THE MATHEMATICS DEVELOPMENT OF THE BOOK SEA MIRROR OF CIRCLE MEASUREMENTS (CEYUAN HAIJING)

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ABSTRACT

The book "Sea mirror of Circle measurements" (Ceyuan Haijing) is completed in 1248 by Li Zhi. The mathematics of the book is to investigate cases of circles inscribed in a right angle triangle. The circle in the book represented a circular fort, with a diameter set at 240. Different information on the sections of the triangles are given in order to calculate the diameter using Tian Yuan Algebra (setting up equations). If a, b, c are respectively the three sides the right angle triangle and the diameter of the inscribed circle is d, then we have the following two conditions: (i) $a^2 + b^2 = c^2$ (ii) $a + b = c + D$ (D as diameter of the inscribed circle). There are 12 chapters in the book Ceyuan Haijing, and each questions in the book is deal with individually in setting up equation to find the diameter. In this study, the mathematics contents of chapter 3 to chapters 6 could be summarised into a general type of mathematics problem based on the above two conditions. For example, questions in chapter 3 can be summarised as question satisfying the following two conditions (i) $\langle c + b\rangle(b) = p$, and (ii) $\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constant (here $\langle a\rangle$ denotes a ratio representation, for example $r = \frac{ab}{a+b+c}$ is denoted by $r = \langle a\rangle(b) = \langle b\rangle(a) = \langle ab\rangle$), As questions in chapter 3 and chapter 4 are symmetric, the process on setting up equation in chapter 3 and 4 is very similar. The same process applies to questions in chapter 5 and chapter 6, as questions in these two chapters are also symmetric. The exploration of this general method will help to relate the work in mathematics in each chapterof the book and discover the possible relation among the thinking of the problems.

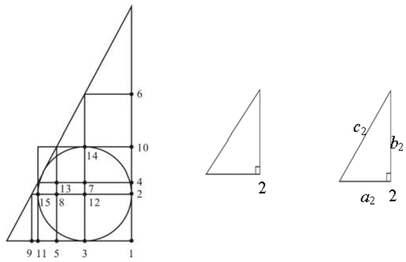
1 Introduction

In the book of sea mirror, there are a total of 12 chapters and 170 mathematical questions. All questions required to find the diameter of a circular fort inscribed in a right angle triangle, and in the following question format.

[Assume there is a circular fort of unknown diameter and circumference.]
One person walks out of the south gate 135 steps and another person walks out of the east gate 16 steps, and then they see each other.
[What is the diameter of the circular fort?]

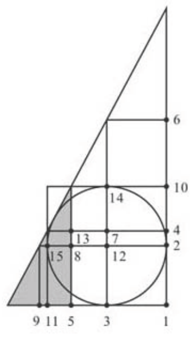
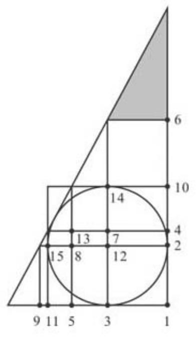
The statement in the squared bracket is the condition and question tag of the problem.

The condition of the question could be described by the following diagram. The point 2, 14, 15, and 3 in the diagram represent the four gates according to the following 2 (west gate), 14 (south gate), 15 (east gate), 3 (north gate). And in the diagram, the numbered point is the vertex of a right angle triangle. The three sides are denoted by a (gou, the base), b (gu, the height), c (xian, the hypotenuse). There are 15 triangles in the diagram. The triangle are numbered, for example, triangle 2 is the triangle with the right base vertex be numbered 2, and then the three sides corresponding are a_2, b_2, and c_2. The book aimed to establish, for each question, equation that will lead to the answer of the diameter of the fort (which is always 240). The unknown variable of the equations studied is given the name Tian-yuan (celestial element).



However, though there are some general formula mentioned in chapter 1 for solving the equation, there is no one general method of solving the equation in each chapter. The equation in each chapter is deal with individually. For example the process of setting equations of the 17 quetsions are quite different.

