Plenary Lecture

PHILOSOPHIES AND THEORIES BEHIND HISTORY AND EDUCATION: THIRTY YEARS AFTER HANS FREUDENTHAL

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This paper goes back to a paper of the Dutch mathematician and philosopher Hans Freudenthal. We analyse and develop two purposes of this paper of 1983: the idea to not separate history and education in the reflection on mathematical education and the notion of anti-didactical inversion where this idea is active. We will examine four situations (1) Philosophy or theory behind History and Education (2) Didactics and History of Mathematics (3) Philosophy and Theory behind using of History in classrooms (4) Curricula, Didactics and History. We will continue with the notion of anti-didactical inversion to examine two orders of knowledge: historical and didactical orders. From this, we question the role of history of mathematics in the reflection on the curricula in mathematics.

INTRODUCTION: THE PAPER OF HANS FREUNDENTHAL (1983)

In the ICM Conference of 1983, Freudenthal presented a paper, titled « The Implicit Philosophy of Mathematics, History and Education ». He called "philosophy of history" "what we can learn from the history of old mathematics for the sake of teaching people […], one philosophy behind both history and education, or if they are two, that one is common to both" (Freudenthal, 1984, 1695). He stressed the relations between history and education, but more, he did not want to separate them in his reflection. He considered that the historical course could be used in teaching, but "people who teach mathematics as a ready-made system prefer anti-didactical inversion". He also noticed about the use of history of mathematics for teaching: "In fact we have not yet understood the past well enough to really give them [young learners] this chance to recapitulate it [the historical learning process]" (Freudenthal, 1984, 1696). Indeed, the history of mathematics is not an easy subject if we want to use it as a tool for teaching. In 1937, the historian of mathematics Gino Loria wrote: "always I did my best to prove to my students [future teachers] that history of mathematics is a very serious subject; which has to be studied very seriously" (Loria, 1937, 275). In a recent paper on the historical dimension in teaching, Niels Jahnke stressed: "History of maths is difficult!" (Jahnke, 1994, 141).

Freudenthal asked the question of the existence of either a philosophy or a theory behind history. For Imre Lakatos, "history without some theoretical 'bias' is impossible" (Lakatos, 1970, 107), while for the historian Paul Veyne "history has neither structure nor method and in advance it is certain that any theory in this domain is still-born" (Veyne, 1971, 144). Lakatos and Veyne represent two opposite conceptions, which do not lead to the same kind of history. In the first case, it is "a rational reconstruction of history" (Lakatos 1970), as Lakatos wrote, which explains features or reinforces a theory. In the second case, the history tells "an intrigue" to understand facts. Veyne criticized the introduction of theories or ready-made frameworks to write history. I introduced the idea of an *histoire dépaysante* in a paper[i] of 1991 (Barbin, 1991), where I quoted Veyne who wrote that "the event is difference and the characteristic effort of the historian's profession and what gives it its flavor are well known: astonishment at the obvious" (Veyne, 1971, 7). A "rational history" can be written with several kinds of theories: mathematics, didactics, sociology, psychology, etc. In this paper we will meet some didactical theories: theory of conceptions, realistic mathematics education, theory of beliefs and radical constructivism.

In his paper, Freundenthal discussed the notion of anti-didactical inversion, which he had written about in a book edited ten years before, *Mathematics as an educational task* (1973), and later in *Didactical phenomenology of mathematical structures* (1983). In this last book, opposing the mental objects to the mathematical concepts, he wrote:

> Children learn what is number, what are circles, what are adding, what is plotting a graph. They grasp them as *mental objects* and carry them out as *mental activities*. It is a fact that the concepts of number and circle, of adding and graphing are susceptible to more precision and clarity than those of chair, food and health. Is this the reason why the protagonists of concept attainment prefer to teach the number concept rather than number, and, in general, concepts rather than mental objects and activities? Whatever the reason may be, it is an example of what I called the *anti-didactical inversion*. (Freudenthal, 1999, x)

Teaching a concept rather than a mental object is an anti-didactical inversion. Here, this inversion reverses the convenient didactical order, which is the phenomenological one. The question of the order of knowledge in general had been a constant concern in mathematical teaching from the 17th century to the Reform of modern mathematics.

To examine philosophies or theories behind history and education, in each part of this paper, we will compare two authors – historians, philosophers, teachers or researchers in didactics – about history and didactics, use of history in classrooms, curricula and history. These authors had been chosen, to focus on the teaching of curve, tangent and function and the order of their knowledge. Many of them wrote on the methods of tangents of the 17th century, so we begin by presenting original texts written by Pierre de Fermat, René Descartes, Gilles de Roberval and Isaac Barrow.

METHODS ON TANGENTS IN THE 17TH CENTURY

There exist propositions on tangents in Greek Antiquity, but the authors didn't explain how they found the result and they prove them by reductio ad absurdum. The geometer of the 17th century researched direct methods to find the tangents: these are called methods of invention.

Fermat's method

Fermat's method of tangents appeared in a text of 1636, entitled "Method for maximum and minimum", and is an application of this last method.

