BOOK XIII OF THE *ELEMENTS* :

Its Role in the World's Most Famous Mathematics Textbooks

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ABSTRACT

Euclid's *Elements* is the prototype geometry textbook. Yet, what this means for us and what it may have meant for a student in antiquity are not necessarily the same. A sign of this difference is Proclus' claim that Book XIII of the *Elements* concerning the construction of the regular solids and their mutual relations, which is usually left out of geometry textbooks—even those closely based on the *Elements*—is one of the two main goals of Euclid's work. It is argued here that what makes the *Elements* an ideal textbook for students, in Proclus' view, a view fitting the ancient idea of *paideia*, is its ability to turn students towards the kind of ordered cosmos that Plato sets out in the *Timaeus*. Conversely, the manner in which wholes are constructed in Book XIII, both the individual solids and their *taxis*, also turns one towards the wholeness of the entire treatment of geometry in the *Elements*.

Keywords: Euclid's *Elements*, Book XIII, Proclus, textbook

Introduction

Euclid's *Elements* is the prototype geometry textbook. The format of Euclid's *Elements* has certainly served as the inspiration for geometry textbooks in the past, and, in fact, at least until the end of the 19thcentury geometry textbooks tended to be directly based on the *Elements* itself, or a version of it, or a reaction to it.[1] One might even go so far as to say that what it meant to write a new geometry textbook then was to repackage Euclid in a more appealing, more pedagogically sound, or more modern or rigorous form.

In the case of Simson's *The Elements of Euclid* (1781/1804) and Playfair's *Elements of Geometry* (1795/1866), this meant keeping the development bound to Euclid's text. Both introduce modifications to be sure: Playfair, for example, points out the usefulness of algebra, adding though that it is in a simple form and does not depart from the nature of the reasoning (Playfair, 1795/1866, preface, page numbers absent). Still, the first sentence of Playfair's preface provides the main message: "It is a remarkable fact in the history of science, that the oldest book of *Elementary Geometry* is still considered as the best, and that the writings of Euclid, at the distance of two thousand years, continue to form the most approved introduction to the mathematical sciences." And Playfair goes on to give the reasons for this: "This remarkable distinction the Greek Geometer owes not only to the elegance and

[1] The overwhelming presence of Euclid's *Elements*, either as a model or as an *anti*-model (!) for mathematics textbooks, can be seen, for example, in Roméra-Lebret (2009), Giacardi (2006), Sinclair (2008), and Carson & Rowlands (2006).

correctness of his demonstrations, but to an arrangement most happily contrived for the purpose of instruction...” Even Legendre, whose Elements de Géométrie departs significantly from Euclid, does not fail to acknowledge Euclid, saying that while his goal is to improve on the rigor of Euclid he stays fairly close to the method of the Elements of Euclid as well as that of Archimedes’ *On the Sphere and the Cylinder* (Legendre, 1794, p.v). And it is significant that the epigraph for the book is, “Si quid novisti rectius istis, candidus imperti,” “If you know better than this, tell me candidly”(Horace, *Epistles* I:6): whether it is Legendre boasting about his own improvements of Euclid or about Euclid himself, it is a challenge that almost certainly one way or the other is directed towards Euclid.

As textbooks of geometry, all of these textbooks based on Euclid—Legendre’s as well—emphasized principally the subjects contained in Books I–VI, XI, XII of the *Elements*, that is, the basic propositions about triangles, rectangles, and parallelograms; circles and inscribed and circumscribed figures; ratio, proportion, and similar figures; and elementary solid geometry. This was indeed the core of the school course. In the case of higher education at the start of the 19thcentury, Nathalie Sinclair (2008) summarizes that situation, saying, “Typically, the geometry curriculum [she is specifically referring to that] would include the first six books of Euclid, and perhaps...Books XI and XII” (p.15).