This paper aimed to give a general solution for each of the chapter 3, 4 5, and 6. For example, in chapter 3 of the book. Two conditions are given (uaually the length of two segments of the triangle in the 17 questions. For chapter 3, one condition is always the height b_2 of the triangle, and for chapter 4, one variable is always the base a_3.

Chapter 3	Chapter 4
	

The following is the list of the 17 questions in chapter 3 and chapter 4.

The following is the explanation of solving of the problem in the book from chapter 3 to chapter 6. We will use chapter 3 as an example to illustrate the process of setting up the equations. And provide the basic formula and proof for the remaining chapters, while the details of the all the examples (70 in total) are not included here.

b_2(chapter 3)																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_4	a_{11}	b_{11}	a_{15}	a_{14}	c_{10}	c_2	c_1	c_6	b_{14}	a_{10}	c_{15}	c_{14}	c_6	c_8	$b_{14} + c_{14}$	$a_{15} + c_{15}$
b_2(chapter 3)																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
c_5	b_{10}	a_{10}	b_{14}	b_{15}	c_{11}	c_{13}	c_1	c_9	a_{15}	b_{11}	c_{14}	c_{15}	c_9	c_7	$a_{15} + c_{15}$	$b_{14} + c_{14}$

2 The setting up of the generalised method

We will start with Chapter 3 (the 17 questions for $\mathbf{b_2}$). Problems in chapter can be generalised into the following format, finding the diameter of the circle with the following two conditions.

i $\langle c+b\rangle(b) = p = b_2$, the value of p equal to the length of leg (gu) b_2,

ii $\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$

where α_0, α_1, α_2, α_3, α_4, α_5, $p = (b_2)$, q are constants.

In order to do so, we set up the following proof.

Formula	Proof (by i. it could be proved)
❶ $\langle b^2 \rangle = \frac{p^2-r^2}{2p}$	By i. $\langle c+b \rangle (b) = p$, $\langle cb \rangle = p - \langle b^2 \rangle$ $\langle cb \rangle^2 = (p - \langle b^2 \rangle)^2$ $\langle c^2 \rangle \langle b^2 \rangle = (\langle a^2 \rangle + \langle b^2 \rangle)\langle b^2 \rangle$ $= p^2 - 2p\langle b^2 \rangle + \langle b^2 \rangle^2$ $\Rightarrow \langle a^2 \rangle \langle b^2 \rangle = r^2 = p^2 - 2p\langle b^2 \rangle$ $\Rightarrow \langle b^2 \rangle = \frac{p^2-r^2}{2p}$ ❶
❷ $\langle cb \rangle = \frac{p^2+r^2}{2p}$	By i. $\langle c+b \rangle (b) = p$, then $\langle cb \rangle = p - \langle b^2 \rangle$ Using ❶ and substitute into RHS : $\langle cb \rangle = p - \frac{p^2-r^2}{2p}$ $\Rightarrow \langle cb \rangle = \frac{p^2+r^2}{2p}$
❸ $\langle a^2 \rangle = \frac{2pr^2}{p^2-r^2}$	As $\langle a^2 \rangle \langle b^2 \rangle = r^2$, that is $\langle a^2 \rangle = r^2 / \langle b^2 \rangle$ Using ❶ and substitute into RHS : $\langle a^2 \rangle = \frac{2pr^2}{p^2-r^2}$
❹ $\langle ca \rangle = \frac{p^2+r^2}{p^2-r^2} r$	$\langle ca \rangle = \langle ab \rangle \langle cb \rangle / \langle b^2 \rangle = r\langle cb \rangle / \langle b^2 \rangle$ Using ❷ and ❶ and substitute into RHS : $\langle ca \rangle = \frac{p^2+r^2}{p^2-r^2} r$
❺ $\langle c^2 \rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$	$\langle c^2 \rangle = \langle a^2 \rangle + \langle b^2 \rangle$ Using ❸ and ❶ and substitute into RHS : $\langle c^2 \rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$

Putting ❷, ❹, ❸, ❶ and ❺ into ii. We will get the equation of the diameter of the circle.