> Let us consider, for example, the parabola with vertex D and diameter DC; let B be a point on it which the line BE is to be drawn tangent to the parabola and intersecting the diameter at E. We choose on the segment BE a point O where we draw the ordinate OI; we also construct the ordinate BC of the point B. We have then $CD / CI > BC^2 / OI^2$, since the point O is exterior to the parabola. But $BC^2 / OI^2 = CE^2 / IE^2$, in view of the similarity of the triangles. Hence $CD / CI > CE^2 / \mathrm{IE}^2$.

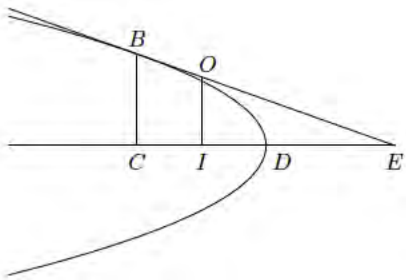

Figure 1. Fermat's tangent of a parabola

> Now the point B is given, consequently the ordinate BC, consequently the point C, hence also CD. Let $CD = d$ be this given quantity. Put $CE = a$ and $\mathrm{CI} = e$; we obtain: $d / (d – e) > a^2 / (a^2 + e^2 – 2ae)$. Removing the fraction: $da^2 + de^2 – 2\,dae > da^2 – a^2e$. Let us then adequate, following the precedent method; by taking out the common terms we find: $de^2 – 2\,dae \approx – a^2e$," or, which the same, $de^2 + a^2e \approx 2\,dae$. Let us divide all terms by e: $de + a^2 \approx 2\,da$. On deleting de, there remains $a^2 \approx 2\,da$, consequently $a = 2d$ (Fermat, 1891, 122-123).

Fermat considered a point B on a parabola, the tangent BE and he choose a point O on this tangent. He knew the relations established by Apollonius to characterize the points of a parabola. The point O is exterior to the parabola, so by similarity of triangles BCE and OIE, he obtained the first inequality. He introduced letters and he transformed the previous inequality by another one between algebraic expressions. Then he applied the rules of his method of maximum and minimum. Now the inequality became what he called an *adequation*, He divided the two members by e and then deleted e. So he obtained an a*dequation* without e and transformed it in an equation that gives the result: CD is equal to DE.

Descartes'method

In his *Geometry* of 1637, Descartes gave a method to find a normal CP to a curve. The normal is the perpendicular to the tangent.

> Let CE be the given curve, and let it be required to draw through C a straight line making right angles with CP. Suppose the problem solved, and let the required line be CP.

Produce *CP* to meet the straight line *GA*, to those points the points of *CE* are to be related. Then, let *MA* = *CB* = *y*; and *CM* = *BA* = *x*. An equation can be found expressing the relation between *x* and *y*. I let *PC* = *s*, PA = *v*, whence PM = *v* – *y*. Since PMC is a right angle, we see that s^2, the square of the hypotenuse, is equal to $s^2 = x^2 + v^2 - 2vy + y^2$, the sum of the two squares. […]

For example, if *CE* is an ellipse, we have $x^2 = ry - (r/q)\, y^2$. By means of these last two equations, I can eliminate one of the two quantities *x* and *y* from the equation expressing the relation between the points of the curve and those of the straight line *GA*. Eliminating x^2 the resulting equation is $y^2 + (qry - 2qvy + qv^2 - qs^2) / (q - r)$. […]

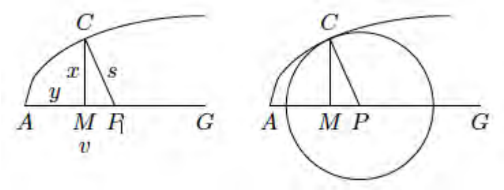

Figure 2. Descartes' method of tangents

Observe that if the point *P* fulfils the required equations, the circle about *P* as centre and passing through the point *C* will touch but not cut the curve *CE* [...]. It follows that the value of *x*, and *y*, or any other such quantity, that is, will be two-fold in this equation, that is the equation will have two equal roots. Furthermore, it is to be observed that when an equation has two equal roots, it must be similar in form to the expression obtained by multiplying by itself the difference between the supposed unknown quantity and a known quantity equal to it [...]. This last step makes the two expressions correspond term by term. For example, I say that the first equation found in the present discussion, […] must be of the same form as the expression obtained by making *e* = *y* and multiplying *y* – *e* by itself, that is $y^2 - 2ey + e^2 = 0$. We may then compare the two expressions term by term (Descartes, 1925, 342-348).

Descartes introduced letters for the coordinates of the point *C*, for *CP* and *PA*. With Pythagoras, he obtained a first equation. He took the example of an ellipse, for which the equation has two parameters *r* and *q*. He eliminated *x* from the two equations and obtained a new equation. Then he examined the situation where *CP* is the normal to the curve. In this situation, the circle about *P* as centre and passing through the point *C* will touch but not cut the curve *CE*. Thus, the last equation must have two equal roots. Indeed this equation is satisfied for points both belonging to the curve and to the circle. Then Descartes observed that when an equation has two equal roots, it must be similar to certain expression. For the example, the equation has to be similar to an equation that has two roots equal to e. By comparing the two equations term by term, Descartes obtained the unknowns *v* and *s*, and so the position of the normal *CP*.

Roberval's method

Roberval invented his method around 1635, but his "Observations on the composition of movements and on the means to find the tangents to curves" were edited in 1693.