None of this is shocking: whether or not we use Euclid as a geometry textbook, we recognize the contents of Books I–VI to constitute the *sin qua non* material for geometry instruction. In fact, if we think of the *Elements* as a textbook, we are likely to think mostly of the *first book only* of the *Elements*, the book where the definitions, postulates, and basic theorems of geometry are laid out. Legendre’s *Elements* does treat the regular solids, the content of the last book of the Elements; however, he does so in an appendix. Thus it is with considerable surprise that we discover when we read Proclus’ *Commentary on the First Book of the Elements* that the real point of the whole work, in his view, was in fact the last book, Book XIII on the five regular, or cosmic, solids, precisely the book that for the most part failed to find its way into the geometry curriculum, and still does not. What is this book about? Why might Proclus think that it is at the heart of the *Elements*? And since, as we shall see in a moment, Proclus’ interests were deeply educational, what does this say about the *Elements* as an educational text and how Proclus, at least, understood the goals of mathematics education? These are the questions that I would like to consider in this paper. I am not sure I shall succeed in answering them definitively; however, I will be pleased if they bring out the great difference that may exist between how we, since the eighteenth century, conceive the intent of a geometry text and how that was conceived in antiquity.

1 Proclus’ View of Book XIII

Before giving a brief sketch of Book XIII of the *Elements*, I want to amplify Proclus’ estimate of its importance to the whole of the *Elements* and to make it clear that Proclus’ views here are held with an eye to the *Elements* as a textbook, that is, as a book for learners.[2]

Now, if there is any doubt whether Proclus meant what he said about Book XIII, it is enough to remark that Proclus tells us *no less than four times*[3] that of the two aims possessed by the entire work one is the investigation of the cosmic figures, the five regular solids. This claim flies in the face not only of

[2]Much of what follows in this and the remaining sections appear in an expanded form in Fried (in press).

[3]*In Eucl.* (Friedlein). pp. 68, 70, 71,74 (see also pp. 82–83). The English translations are taken from Morrow (1970); however, I will always use Friedlein’s page numbers, which, being included also in Morrow’s text, will allow the reader to

what modern mathematicians, among whom I include Simson, Playfair, and Legendre, consider the core of the *Elements*, but also of what historians of Greek mathematics, raised on this more modern Euclidean tradition, often have thought. When Thomas Heath comes to this claim by Proclus, for example, he says confidently that it is "obviously incorrect," expanding thus: "It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work" (Heath, 1956, I, p.2). Heath's skepticism[4]of course is not out of place. It is indeed difficult to see how Proclus can put aside the immense wealth of other mathematical work in the *Elements* and single out so pointedly one of its shortest books, Book XIII on the regular or Platonic solids—Book XIII, in this view, must take precedence over Book X with its 115 propositions on incommensurables, or Book V which sets out the theory of proportion, or Books VII–IX on the theory of numbers, not to speak of Books I, III, IV, and VI, which make up every modern student's own elements of geometry. Nor does the deductive structure of the book give credence to Proclus' claim, for while the propositions in Book XIII do rely widely on propositions from other parts of the *Elements*, they do not draw from all parts of the *Elements*, as Heath points out, and, when they do, it is not always in a very striking way.[5]

As for the Elements being a kind of textbook, the second of the two aims of the *Elements*, as Proclus sees them, refers explicitly to learners, to *manthanontes*: it is that with Euclid's work at hand learners will have a treatment of the elements before them and the means to perfect their understanding of the whole of geometry (*In Eucl*. p.71). On the face of it, this seems to be a different kind of aim than the first. Heath unsurprisingly takes this aim more seriously than that concerning the Platonic solids, which he attributes only to Proclus' Platonic loyalties. Proclus himself, according to Heath, had difficulty reconciling the two aims and that, "To get out of the difficulty. . ." (p.2) Proclus delegated one aim to learners and the other to the subject. But, if this really is a difficulty for Proclus, it is hard to see how it is resolved by referring one aim to the learner and the other to the subject; can learners perfect their knowledge about the whole of geometry and yet exclude its subject? It is more likely that not only did Proclus not see any difficulty here but that he also saw these two aims as essentially complementary.

As for Proclus' Platonic loyalties, that is not to be denied—Heath is certainly right on that count. Moreover, Proclus does associate Euclid with Plato; indeed, he does so explicitly just when he first tells us that Book XIII was the aim of the *Elements*.[6] Proclus was in fact one of the last heads of the Platonic academy, and that should be kept in mind not only because the academy was a place of learning but also that education itself was at the center of Plato's thought. In that light, Proclus' assertion that "Euclid belonged to the persuasion of Plato. . .," meant, in effect, that Euclid's Elements had a place among the studies of the academy, that it had educational value, or, to use the far more subtle Greek term, that it contributed to *paideia*. To understand how Book XIII can be conceived as the aim of the *Elements*, then, one must view it in terms of *paideia*—not so much what facts or skills it affords the learner as what view of the world it gives the learner and what it does to the learner. This, in a nutshell, is the thesis of my entire paper. But to see how Book XIII serves this purpose, we do need to

follow the Greek text or Morrow's translation.

