical potential of the artefact is related to the mediation oriented towards the individual. The use of an artefact for accomplishing a task may trigger the students' development of personal meanings concerning the object of the artefact-mediated action, that are potentially coherent with historically established mathematical meanings. In educational settings this process is not spontaneous but mediated by the teacher (Bartolini-Bussi & Mariotti 2008, Maracci & Mariotti, 2013).

Summing up, this general perspective contributes to understand the role that artefacts may play in the mathematical research, teaching and learning process, illuminating the aspects of continuity between computer and non-computer technology. Even if the use of computer technology is bringing undeniable shifts in work processes of students, teachers and researchers of mathematics, there are still aspects of continuity between computer technology and non-computer technology and between their use and roles in mathematics education. The use of computers in mathematics is an extension of a practice that goes back a long time.

HOW DO DIFFERENT THEORIES DESCRIBE DOING AND LEARNING MATHEMATICS WITH COMPUTER TECHNOLOGY?

So far we have mainly focused on interactional theories such as distributed cognition and the theory of semiotic mediation. Such theories provide an important starting point for developing our understanding of the use of computers in *mathematical practice*. However, theoretical constructs have different centres of gravity proposing different issues and problems. Being aware that the complexity of the issue at stake requires us to view the problem from different angles, we are left with the question of how to approach the issue of comparing theoretical perspectives.

Mediating concepts and questions

We can seek inspiration in two European research projects TELMA and ReMath. In these projects one of the aims was to investigate the role of theoretical frameworks in the design and in the analysis of the educational use of computers for mathematics education. With this focus these projects investigated how different theories drive the design and analysis in different ways. Hence theoretical constructs studying mathematics education can be compared through specific attention on three interrelated poles (Cerulli et al. 2006):

1. a set of features/characteristics of the tool;
2. a specific educational goal; and
3. a set of modalities of employing the tool in a teaching/learning process with respect to the chosen educational goal

Different theories contribute differently to analyse these poles and their relationship, some theories are more sensitive to issues related to one pole and leave the others in the shadow. For instance, when considering the educational goals that can be pursued through the use of artefacts, one can (or not) focus on epistemological issues concerning specific mathematical contents or practices, express the educational goals in terms of cognitive processes possibly considering specific cognitive difficulties, address the process of construction of knowledge as a social or an individual process, be concerned about institutional expectations, and so on.

Let us examine three theoretical approaches in that respect. The instrumental approach (Rabardel 1995, Guin & Trouche 1999) raises the crucial importance of considering the process through which students develop the “utilization schema” of an “instrument”. That draws the attention on the pragmatic/operational side of the knowledge developed by students, involving both knowledge of the artefact and mathematical knowledge. The theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) explicitly raise the epistemological issue of the relationship between the meanings which individuals autonomously develop when using an artefact and the culturally established mathematical meanings, and addresses it through a semiotic lens. The anthropological theory of didactics (Chevallard, 1992), on its side, explicitly address the question of the institutional expectations and of the compatibility of the forms and contents of the activity mediated by the artefact and those valued by the educational institutions.

The above summary is not meant to compare or evaluate the three mentioned theories, but simply to point out that different theories offer specific theoretical tools, which inevitably can address only part of the complexity of mathematics teaching and learning with artefacts. Analogously, we could attach several dimensions even to the other poles of the construct of didactical functionality: the features of an artefact, and their modalities of use.

What we have sketched above is in fact the so-called Concern Methodological Tool (elaborated within the TELMA project, Artigue et al, 2009, and refined in the ReMath project, Artigue et al.2006, Mariotti et al. 2007) which is meant to express (some of) the main different dimensions and sensitivities through which different theories contribute to conceptualize the *features of the tool*, the *educational goal* which can be pursued through the use of these feature, and the *modalities of employing the tool* in a teaching/learning process with respect to the chosen educational goal (Artigue et al., 2009).

INTERCONNECTED CHANGING PRACTICES

To conclude this discussion of how we should conceptualize the use of computer technologies in mathematics education, we have suggested that the influence should be studied with different theoretical lenses (interactional, cognitive, curricular) and different focus points. One important focus point is the actual artefacts (e.g. a computer algebra system) used by students, teachers and mathematicians, as well as the direct influence that such artefacts has on practices. And as we have shown, different practices are influenced in different ways. If we return to our initial questions, we have addressed how the use of computer technology change mathematical work and learning. We have done so by looking at mathematics as an essentially tool-driven practice. This has given us the insights that the use of tools is a necessary part of the mathematical practice and that the introduction of new tools is a common event both in mathematics research and in education. New tools act as drivers for the development of mathematical research. From this perspective the introduction of computers is not a special event but is in continuity with the development and practice of mathematics. The introduction of new cognitive tools however, change the cognitive landscape and consequently force us to reconsider what mathematical tasks we consider important and worth learning and what problems and learning situations we should design in order to teach these tasks in a meaningful way. The problem of blackboxing described above, illustrates this process well. If a new cognitive tool, such as a CAS-system, effectively hides the intermediate steps in a task and turns the task into the use of a simple solve function we should ask whether the task is worth teaching anymore, and if it is, we should also ask how to do that in a meaningful way.

The question of whether we should view computer technology in mathematics (and mathematics education) as in continuity with or as a break away from the use of non-computer technology is almost answered by our approach to the first question. Considering mathematics as essentially a tool driven practice, puts the tool in the centre of the activity and almost forces the continuity perspective. If we say that the tools that people use have always significantly affected mathematics, and that these tools always have changed over time, then computational tools are just a natural and continuous development. However, we are able to see some accelerated changes in mathematical practices as a consequence of computer technology. These changes relate to the practices of both teachers, students and researchers of mathematics, as described in the paper.

The observation that our view of mathematics as a tool driven practice, at least to some extent, forces a view of computer technology and mathematics that are in continuity with other tool uses in mathematics, does give some insights to the last question. A different conception of mathematics, for instance a realist one, considering tools as mere means to obtain pure mathematical insights, could legitimate other answers to our questions, and hence prescribe other reasonable views and practices on the use of computer technology in mathematics education. We have seen that different theoretical lenses construct the use of tools in mathematics education differently, and that these theoretical lenses can be compared by how they construct the tool, the learning goal and the modes of using the tool (Cerulli et al. 2006). However, we should also be aware that philosophical construction of what mathematics is, what technology is, and what education is, can play a role for how the questions put up in this panel will be answered.

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