> Axiom or principle of invention. The direction of a movement of a point, which describes a curve, is the tangent of the curve in each position of this point.
>
> General rule. From the specific properties of the curved line (which you will be given) examine the different movements, which the point describes where you wish to draw a tangent: from all these movements compose one single movement, draw the direction of that movement, and you will have the tangent to the curved line.
>
> First example of the tangent to the parabola. It is clear by the above description that the movement of *E* which describes the parabola is composed of the movements of two equal straight lines, the one is the line *AE*, the other is the line *HE* on which it moves with the same velocity than the point *I* in the line *BA*, which is the same than the one of the line *AE* by construction, since always *AE* is equal to *BI*. Accordingly, since the direction of the equal movements is known, that is along the given straight lines *AE*, *HE*, if you divide the angle *AEH* in two [equal] parts by the line *CE*, [...] the line *EC* is the tangent (Roberval, 1693, 80-81).

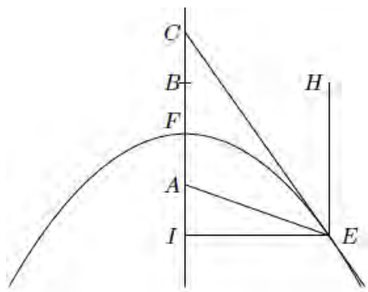

Figure 3. Roberval's tangent to the parabola

Roberval's general rule to find tangents to a curve contains three steps: to examine the different movements of the point describing the curve, to compose them in one single movement and to draw the direction of this movement. Roberval's first example is a parabola. He knew the characteristic of the point of a parabola given by the equality of the distances *EA*, of E to the focal A, and *EH*, of *E* to the perpendicular at the axis passing by B. He concluded that the movement, which describes a parabola, is composed of two movements, one in the direction of *EA* and the other one in the direction of *EH*. Since the segments are equals, the bisector is the tangent.

Barrow's method

Barrow introduced "indefinitely small" parts of tangent and curve in his *Lectiones geometricae* of 1670:

> Let *AP*, *PM* be two straight lines give, in position of which *PM* cuts a given curve in *M*, and let *MT* be supposed to touch the curve at *M*, and to cut the straight line at *T*. "In order

> to find the length of the straight line *PT*, I set off an indefinitely small arc, *MN* of the curve; then I draw *NQ*, *NR* parallel to *MP*, *AP*. I call *MP* = *m*, *PT* = *t*, *MR* = *a*, *NR* = *e*, and other straight lines, determined by the special nature of the curve, useful for the matter in hand, I also designate by name; also I compare *MR*, *NR* (and through them, *MP*, *PT*) with one another by means of an equation obtained by calculation; meantime observing the following rules.
>
> I omit all terms containing a power of *a* and *e*. I reject all terms which do not contain *a* and *e*. I substitute *m* for *a* and *t* for *e*. So *PT* is found and the tangent is obtained.
>
> 1. In the calculation, I omit all terms containing a power of *a* and *e*, or products of these (for these terms have no value).
>
> 2. After the equation has been formed, I reject all terms consisting of letters denoting known or determined quantities or terms which do not contain *a* or *e* (for these terms, brought over to one side of the equation, will always be equal to zero).
>
> 3. I substitute *m* (or *MP*) for *a*, and *t* (or *PT*) for *e*. Hence at length the quantity of *PT* is found (Barrow, 1670, 80-81).[ii]

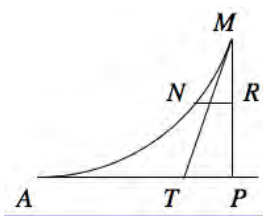

Figure 4. Barrow's method

Barrow considered an indefinitely small arc *MN* of the curve. He associated letters to the segments of the figure: *NR* is called *e*. He compared the sides *MR* and *NR* of the triangle *MNR* to the sides *MP*, *PT* of the triangle *MPT*. Like in Fermat's method we have to observe rules. As *NM* is indefinitely small, he considered it as a straight line and used similar triangles of the figure.

PHILOSOPHY OR THEORY BEHIND HISTORY AND EDUCATION

We begin by comparing two authors concerning the philosophy or theory behind history. We then continue with two authors often quoted in research in didactics.

Histoire dépaysante against rational history

Léon Brunschvicg was a French philosopher who wrote a book on "the steps of the mathematical philosophy" in 1912. Derek Whiteside was an English historian who wrote a paper on the "patterns of mathematical thought" in 17^{th} century in 1962.

Brunschvicg used the experience of history against a "pedagogical tradition of the philosophy" and the "dogmatic systems", he wanted to write the history of a "collective acquisition of knowledge between incidents of the invention and forms of the discourse" (Brunschvicg, 1912, 459). Whereas Whiteside wrote "a study of the particular mathematical forms which developed in the 17th century with emphasis on their interconnections rather than on their philosophical aspects", and wanted "to isolate significant trends of development" (Whiteside, 1962, 179). The results are very different, but we will compare them on two points only. Brunschvicg wrote a *histoire dépaysante,* where he gave the words exactly used by the authors and long quotations for Fermat and Descartes. It is also a history oriented on the research of differences. Brunschvicg compared the mathematical materials used and the intentions of the geometers, examined the disputes between them about the value of the methods.