[4]The skepticism is also shared by Morrow (1970, p. l)

[5]Mueller is less dismissive of Proclus' claim, even though Mueller, like Heath, places great weight on deductive structure in judging the relative importance of parts of the *Elements* (see Mueller, 1981, p. 303).

[6]*In Eucl*. p. 68: "Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures".

take at least a brief look at the contents of the book.

2 A Brief Sketch of Elements, Book XIII

Book XIII can be divided into four parts. The first part spans propositions 1 to 6; the second, propositions 7 to 12; the third, propositions 13 to 17; while the last part contains just the single proposition, proposition 18.

Propositions 1–6, making up the first part of Book XIII, contain aspects of the extreme and mean ratio (*akron kai meson logon*), what is popularly called the "golden section." With one exception, XIII.2, these propositions serve as lemmas used in the rest of the book[7], particularly in propositions 16 and 17 in which the icosahedron and dodecahedron are constructed. The fact that there is an exception at all, however, makes it clear that these are not only a set of tools but also a subject for the book. Viewed this way, Euclid would be underlining that the regular solids are born in the realm of proportion; the proportion determined by the extreme and mean ratio, moreover, has a special place, being a continuous proportion (*sunexēs analogia*) whose three terms are the segments of a line and the whole line.[8] And it is at least suggestive that Plato calls a continued proportion, which ". . .makes itself and the terms it connects a unity in the fullest sense. . .," the "most beautiful of bonds" (*desmōn kallistos*).[9]

The way in which the segments of a line divided according to the extreme and mean ratio are classified and compared in XIII.6 and the other propositions in the first part of Book XIII hints, perhaps, at the way the sides of the cosmic figures will be compared and classified later in the constructions and especially in the final part, proposition 18, where all of the sides of the various figures are brought together and described in one diagram. But in propositions XIII.1–6 that hint is at best vague and visible at all only on hindsight. In the next set of propositions, propositions XIII.7–12, the prefigurement of the constructions, however, is more cogent.

Propositions XIII.7–11 all concern, one way or another, properties of regular pentagons or of the decagons derived from them, while proposition 12 concerns an equilateral triangle inscribed in a circle, telling us, in particular, that the square on the side of the triangle is triple the square on the radius of the circle. Indeed, except for XIII.7, all of these propositions consider the regular figures inscribed in circles, just as cosmic figures, the regular solids, are inscribed in spheres.

Thus, taken together, propositions XIII.1–12, what I have framed as the first two parts of Book XIII, prepare us for the constructions of the regular solids not only by providing necessary lemmas, that is, prepare us in a deductive sense, but also it prepares us, one might say, in a thematic sense. With that, let us move on to the third part of Book XIII, the constructions themselves.

To start, it is worth noting that, except for the tetrahedron, the definitions of the regular solids are not given in Book XIII, but in Book XI.[10] One arrives to Book XIII, therefore, already knowing what the solids are: the job of Book XIII, rather, is the construction and ordering of the solids. The construc-

[7]Noted also by Heath (1956) and Mueller (1981).

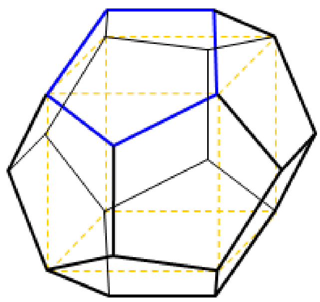
[8]Just as a reminder regarding the meaning of these terms. The quantities A, B, C are in a *continuous proportion* whenever, A:B::B:C. A segment S is divided into the *extreme and mean ratio* whenever the two parts of the segment and the whole segment are in a continuous proportion, that is, if S is divided into two segments A and B (where B is the greater segment), then A:B will be an *extreme and mean ratio* if A:B::B:S, that is, A:B::B:(A+B).

[9]*Timaeus*, 31c, translation by M. Cornford (1975), with some minor modification by myself.