With the above 5 formula, the general procedural could be explained through examples in the problems of chapter 3.

Example 1: (problem 1 of chapter 3) Find the diameter with the information of the length b_2, c_4. Known i. $\langle c+b \rangle (b) = p = b_2$ ii. $\langle 2b \rangle (c) = q = c_4$
Original question and its method One person walks south out of the east gate without knowing the number of steps and stand still. And another person walks south from the west gate 480 steps, and then they see each other. Then he walk 510 steps to meet the first person. Method in the book To find D with length of b_2 and c_4. : $c_4 - b_2 = b_{15}$, and so $\sqrt{2(c_4 - b_2) \cdot 2b_2} = D$

In this question, the working of the generalised method is as follow:
By ii. $\langle cb \rangle = q/2$, using ❷ and substitute, we have $\frac{q}{2} = \frac{p^2+r^2}{2p}$, the equation $r^2 = pq - p^2$ is obtained after simplification, which is $r^2 = b_2c_4 – (b_2)^2$, the same as $\sqrt{2(c_4 - b_2) \cdot 2b_2}$ in the book.

Example 2: (problem 2 of chapter 3) Find the diameter based on the information of the length b_2, a_{11}. Known: i. $\langle c+b\rangle(b) = p = b_2$, ii. $\langle c-b+a\rangle(a) = q$
The original question A person leave the west gate and walk south 480 steps. Another person start from point 11, and walk 80 steps eastwards. They then see each other. Method of the Book: Let $x = D$, $2b_2 - x = 2b_{10}$ and $a_{11}(2b_2 - x) = D^2$ With $x^2 = D^2$, then $x^2 = a_{11}(2b_2 - x)$ and $-x^2 - a_{11}x + 2a_{11}b_2 = 0$

Using our format, the process of setting up equation can be described as follow: From ii. $\langle c-b+a\rangle(a) = q$, i.e. $\langle ca\rangle - r + \langle a^2\rangle = q$

Using the formula and substitute, we have $\frac{p^2+r^2}{p^2-r^2}r - r + \frac{2pr^2}{p^2-r^2} = q$,

simplify the equation and the equation of the diameter of the circle is $2r^2 + qr - qp = 0$

As $p = b_2$, and $q = a_{11}$, we have $2r^2 + a_{11}r - a_{11}b_2 = 0$. which is the same equation $-x^2 - a_{11}x + 2a_{11}b_2 = 0$ set up by the method of the book.

Example 3: (problem 3 of chapter 3) Find the diameter based on the information of the length b_2, b_{11}. Known: i. $\langle c+b\rangle(b) = p = b_2$, ii. $\langle c-b+a\rangle(a) = q = b_{11}$
The original question A person leave the west gate and walk south 480 steps. A person walk south from point 11 for 150 steps. They then see each other. Method(in the Book): Let $x = r$, $b_{11} - x = b_{15}$(area of trapezium) $b_2(b_{11} - x) = r^2$ and $x^2 = r^2$ Which implies $-x^2 - b_2x + b_2 \cdot b_{11} = 0$

Base on the general method here, the process of setting up the equation is as follow: From ii. $\langle c-b+a\rangle(b) = q$, i.e. $\langle cb\rangle - \langle b^2\rangle + r = q$.

Through substition, we have $\frac{p^2+r^2}{2p} - \frac{p^2-r^2}{2p} + r = q$, simplify the equation and we obtain the equation of the diameter of the circle: $r^2 + pr - pq = 0$. With $p = b_2$, $q = b_{11}$, the equation becomes $r^2 + b_2.r - b_2b_{11} = 0$ which is the same equation obtained in the book.

Through the above examples, it can be proved that other similar problems in chapter 3 can be solved by using the above method.