Whiteside didn't research differences, but similarities. Thus, he pointed to the "slight differences of treatment required in the two approaches" of Fermat and Descartes. To obtain this result, he translated the texts into the modern language of limits, which leads to a none disorienting reading of the texts. He concluded his paper with a continuous, recurrent and limited view of history: " In fact – and in summary – what was done in 17th century mathematics [...] was sufficient to provide rich pickings for 18th century mathematicians seeking a lead into the unknown" (Whiteside, 1962, 384).

Philosophy behind History and Education

Raymond Louis Wilder was an American mathematician interested by philosophy, he wrote in 1972 a paper "History in the Mathematics Curriculum: Its Status, Quality, and Function", also *Evolution of mathematical concepts* (1969) and *Mathematics as a cultural system* (1981). Gaston Bachelard was a French philosopher, he wrote many books, and two has been translated into English: *The formation of the scientific mind* (1938) and *The new scientific spirit* (1934).

In his paper of 1972, Wilder researched the necessary conditions to introduce history of mathematics in curriculum and he wrote:

> Actually, the standpoint from which I believe we should present the history of mathematics is at an even higher level than mathematics. By this I mean, to take a broad view of mathematics as a living, growing organism, which is continually undergoing evolution; in short we should study it as a culture (Wilder, 1972, 483).

He described this evolution by giving the stages in evolution of geometry, of real number system, aspects of reality, etc. He described the "forces of mathematical evolution" like "environmental stress", "hereditary stress", etc. and he explained the evolution inside these frameworks.

The purpose of Bachelard was not to establish a Curriculum, but he thought that history of sciences could help students "to learn to invent":

> Teaching about the discoveries that have been made throughout the history of science is an excellent way of combating the intellectual sloth that will slowly stifle our sense of mental newness. If children are to learn to invent, it is desirable that they should be given the feeling that they themselves could have made discoveries (Bachelard, 1991, 10).

It is also meant to disorientate (*dépayser*) the teachers: “we must also disrupt the habits of objective knowledge and make reason uneasy. This is indeed part of normal pedagogical practice”(Bachelard, 1991, p.245). Bachelard stressed on the polemical character of knowledge. For him, “scientific operation is always polemical; it either confirms or denies a prior thesis, a pre-existing model, an observation protocol; [...] it reconstructs first its own models and then reality” (Bachelard, 1984, 12-13).

His epistemology is inscribed in a negative philosophy, an open philosophy which struggles against the tendency to systems, against positivism and empiricism, like we read in his book *Philosophy of no*. It is an epistemology of the difference and of rupture: “Specifying, rectifying, diversifying: these are dynamic ways of thinking that escape from certainty and unity, and for which homogeneous systems present obstacles rather than imparting momentum”(Bachelard, 1991, 27). It is both a constructivist and historical epistemology, where Bachelard introduced the notions of epistemological obstacle and rectification of knowledge, and stressed the role of problems in the historical construction of the sciences.

DIDACTICS AND HISTORY OF MATHEMATICS

Maggy Schneider is a Belgium researcher in didactics. In her thesis of 1988, she examined the difficulties of students to find the tangent from the calculus of the slope. Michèle Artigue is a French researcher in didactics, who wrote in 1990 a paper on relations between epistemology and didactics.

A question of order: comparing Fermat’s and Barrow’s methods for tangents

To explain the difficulties of students to obtain tangent from the calculus of the slope, Schneider explained that for the students, the tangent is a “mental object” linked with the idea of slope, while the infinitesimal calculus begins with the derivate number (Schneider, 1988, 291-292). Thus, the phenomenological order goes from the notion of tangent to the notion of slope, while the anti-didactical order (which is the scholarly order) goes from the calculus of the slope to the tangent. It is a case of an anti-didactical inversion. For Schneider, history helps to understand the difficulties of the students by comparing Fermat’s and Barrow’s methods for tangents.

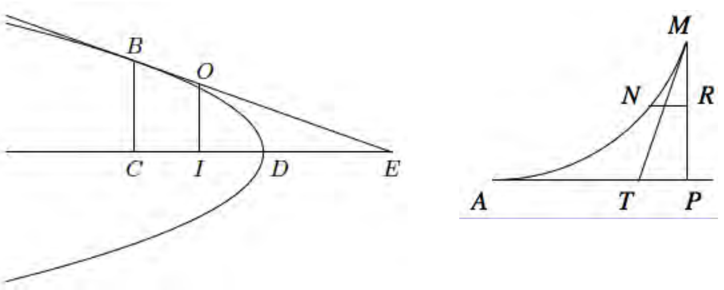

Figure 5. Fermat’s and Barrow figures of tangents

She explained the difficulties by the fact that “the pupils seem nearer to Fermat”. Indeed, Fermat doesn’t use the slope, while in the procedure of Barrow, the triangle MNR gives the slope. Schneider didn’t use the Barrow’paper but a simple and short explanation given by the historian Morris Kline (1972). So, she didn’t mention that the geometer introduced “indefinitely small” parts of tangent and curve (see above). Thus, she did not examine the relative questions: what is a curve for Fermat and Barrow? Or for the students?

Nine conceptions on tangent of a curve

In her paper, Artigue re-situated “the trajectory of the notion of conception” in the French didactical community in ten pages, with the purpose of grouping “in relevant class for didactical analysis” the multitude of conceptions on a given object (Artigue, 1990, 265). For her, a historical analysis can show the diversity of the “points of view” on the “object” of tangent. In consequence, she gave a catalogue of nine conceptions on tangent and the names of the mathematicians associated which them.