[10]The definitions are precise; yet they do not allow one to visualize the solids easily. For example, the dodecahedron is defined as a "solid figure contained by twelve equal equilateral, and equiangular pentagons." The constructions in Book XIII also leave to the reader much of the work of visualizing the solids, but enough guidance is given to make that possible.

tions appear in the book as follows. The tetrahedron composed of four equilateral triangles, three triangles meeting at each vertex, is the first solid constructed in XIII.13. Next, the octahedron composed of eight equilateral triangles, four triangles meeting at each vertex, is constructed in XIII.14. The cube is constructed in the next proposition. The icosahedron composed of twenty equilateral triangles, five triangles meeting at each vertex, is constructed in XIII.16. Finally, in XIII.17 the dodecahedron composed of twelve regular pentagons is constructed. The form of each of these propositions follows the same general pattern: each requires the figure be constructed, inscribed in a sphere, andits side compared to the diameter of the sphere. The constructions themselves, however, are completely distinct—there is no master scheme for the constructions; there are five procedures for the five solids.

Thus for example, in XIII.17 where Euclid describes the construction of the dodecahedron, he begins with another previously constructed figure, the cube, and then literally builds the dodecahedron around the cube (see figure 1), showing in the course of that that the ratio of the side of the dodecahedron to the side of the cube is the extreme and mean ratio. It is worth noting in this connection that what Euclid actually constructs in XIII.17 is just one face of the dodecahedron (see fig. 2) and simply tells the reader that, "Therefore, if we make the same construction in the case of each of the twelve sides (*pleurai*) of the cube[11], a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron." In other words, the reader, the learner, is not only asked to complete the details of how the construction is to be completed for the other sides of the cube, but also to imagine the shape of the finished figure.[12]



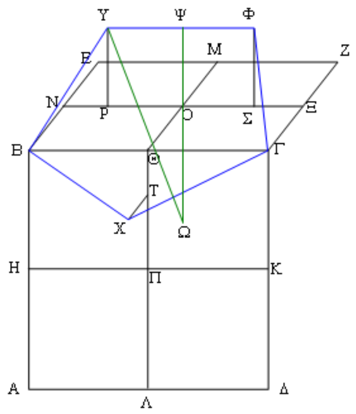
XIII.17: The completed dodecahedron

While this task is not impossibly difficult given the head start Euclid has already provided, it is not effortless. It requires an act of the imagination more than a concrete execution with, say, compass and straight edge; the construction is distinguished in this way from constructions such as that of the equilateral triangle, the first regular figure, which opens the very first book of the *Elements*.

The next parts of the proposition are 1) to show how the dodecahedron can be inscribed in a sphere and 2) that the side of the dodecahedron is an "apotome" (*apotomē*). This is a term from Book X of the *Elements*: an apotome is the difference between two rational lines that are commensurate in square only, that is two lines a and b such that sq.a: sq.b::m:n where m and n are whole numbers, for

[11] That is, the edges of the cube.

[12] A comparison might be made here to the constructions ending Book I of Apollonius' *Conica* (for example, Conica, I.52). Having provided the vertex and base circle, Apollonius asks us there to "imagine a cone"(*noesthō kōnos*), that is, to complete the picture.

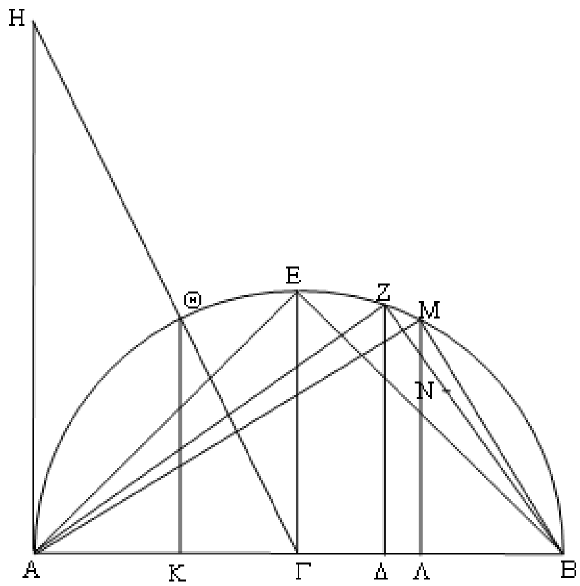


Proposition XIII.17: Construction of the face of the dodecahedron and finding the circumscribing sphere

example, the difference between the diagonal of a face of a cube and the diagonal of the cube itself. It is fair to ask, why is it important to show that the side of the dodecahedron is an apotome? It is not hard to see why one should want to know that the ratio of the side of the dodecahedron to the side of the cube is the extreme and mean ratio, for, together with the fact that the square on the diameter of the sphere circumscribing the cube is three times the square on the side of the cube, this fact allows one to construct the side of the dodecahedron given the side of its circumscribing sphere. But why must we know that it is an *apotome*? I think the only reasonable answer is that Euclid wanted to show how the ordering of the solids fits with the other great taxonomic book in the *Elements*, namely, Book X.[13]