3 Chapter 4(17 questions frome on a_3, the base of the triangle 3)

Problems in chapter 4 can be generalised into the following format.

Known : i. $\langle c+a\rangle(a) = p$

ii.$\alpha_0\langle ab\rangle + \alpha_1\langle ca\rangle + \alpha_2\langle cb\rangle + \alpha_3\langle a^2\rangle + \alpha_4\langle b^2\rangle + \alpha_5\langle c^2\rangle = q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constants

As the process of proof is similar to those in chapter 3, we only list the results of the formula for comparsiion purpose.

Formula on Chapter 3	Formula of chapter 4
$\langle a^2 \rangle = \frac{2pr^2}{p^2-r^2}$	$\langle a^2 \rangle = \frac{p^2-r^2}{2p}$
$\langle cb \rangle = \frac{p^2+r^2}{2p}$	$\langle cb \rangle = \frac{p^2+r^2}{p^2-r^2} r$
$\langle b^2 \rangle = \frac{p^2-r^2}{2p}$	$\langle b^2 \rangle = \frac{2pr^2}{p^2-r^2}$
$\langle ca \rangle = \frac{p^2+r^2}{p^2-r^2} r$	$\langle ca \rangle = \frac{p^2+r^2}{2p}$
$\langle c^2 \rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$	$\langle c^2 \rangle = \frac{(p^2+r^2)^2}{2p(p^2-r^2)}$

4 Generalised process in chapter 5 and chapter 6

In chapter 5, there are 18 questions based on the information of the length of b_1. And in chapter 6, there are 18 questions based on the information of the length of a_1. We list the conditions and formula below, and provide the proof of formula for chpater 5. One example for this method is included.

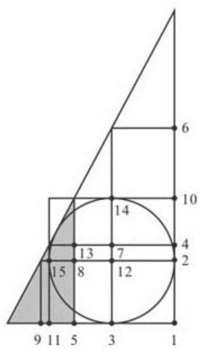
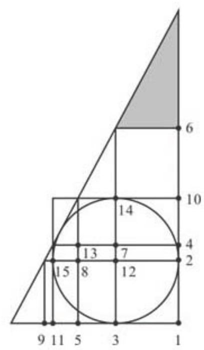
Problems in chapter 5 can be generalised into the following format.

Known : i. $\langle c+b+a \rangle (a) = p$

ii.$\alpha_0 \langle ab \rangle + \alpha_1 \langle ca \rangle + \alpha_2 \langle cb \rangle + \alpha_3 \langle a^2 \rangle + \alpha_4 \langle b^2 \rangle + \alpha_5 \langle c^2 \rangle = q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constants and problems in chapter 6 can be generalised into the following format.

Known : i. $\langle c+b+a \rangle (a) = p$

ii.$\alpha_0 \langle ab \rangle + \alpha_1 \langle ca \rangle + \alpha_2 \langle cb \rangle + \alpha_3 \langle a^2 \rangle + \alpha_4 \langle b^2 \rangle + \alpha_5 \langle c^2 \rangle = q$, where α_0, α_1, α_2, α_3, α_4, α_5, p, q are constants

Formula on Chapter 5	Formula of chapter 6
	
A. $\langle a^2 \rangle = \frac{2r^2(p-r)}{p(p-2r)}$	A. $\langle a^2 \rangle = \frac{p(p-2r)}{2(p-r)}$
B. $\langle cb \rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$	B. $\langle cb \rangle = \frac{p^2-2rp+2r^2}{p(p-2r)} r$
C. $\langle b^2 \rangle = \frac{p(p-2r)}{2(p-r)}$	C. $\langle b^2 \rangle = \frac{2r^2(p-r)}{p(p-2r)}$
D. $\langle ca \rangle = \frac{p^2-2rp+2r^2}{p(p-2r)} r$	D. $\langle ca \rangle = \frac{p^2-2rp+2r^2}{2(p-r)}$
E. $\langle c^2 \rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$	E. $\langle c^2 \rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$