For instance she wrote that for Euclid a straight line is tangent to a curve when having a common point with the curve, we cannot lead any straight line between the curve and the tangent at this point. Here, we recognize a result proven in the *Elements* for the tangent of the circle, but it is not the definition of the tangent. She wrote also that for Descartes, a straight line is tangent to a curve if it has a common point with the curve and is perpendicular to the normal in this point. Here the question became to know what is a normal for the geometer. She added that “this generalizes the notion of tangent to a circle via the osculatory circle” and so Descartes finds the tangent of a cycloid in Book II of his Geometry (Artigue, 1990, 275). It is a very modern reading of the original text, which causes confusion, since the Book II treats only the algebraic curves and so the cycloid cannot be there. For Roberval, Artigue wrote: “the tangent to a curve in a point M is the vector velocity in M of a moving point describing a curve”. The notion of vector velocity arrived only in the end of the 19^{th} century, so the purpose is not to render a comprehensive or disorienting history. The purpose is situated in the field of the theory of didactics, and the researcher concluded that the notion of “conception” corresponds to an “intermediary level in the operational effectiveness of the didactical analysis”.

Nicolas Rouche, the director of the thesis of Schneider, followed Freudenthal when he asked: what can we learn from educating the youth for understanding the past of mankind? This idea is the contrary of the usual one, which is that we can learn from the past for education. Bachelard is close to this when he remarked: “the idea of the epistemological obstacle can be examined in the historical development of scientific thought and also in educational practice” (Bachelard, 1991, 27). For them, the purpose was not to separate history and education. On the contrary, by referring to the theory of Yves Chevallard, Artigue separated epistemology and didactics:

> The student cannot be reduced to an epistemic subject or to a cognitive subject. His behaviour is also and almost determined in priority by his status of didactical subject. [The epistemological analysis] shows all that separates these two fields: the epistemological one and the didactical one. This is this fact, which is at the centre of the theory of Y. Chevallard already quoted (Artigue, 1990, 278).

HISTORY OF MATHEMATICS IN CLASSROOMS

Laurent Vivier is a French teacher, who wrote a paper in 2010, on "a theoretic background on the notion of tangent in the secondary teaching" (Vivier, 2010), he proposed to solve a problem on the teaching of tangent by the Descartes' method on tangents. Evgenios Avgerinos and Alexandra Skoufi are teachers in Rhodes. In a paper of 2010 "On teaching and learning calculus using history of mathematics: a historical approach of calculus", they used the Fermat's method on tangents.

Descartes' method in classroom: an adaptation

Laurent Vivier tried to solve one problem of teaching, which is how to introduce the notion of tangent before the notion of derivative? It is a question on an anti-didactical inversion since in the French Curriculum, the derivate calculus is presented before the tangent and used to find tangents. He considered that a historical light would permit to define an alternative teaching: thus, history is used against an anti-didactical inversion. For this purpose, he compared Descartes and Fermat's methods from the point of view of a teaching approach. For him, Descartes' method has the advantages to correspond to a properly defined class of curves, to be an entirely algebraic method and to be easily understood. He remarked that it could be adapted to find a straight line and not a circle which is tangent to a curve. While, Fermat's method permits us to find tangents to algebraic curves easily, it has the disadvantages to be difficult to explain and "it is already in analysis". Moreover, Fermat didn't give a class of curves for which the method works.

Vivier adapted the Descartes' method by intersecting the curve by a straight line, here a parabola. We can note that Descartes used also this method in his correspondence. He proposed a problem to his students where he considered a parabola $y = x^2$, a point A with coordinates (a, a^2) and the secants passing through A whose the equations are $y = k\ (x - a) + a^2$. The question is to find the tangent among the secants. Vivier concluded his paper by this question: what is a curve?

Fermat' s method in classroom: a rational re-construction

Avgerinos and Skoufi introduced the teaching of differential calculus inspired by the principles of the theory of Realistic Mathematics Education of Koeno Gravemeijer, which promotes real situations in teaching. They wrote:

> Fermat discover how applies the [method of maximum and minimum] before in extrema process of neighbouring points, using the mysterious *E,* for finding tangent line of a curve

> $y = f(x)$. Let P (a, b) is a point of parabola and P' a neighbouring point in curve with coordinates $(a + E, f(a + E))$. If the P' be found too much near the P then could one say that the secant PP' coincides with the tangent in the P (Avgerinos & Skoufi, 2010, 94)

The authors proposed a rational re-construction of Fermat's method of tangent, where they used coordinates and function symbolisms. Further, they used the slope of the tangent, trigonometry and finally the notion of limit. It is not a *histoire dépaysante,* in despite or because they had been disoriented by the "mysterious E" of the method. They guided students to apply the method to the function $f(x) = -x^3$. They considered a point P of the curve with coordinates (x, y) and a neighbouring point P' with coordinates $(x + E, f(x + E))$, T the section of tangent with x-axis and $TQ = c$. The students are asked to calculate the ratio $f(x) / c$, then "to set inside" $E = 0$ to find the result. In this re-construction, the difference between a curve and a function is not examined, nor the history of the concept of function.