Indeed, reading proposition 18, the closing proposition of the book, one sees that a taxonomy of the solids and their sides was truly on Euclid's mind. Proposition XIII.18, which I have deemed a distinct part of Book XIII, comprises two completely independent parts. The first part of this proposition is a synopsis of the relations between the sides of the various solids and the diameter of the sphere circumscribing them. It is unique in the *Elements* in that while it contains geometrical arguments and there are claims proved along the way, its main goal is not to prove a particular claim or solve a particular problem; what it does is "to set out and bring into comparison with one another" (*ekthesthai kai sunkrinai pros allēlas*) various lines that have already been determined. The sides of the five constructed figures are laid out all together in a half circle whose diameter is the diameter of the sphere circumscribing them. Thus, in figure 3, AB is the diameter of the sphere, AZ is the side of the tetrahedon, BZ is the side of the cube, BE is the side of the octahedron, MB is the side of the icosahedron, and BN is the side of the dodecahedron. The point Γ divides AB into half; Δ divides AB such that AΔ is twice ΔB or two thirds AB; Λ is such that $\Gamma\Lambda = \Lambda K$ where K is determined by the intersection of the circle with HΓ drawn from the extremity of AH, which is equal to AB and at right angles to it; and N divides BZ, the side of the cube, in the extreme and mean ratio. Little new is proven in this first part of XIII.18: all have been established in the constructions themselves. As the

[13]The point remains even if one fully accepts that Theaetetus was behind both Book X and Book XIII. Euclid was an editor, if not more than that: he had to concur that the connection between the theory of irrational lines and the regular solids that might have been in Theaetetus' own work (assuming that a written work existed) was essential for the flow of the work.



The setting out and comparison of the sides of the five regular solids

diagram itself suggests, what Euclid does here is to bring the cosmic figures together in the text and compare them as parts of a cosmos: one is brought together with the other like the clicking of glasses at a meeting of friends.

The second part of XIII.18 shows that no other regular solids exist besides the five constructed in the previous propositions. This Euclid shows quite simply using the fact proved in Book XI (prop. 21) that a solid angle is contained by (three or more) plane angles which together must be less than four right angles. Therefore, the solid angles of a regular solid can be contained by three, four, or five equilateral triangles, three squares, or three pentagons. The second part of XIII.18 is, in a way, a limit of possibility, a *diorismos*, but it is not a typical diorismos. Rather than bounding an unlimited number of possibilities, it shows that the five figures constructed in propositions 13–17 cannot be exceeded, that is, they form a totality. The first part of XIII.18 shows the relations between the cosmic figures with respect to a single sphere that circumscribes them: now we know that that sphere circumscribes a true cosmos, an ordered whole (as in the sense of the Greek, *kosmos*). Obviously there is more one can say about the regular solids, but, unlike any other topic in the *Elements*, this concerns a set of *objects* that cannot be further extended. There is no other book in the *Elements* that presents a *whole* in this sense and a true ending.

This display of wholeness, I believe, is the key to understanding the special status Proclus attaches to Book XIII as a *telos* and to understanding its special educative value. But to complete that argument, we need to consider first what being educative might have meant for a thinker like Proclus, and that means, among other things, considering what place mathematics occupied in education in Classical times.

3 Book XIII as an Educational Text

We have grown used to a mathematics education that begins with elementary school mathematics and continues by stages to university level mathematics. One expects to find a kind of educational program in the classical world that led to mathematical work such as Archimedes' *On the Sphere and Cylinder*, or Apollonius' *Conica*, or Euclid's *Elements*. But one's expectations here are frustrated: there is no clear path from very rudimentary mathematical training to studies in higher mathematics at institutions of advanced learning, such as the *Museum* in Alexandria and the *Academy* in Athens; there appears to be a gap.[14] Part of the problem may well be that we think of mathematics today as an entire somehow self-contained domain. The steps towards mathematics are, therefore, also mathematical in character: numbers lead to arithmetic; arithmetic leads to algebra; algebra leads to analysis, etc. There is good reason to think, however, that mathematical inquiry in classical times was not so far from philosophical inquiry (see Fried, in press). What steps lead towards that kind of inquiry? What kind of training is required? What kind of textbook? We may be looking in the wrong direction in trying to see why Euclid might serve as a textbook. Perhaps we should look more in the direction of philosophy or rhetoric, which was a central locus of educational practice in the classical world (see Bernard, 2003a, 2003b).