5 The following is the proof for fomule in chapter 5(questions with length b_1)

Formula on Chapter 5	Proof(by i it could be proved)
A. $\langle b^2\rangle = \frac{p(p-2r)}{2(p-r)}$	From i. $\langle c+b+a\rangle(b) = p$, we have $\langle cb\rangle = p - \langle ab\rangle - \langle b^2\rangle = p - r - \langle b^2\rangle$ $\langle cb\rangle^2 = (p - r - \langle b^2\rangle)^2$ $\Rightarrow \langle c^2\rangle\langle b^2\rangle = (\langle a^2\rangle + \langle b^2\rangle)\langle b^2\rangle = (p-r)^2 - 2(p-r)\langle b^2\rangle + \langle b^2\rangle^2$ $\Rightarrow \langle a^2\rangle\langle b^2\rangle = r^2 = (p-r)^2 - 2(p-r)\langle b^2\rangle$ $\Rightarrow \langle b^2\rangle = \frac{(p-r)^2-r^2}{2(p-r)} = \frac{p(p-2r)}{2(p-r)}$
B. $\langle cb\rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$	From i. $\langle c+b+a\rangle(b) = p$, we have $\langle cb\rangle = (p-r) - \langle b^2\rangle$ using A and substitute into RHS: $\langle cb\rangle = (p-r) - \frac{p(p-2r)}{2(p-r)}$ $\Rightarrow \langle cb\rangle = \frac{p^2-2pr+2r^2}{2(p-r)}$
C. $\langle a^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$	$\because \langle a^2\rangle\langle b^2\rangle = r^2$ $\langle a^2\rangle = r^2/\langle b^2\rangle$ using A and substitute into RHS: $\langle a^2\rangle = \frac{2r^2(p-r)}{p(p-2r)}$
D. $\langle ca\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$	$\langle ca\rangle = \langle ab\rangle\langle cb\rangle/\langle b^2\rangle = r\langle cb\rangle/\langle a^2\rangle$ using both A and D and substitute into RHS: $\langle ca\rangle = \frac{p^2-2rp+2r^2}{p(p-2r)}r$
E. $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$	$\langle c^2\rangle = \langle a^2\rangle + \langle b^2\rangle$ using both A and B and substitute into RHS: $\langle c^2\rangle = \frac{4r^4-8pr^3+8p^2r^2-4p^3r+p^4}{2p(p-r)(p-2r)}$ $= \frac{(p^2-2rp+2r^2)^2}{2p(p-r)(p-2r)}$

We provide an example in chapter 6, and putting A, B, C, D and $\langle ab\rangle = r$ into ii. We will get the equation of the diameter of the circle after simplification.

Example 1: (problem 1 of chapter 6) Known: i. $\langle c+b+a\rangle(a) = p = a_1$, ii. $\langle c-b\rangle(a) = q = a_{15}$
Question A person leave from east gate and walk straight for 16 steps. Another leave from vertex 1, walk east for 320 steps. They then see each other (the sight line is tangent to the fort).

Using the representation, and from ii. $\langle ca\rangle - r = q$, Using Equation A and substitute $\frac{p^2-2pr+2r^2}{2(p-r)} - r = q$, the equation of diameter: $-4r^2 + (4p-2q)r + 2pq - p^2 = 0$ is obtained after simplification.

6 Conclusion

The generalised method for solving the work of setting up the equations show that Li Zhi may notice the symmetric property of the triangle and this properties can bed used to solve symmetric problem. This may be the reasons that the 17 questions in chapter 3 and the 17 questions in chapter are linked. So does the 18 problems in chapter in chapter 5 and 6. In working out with all possible equations, we can understand the hard work and the difficult point of the mathematics problem at that time. Also, the working of a generalised method is only possible through using good symbols.

The possible extension of using the generalise formula may also help us to understand how the degree of the equations obtained be mininized. In most of the equation provided by Li Zhi, the degree of the equations are not more than 4 and this is not easy at that time without such symbols that we used today.

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