In contrast to Avgerinos and Skoufi, Schneider used Fermat's method to understand the students (see above), because there is no slope in the procedure of Fermat. We have two completely different readings of Fermat's method. In the historical reading of Schneider, the method is linked with the "mental object" tangent of the students. So the students are nearer to Fermat, because for them the notion of slope is not associated with the notion of tangent. In her study, the history comes against an anti-didactical inversion because it permits to understand the difficulties of the pupils with the order of the Curricula, which goes from the analysis to the slope. While the modern interpretation of Avgerinos and Skoufi obeys and reinforces the anti-didactical inversion.

CURRICULA, DIDACTICS AND HISTORY

Anna Sierpinska is a researcher in didactics who works on understanding. In one of her first papers "On understanding the notion of function" of 1992, she wrote on the relations between history and didactics. David Dennis is a researcher in mathematics and science education, he wrote in 1995 a thesis on historical perspective for the curriculum titled "Historical perspective for the Reform of Mathematics curriculum geometric curves drawing devices and their role in the transition to an algebraic description of functions".

From « epistemological obstacles » to a theory of « beliefs »

The purpose of Sierpinska' paper concerns the evaluation of teaching: "any evaluation of a teaching design [...] has to be based on a framework that is external to it. We must have some theory about understanding and about understanding functions against which to construct or to evaluate our projects" (Sierpinska, 1992, 25). It became a theoretical problem in scientific didactics. In her paper, she introduced a notion of epistemological obstacle: "If once, we know in a new way, we contemplate our old ways of knowing and what we see are things that prevented us from knowing in a new way. Some of these things may be qualified as epistemological obstacles" (Sierpinska, 1992, 27). Her notion of obstacle is not the same as that of Bachelard, because it is something to avoid, while for Bachelard the obstacles are normal components in the process of knowing. She distinguished three levels to explain obstacles: "attitudes, beliefs and convictions", together with "schemes of thought" and "technical levels". Then she stressed the role of beliefs and schemes of thought, since, as she explained, an obstacle will be overcome if we are able to stand back from our beliefs or scheme of thought, if we see their consequences and are able to consider other points of view.

To develop and reinforce her theory, Siepinska employed the history of mathematics. She wrote that the first definitions of the concept of function presented it as an algebraic expression. Below, we will see that the history is more complicated. Then she gave some definitions of the concept of function, those of Johan Bernoulli, Leonhard Euler (in his *Introductio*), Louis Lagrange and Augustin Louis Cauchy, to conclude that mathematicians have always researched to describe relationships. For her, curves are not interesting by themselves in history but they provided a context in which analytic tools for describing relationships could be developed. She added that Leibniz introduced his calculus and the first definition of function in the context of analytical geometry and that it is in this context that he and Bernoulli coined the term « Function » and came to formulate its first definition, but it is not exact as we will see, since the context was geometrical only.

Sierpinska saw the geometric diagram of a function as an epistemological obstacle. As she explained, students happen to identify functions with the geometric diagrams sometimes used to represent them, some students view the diagrams in "synthetic and concrete way", other students have "a more analytic view of analytical representations of functions" but "the line does not represent the relation" and "rather the line is represented by the relation". For her, the didactical order, which goes from function to curve, is not questioned. Moreover, she thought that it is the historical order from some definitions of the 18^{th} century. The idea that this order would be an anti-didactical inversion does not emerge.

On curves and functions: epistemological versus historical studies of concepts

In a part of the paper of 1992, titled "epistemological studies versus historical studies of concepts", Sierpinska wrote:

> An epistemological study of a concept differs from its history. Histories of a mathematical concept are usually presented as if the concept's development followed a smooth curve with positive gradient. Learning cannot be thus modelled. At greater cognitive depths catastrophe occurs (Sierpinska, 1992, 58).

By these words, she separated history and education, contrary to Freudenthal. The issue is that she did not criticize the few historical works that she read. Yet, as we already saw, the history of mathematics depends on the historian. Probably, she read authors like Wilder but not others, like Brunschvicg. From this point of view, to come back to the Leibniz's texts themselves is interesting, as we have already seen.

Leibniz gave a first definition of function in a paper of 1694 "[On] constructions of a curve from a property of its tangents", but he used the word "function" in 1673 and in 1692 with the same meaning and about the same problem, the inverse problem of tangents. The inverse problem of tangents is a geometrical problem, which consists to find a curve when the tangents in each point are known.

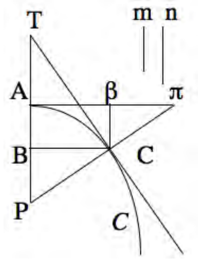

Figure 6. Leibniz's geometrical figure for the definition of function

In 1694, Leibniz wrote: "I call function a finite straight line exclusively determined from straight lines drawn from a fixed point to a given point of the curve. Coordinates *CB*, *Cβ*, tangent *CT*, sub-tangent *BT*, normal *CP* are functions of the point" (Leibniz, 1989, 271). That means, that the context is not algebraic but geometrical. The calculus was invented to solve problems on curves and not problems on functions. More precisely, Leibniz introduced the notion of function to solve a difficult problem, which is to find a curve tangent to a family of circles. For this purpose he used his calculus, and two ways to characterize a curve, which are a differential equation or a series. Twenty years later, Bernoulli gave another definition in a paper "On the isoperimetrics" of 1718: "Definition. We called function a variable magnitude or quantity composed in any manner of this variable magnitude x and of constants F x". He did not indicate the manner, but of course he did not only consider algebraic polynomial equations. In his *Introduction to Analysis of the Infinite* of 1748, Euler called function an analytical expression but he changed, after the controversy on the vibrating strings, to define a function as a dependence between variables.