What I am suggesting is that the propaedeutic for higher mathematics may not necessarily have been a similar but simpler type of inquiry. If one thinks about the way to philosophy as being akin to the way to mathematics, one might consider a more general education. The idea of a general preparation that forms the constant foundation of one's thoughts, actions, and works is close to the classical idea of *paideia*. As an educational ideal, *paideia* was hardly the monopoly of philosophy; the rhetorical tradition claimed that for itself quite as much, if not more, but both aspired to genuine knowledge and a perspective on how one should live. In the philosophical tradition as represented by Plato, possessing *paideia* meant leading a philosophical way of life. As Jaeger puts it in his great three volume work (Jaeger, 1945) entirely dedicated to defining the notion of *paideia*, "[Plato's] *philosophos* is not a professor of philosophy, nor indeed any member of the philosophical 'faculty', arrogating that title to himself because of his special branch of knowledge (*texnudion*)... Although...he uses the word to imply a great deal of specialized dialectical training, its root meaning is 'lover of culture', a description of the most highly educated or cultured type of personality...[Plato's *philosophos*] is averse to all petty details; he is always anxious to see things as a whole; he looks down on time and existence from a great height" (vol. II, p.267).

Seen in the context of a philosophical education, one would have to view mathematics in terms of its potential to make one "see things as a whole" and point one to a higher life; this gives mathematics a place in one's thinking life, in one's *paideia*. The true mathematician, that is, one for whom mathematics is part of *paideia*, is the kind of mathematical learner that Proclus must have in mind when he claims that one of two principal aims of the *Elements* is Book XIII.

What then is it that this philosophico-mathematical learner gains from Book XIII? The main thing a learner gains is what we have already discussed at the end of the last section, namely, that one is

[14] Morgan (1999) writes: "Among non-literary elements of education, our ignorance of mathematics and mathematical education in the Classical period constitutes a problem in a class of its own" (p.52) (See also Mueller, 1991, p.88). For example, one of the only texts referring to older students' studying geometry is a remark by Teles, quoted by Stobeus to the effect that teachers of arithmetic and geometry (and riding!) are among chief plagues of students' life (see Marrou, 1982, p.176; Freeman, 1912, p.160).

given an image of a whole in a way that is more clear than in any other book of the *Elements*. This is both because the five solids are only five and, therefore, form a totality, and because each figure is itself a kind of totality, whose likeness to a sphere is underlined by each being circumscribed within a sphere. This is the time to remind the reader that these five figures are also called the Platonic solids because of their role in Plato's *Timaeus*. The *Timaeus* is very much a dialogue about wholeness: words referring to the whole (to *holon* or the adjective *holos*) or the all occur throughout it with striking frequency. Proclus wrote a commentary on the *Timaeus*, and it is clear that the *Timaeus* was not far from his thoughts when he wrote his commentary on the *Elements*. He refers to the dialogue often. For example, in describing the contribution of mathematics to the theory of nature (*phusikēn theōrian*) he writes:

> It reveals the orderliness (*eutaxian*) of the ratios according to which the universe is constructed (*dedēmiourgētai to pan*) and the proportion that binds things together in the cosmos, making, as the *Timaeus* somewhere says, divergent warring factors into friends and sympathetic companions. It exhibits the simple and primal causal elements as everywhere clinging fast to one another in symmetry and equality, the properties through which the whole heaven was perfected (*ho pas ouranos eteleōthē*) when it took upon itself the figures appropriate to its particular region; and it discovers, furthermore, the numbers applicable to all generated things and to their periods of activity and of return to their starting-points, by which it is possible to calculate the times of fruitfulness or the reverse for each of them. All these I believe the *Timaeus* sets forth, using mathematical language throughout in expounding its theory of the nature of the universe [literally, the "whole"] (*peri tēs phuseōs tōn holōn theōrian*). (*In Eucl*. pp.22–23)