History against anti-didactical inversion

David Dennis applied the ideas of the "radical constructivism" of Jere Confrey, who used epistemological arguments to critique standard historical descriptions of mathematics and used history to describe, examine and legitimise students' conceptions. Dennis saw history as a source of contexts and activities, his intention is to use mathematical history "to create a broad and flexible notion of how language evolved in response to activities and experience". He knew and criticized the use of history of mathematics by Sierpinska because she briefly describes a variety of historical conceptions of functions and gives some details from original sources, but as he wrote, “her overall theory of history and its relations to education, remain progressive-absolutist”. He added:

> History is not seen by her as a source of conceptual diversity, but almost as a set of pitfalls to be avoided or overcome. She suggests that some sense of history can be useful in helping students to overcome these possible obstacles, but there is no indication that student investigations within a given historical conception might offer valuable insights that are obscured by modern conventions (Dennis, 1995, 27).

For him, the purpose of historical and investigations is quite different, in particular:

> Historical discussions of the social and technological history of the scientific revolution would connect such mathematical investigations directly with larger cultural issues, but most importantly these investigations would provide students with more appropriate, dynamic, geometric experience (Dennis, 1995, 200).

Dennis questioned the teaching of the concept of function: “a fundamental goal of mathematics education is for students to develop an understanding of the concept of a function. In mathematics classrooms curves are usually created from algebraic equations or numerical data, and only rarely by physical or geometric actions” (Dennis, 1995, p.198). Like Schneider, he asked that a pedagogical problem should be linked with the curricula. He remarked that, even before algebraic equations are found, one can often determine tangent lines, areas between curves, and arc lengths of curves, all from an analysis of the actions which produced the curves. History shows that, as we saw with the methods of tangents. Thus, his question concerns an anti-didactical inversion and opposes an historical order of knowledge and to a didactical order, which goes from function to curve. Thus, he was interested by curves themselves. Here, history is used against an anti-didactical inversion and as a tool to criticize Curricula:

> What is governing our choice of curriculum? It would seem to be regulated by algebraic convenience. Students are asked to consider many curves that I have never seen in daily life, simply because their equations are tractable.
>
> The role of functions as conceptual tools for the analysis of curve drawing actions reverses the usual epistemic role that they play in current curriculum where functions are used to create curves (Dennis, 1995, 175).

That means that, contrary to Chevallard’s theory, the cognitive subject can be more important than the didactic subject. In his thesis, Dennis gave numerous and various examples of construction of curves in history: curves are the heart of the learning of analysis.

ANTI-DIDACTICAL INVERSION: ON HISTORICAL AND DIDACTICAL ORDERS

We come back to the anti-didactical inversions met until here. We saw that Schneider considered the order from slope to tangent, Vivier the order from calculus to tangent and Dennis the order from function to curve. All these inversions concern the order between notion of function and notion of curve. The historical order goes from the notion of curve to the notion of function, but between them there are constructions of concepts of curve in the years 1630, which are strongly linked with the methods of tangents (Barbin, 1996). Here I distinguish notion and concept in this manner: a notion takes is meaning in relation with problems (to solve them) and a concept takes its meaning in relation with concepts into a theory. For the curve, for instance, we can speak about a notion of parabola as a way to solve the problem of the duplication of a cube, but it appears as a concept in the Apollonius' *Conics*.

Fermat and Barrow proposed a notion of curve in their works on tangents. Following the dispute on tangents between Descartes and Fermat, the latter one felt obliged to give " a foundation" to his method (Barbin, 2015). He wrote in a paper titled "On the same method": "we suppose the tangent already found at a given point on the curve, and we consider by adequality the specific property of the curve, not only on the curve itself, but on the tangent to be found (Fermat, 1981, 141). As we saw, Barrow consider that "an indefinitely small part of the tangent can be substituted for an indefinitely small arc of the curve". That means that in these two methods, a curve can be considered as composed by parts of tangents. This notion of curve permits them to give an account for the procedures of their methods.

Descartes gave two definitions of what he called a "geometric curve". In the second Book of his *Geometry* he characterized them by this way: "they can be conceived as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede; for in this way an exact knowledge of the magnitude of each is always obtainable" (Descartes, 1925, 316). But some lines later, he added: "all points of those curves which we may call 'geometric' that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation". He did not prove that these two definitions, one in terms of motions and the other in terms of equations, are equivalent. But he gave some examples, where he defined a curve by motions and obtained an equation for the points of the curve.

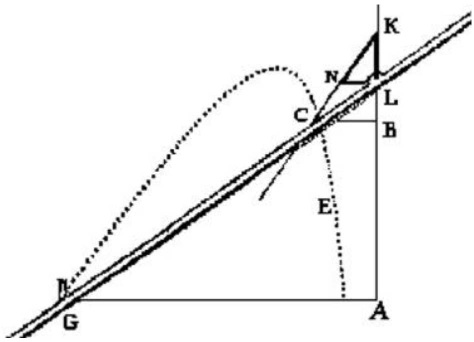

Figure 7. A Descartes' curve defined by motions

In the first paper "Nova methodus" of 1684, where Leibniz introduced his calculus, he explained that his method does not only concern curves associated to algebraic equations, but also the others, what he called "transcendental curves". He wrote:

> It is clear that our method also covers transcendental curves – those that cannot be reduced by algebraic computation, or have no particular degree – and thus holds in a most general way without any particular conditions.
>
> In its principle, to find a tangent consists of drawing a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinitive number of angles, which for me is equivalent to the curve.[...] We can always obtain the value of *dx* : *dy*, the ratio of *dx* to *dy*, or the ratio of the required *DX* to the given *XY* [*dx* : *dy* :: *DX* : *XY*] (Leibniz, 1989, 110-111).

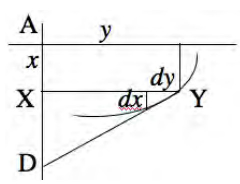

Figure 8. The infinitesimal triangle in the Leibniz's method

A curve can be considered as a polygon with an infinite number of infinitely small sides. Leibniz used the similarity of the infinitesimal triangle, with slides *dx* and *dy*, and the triangle *XDY* to establish the fundamental proportion of his calculus.

Descartes distinguished two kind of curves: the "geometric curves" described with motions, where each motion is completely determined by the others, and the "mechanic curves", which are described by independent motions – like the spiral. Moreover, he announced that the geometrical curves are expressed by algebraic equations. Leibniz also distinguished two kinds of curves: the curves associated with algebraic equations called "algebraic curves", and the others, called "transcendental curves". Thus, the mechanic curves of Descartes are not the transcendental curves of Leibniz, because Descartes always considered the curves as produced by motions, while for Leibniz, all the curves had a "regular rule": algebraic equation, differential equation or series. This distinction is important in a pedagogical context, but also in a historical context, since Leibniz and Newton researched different ways to construct curves, despite of their calculus (Bos, 1986, Knobloch, 2006). The concrete production of curves enlarges the teaching to the studies of motions and optics, to physic problems where the unknown is a curve (Barbin, 2006).

History shows the role of the methods of tangent in the construction of a concept of curve, linked with a concept of tangent, into a theory. In Roberval, a curve is the trajectory of a point in motion, the tangent is the direction of motion and the method is cinematic. The concept of curve takes it's meaning into a theory of the cinematic. In Descartes, a curve is described by an equation, the tangent is obtained thanks to the equation of a circle and the method is algebraic. The concept of curve takes it's place in algebra. In Leibniz' infinitesimal method, a curve has to be conceived as a polygon with infinitively small sides, the tangent is one of its sides and the method use infinitesimal magnitudes. The new theory is the calculus of differences.

Conclusion: Curriculum and anti-didactical inversions

The historical order is not the order proposed in the Curricula, and we observe that anti-didactical inversions is a subject of many works – some of them are examined in this paper. But, accordingly with Freudenthal, a historical course would be used for teaching. Vivier examined this possibility locally by adapting Descartes' method, and Dennis did more radically. But is it possible to use history of mathematics without changing the Curricula? The answer given by Avgerinos and Skoufi consists in a re-construction where the result is a hybridization, not necessarily comprehensive by students and not more efficient than the classical calculus. As we saw also, history of mathematics is used in didactics research, more often to evaluate or to reinforce didactical theories than to construct a teaching method. In this kind of research, history and education are separated, contrary to the Freudenthal' s philosophy.

The question can be also asked in another manner: how teachers and researchers have to advance in face of Curricula, which are producers of anti-didactical inversions? Can history be used and adapted in any Curriculum? What will be the meaning of these changes? What will be the results? Luis Radford is a researcher in science of education who examined these questions in a paper of 1997, he wrote:

> The way in which an ancient idea was forged may help us to find old meanings that, through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of elaboration of teaching sequences [...] in order to reconstruct accessible presentations [of history] for our students (Radford, 1997, 32)

The proposal would be to reconstruct history of mathematics, to render it compatible with curriculum thanks to didactic works. With the examples given in this paper and others, we can imagine the danger of a terrible anti-didactical inversion, which would be the "didactical transposition" of history of mathematics. As Freudenthal wrote in 1986, Chevallard's didactical transposition is "the expression of an anti-didactical conception" (Freudenthal, 1986, 327). But, another axis would be to consider history of mathematics as a source to construct new Curriculum, introducing a most important interdisciplinary and cultural part in teaching. Dennis asked what is governing our choice of curriculum. We can add why should we prefer to see a student as a "didactical subject" rather than as a "cognitive or epistemological subject". Indeed, the question of the order of knowledge is an important one in teaching, linked with epistemological ideas of simplicity and generality, which concerns the comprehension of students inside mathematics, but also in relations with other scientific fields. Freudenthal's paper has the virtue of stressing the role of history of mathematics to examine the anti-didactical inversions but also to propose a reflection on the order of knowledge in Curricula.

Acknowledgment. I thank Leo Rogers very much for his comments and his help to write the paper in English.

NOTES

[i] It was an answer to the researcher in didactics Yves Chevallard, about two manners to make history with "bare hands" or with "hands full", full of didactical concepts.

[ii] translated in Struik, D. J. (1969), p.259.

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