The connection between the regular solids and the construction of the elements described in the *Timaeus* can also be seen in a scholium to Book XIII in which the identification of the tetrahedron, octahedron, cube, and icosahedron with fire, air, earth, and water, and the dodecahedron with the all (*tōi panti*) is repeated.[15] It is not altogether clear when that scholium was written or by whom, but there is good reason to believe it was written prior to Proclus' time;[16]nevertheless, it shows that not only Proclus related the material in Book XIII to what is spoken of in the *Timaeus*. Recalling the *Timaeus* is important also in the way it presents wholes coming to be, namely, as, in some way, being constructed (the nature of which is, of course, one of the points where Neo-Platonism is not clearly in line with Platonism). In the *Timaeus* itself the four solids corresponding to the elements are constructed in a heuristic way, as if only to show that the elements can be constructed rationally the way the *demiourgos* constructs the entire universe, the whole itself. The dodecahedron is not constructed in the *Timaeus*, though it is, as we have seen, in the *Elements*. What one sees then in Book XIII is the precise completion of those constructions, how they might be realized in detail. It ought to be recalled, however, that that detail is meant only to be enough to allow the learner to imagine the completion: the geometrical constructions, as we described in the last section, are a kind of prompt for them to open the eyes of their imaginations.[17]

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[15]See Euclid (1969–77), vol. V., p. 309.

[16]The particular scholium cited here is from Heiberg's *Schol Vat. series P*, which means there is a good chance it was written before Theon's time in the 4th century (see Heath, 1956, I, pp. 64–69).

[17]One might make a comparison here to how Socrates, at the start of the *Timaeus*, asks to see the image created in the previous day's talk to come to life (*Tim*. 19b): learners, reading Book XIII, must bring the cosmic figures to life.

But construction in the *Timaeus* is central not only to the conception of the individual cosmic figures, but also of cosmos considered as a whole, whether or not Plato took that construction literally or metaphorically. Be that as it may, what construction shows is how parts fit together, or, to use the more suggestive Greek word, how they demonstrate *harmonia*. Indeed it is the fitting together into a whole that makes a cosmos a cosmos. Its order is one of organization and place, of taxis, rather than, as Jacob Klein (1985, esp. pp. 30–34) has pointed out, one of *lex*, of law, which marks the modern sense of an ordered universe. It is this kind of order that is exemplified so well in proposition XIII.18 setting out the relationships of the five regular solids. As suggested in that discussion, this is not the only proposition which relates one kind of mathematical object to another—what proposition does not?—but, with the final part of XIII.18 showing that there are no other regular solids, that the five form a totality, the first part becomes a *taxis* indeed of a whole.

Interestingly enough, this brings us to Proclus' second claim regarding the aim of the *Elements*, namely, that the work presents learners with a thorough treatment of the elements of geometry and the means to perfect their understanding of the whole of geometry (*In Eucl*. p.71). Now, we have already addressed the point that learners are those who want to cultivate their *paideia* and therefore are not easily separated from other mathematicians. But what ought to be said in addition is that learners' perfecting their understanding of the whole of geometry may not be very far from their contemplating a cosmos, like that suggested by the material of Book XIII. There is no fine line between the taxis of the material of geometry and its discursive means. Thus Proclus' description of the formal aspects of geometry, of synthesis and analysis, of definitions, hypotheses, and demonstrations, flows seamlessly from his description of the material of geometry moving from the simple to the complex, from points to bodies, and from the complex back to the simple (*In Eucl*. pp.57–58). In this, he may be betraying his Neo-Platonic rather than strictly Platonic outlook. For in struggling with the question of wholeness of The Soul with respect to individual souls, Plotinus seriously considers the image of geometrical science, how, "Each theorem contains accordingly the total science in potency and the total science does not exist one whit the less" (*Enneads*, IV, 3, in O'Brien, 1964), though, it must be added that Plotinus, for his own reasons, ultimately rejects the image.

In any case, what is clear is that to the extent consideration of the five regular figures is consideration of a cosmos, an organized whole, this seemingly special topic can be related to the organized whole of the *Elements* as a science. Proclus' two aims of the *Elements* thus really become one: seeing the whole portrayed by the science becomes at once seeing the whole of the science. As a textbook for a Greek mathematician, then, we begin to see the *Elements* in a different light. Besides providing tools and specific facts, it also turns viewers toward a view of the world, or rather shows them what a world can be. This makes it a very different kind of textbook than the kind Simson or Legendre imagined, but, for one like Proclus, a book very much for students nonetheless.

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