REMARK ON THE NOTION OF GOLDEN RATIO*
—concerning "Divine Proportion" in the Renaissance period—

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ABSTRACT

In this paper, we try to show diverse aspects of the notion of Golden Ratio seen in its historical development. This notion should be discussed not only from a mathematical viewpoint, but also from various viewpoints based on human cultural activities and natural phenomena. Through the discussion we try to suggest that this notion could be understood in wider and deeper context and even to stimulate mathematics teachers' interests in the notion. And especially, we try to point out the fact that this notion was called "divine proportion" in the Renaissance period and that this name (term) would be derived from the collaborative study of Leonardo da Vinci and Luca Pacioli. Leonardo da Vinci might be the first to introduce this terminology in the "*Paragone*" of his book "*Libro di Pittura*" (on Painting). This fact is considered to be an aspect in which mathematical notions and human activities have a close relation to each other. Mathematics is a product of human wisdom, to be sure, and therefore mathematical notions should be the product of human activities. But nowadays, we are apt to perceive mathematics to be a conceptual discipline, which has been formed by cutting off many concrete human-cultural parts and by rearranging the remaining conceptual parts into a logical and concise system. Thinking so, Leonardo da Vinci's "divine proportion", as well as the notion of Golden Ratio, could be considered to be one of the most appropriate examples to mathematics education, in order to understand the original features of mathematics related to human life and culture. This kind of fact might be profitable for teachers who want to encourage the students in the mathematics classroom.

1 Introduction

The Golden Ratio might be one of the most fecund notions in the history of mathematics, and therefore, it should be also one of the most applicable notions to mathematics education in classroom. In fact, we could find various aspects concerning the Golden Ratio in historical steps of human activities; in the development of mathematics, especially geometry, in human cultural activities, for example the fine art of the Renaissance period, and even in the human efforts to explain some principles seen in Nature. The various factors of these kinds might give us lots of topics, which are interesting and even sometimes mystic, for mathematics education.

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Nevertheless, despite of these possibilities to apply this notion into mathematics classrooms, many mathematics teachers seem to be apt to depend on fragmentary topics concerning this notion. Actually, so many books and articles are published, to report and discuss diverse aspects of the Golden Ratio, on the one hand. Much of them should be written with the quite precise and steadfast intention of discussing various factors related to this notion as mentioned above. But, on the other hand, there is no denying that many topics can be taken up without full context but separately and fragmentarily in mathematics classroom. It is because the fragmentary topics on the Golden Ratio might be susceptible for students' understanding and also enable teachers to encourage easily the students in their interests and curiosity. This can be true to a certain extent, and however, it is better and even effective for mathematics education that teachers should try to understand this notion as well as possible; for, from a historical viewpoint, the notion of Golden Ratio should be deeply related to human thoughts and activities for the objects in the world. This is also related essentially to the aim and the significance of mathematics education.

In this paper, we try to show diverse aspects of the Golden Ratio seen in its historical development, to suggest that this notion should be discussed in wider and deeper context and even to stimulate mathematics teachers' interests in the notion. And especially, we try to point out the fact that this notion was called "divine proportion" in the Renaissance period and that this name (term) would be derived from Leonardo da Vinci's "*Paragone*" of his book "*Libro di Pittura*" (on Painting).

2 Brief history of the Golden Ratio

The Table 1 shows an example of the brief history of the Golden Ratio. Especially, this notion became related to Fibonacci number; it originated in Fibonacci's "Liber Abaci" (The Book of Abacus) written at the beginning of the 13^{th} century, but the close linkage between the two notions had been found and discussed from the 18^{th} century onward.

Considering the history of human efforts to investigate the Golden Ratio, we can find, from a viewpoint of harmony or mystery possessed in the notion, the three categories of human understanding, as follows:

(a) - harmony and mystery in mathematics

(b) - harmony and mystery in human cultural activity(art, etc.)

(c) - harmony and mystery seen in Nature.

(a) Harmony and mystery in mathematics The Golden Ratio is, of course, a mathematical notion, and many studies have been done over its historical development. It is because this notion possesses various properties, which are mathematically interesting and even mystic. For example, Pythagorean School is said to take notice of a regular pentagon and to adopt a pentagram as their symbol. It is well-known that the ratio between the diagonal and the side of the regular pentagon is equal to the

[1]Cf. Sculptures: "*Diadumenus*"(B.C. 430, "*Man Tying on a Fillet*"), and "*Doryphorus*" (c. B.C. 450, "Spear Bearer")

[2]The book contained the following recreational problem: "How many pairs of rabbits can beproduced from a single pair in one year if it is assumed that every month each pair begets a new pair which from the second month becomes productive?"

[3]For two consecutive Fibonacci numbers, f_n and f_{n+1}, the ratio f_{n+1}/f_n approaches $\phi(\phi = 1.6180\ldots$, the golden ratio) as n becomes larger.

B.C. 6C.–5C.	- Pythagorean School: Study on Pentagon (Pentagram) and on the notion of incommensurability. - Polykleitos : "*Canon of human proportion*," which is considered as a theoretical work that discusses ideal mathematical proportions for the parts of the human body[1].
c. B.C. 300?	Euclid: "*Elements*." Introduction of the notion of "*extreme and mean ratio*," which signifies the now called Golden Ratio.
B.C. 1C.–1C.	Vitruvius: "*De architectura*" (On Architecture), in which Vitruvius discussed the human proportion by employing a square and a circle as a frame of an ideal human body.
1202	Fibonacci: "*Liber Abaci*" (The Book of Abacus). Introduction of the Fibonacci numbers[2].
c. 1482	Piero della Francesca: "*De quinque corporibus regularibus*" (On the Five Regular Bodies), a manuscript which was not published at the time.
c. 1490	Leonardo da Vinci: Sketch of "*Vitruvian human proportion*." Leonardo da Vinci: "*Libro di Pittura*" (On painting).
1498	Luca Pacioli: "*De Divina proportione*" (Divine proportion), a manuscript. But the version published in 1509 was in the form of printing.
1525	Albrecht Durer: "*Underweying der Messung mit dem Zirckel und Richtschyt*" (Treatise on measurement with compass and ruler). Dürer introduced implicitly the notion of the logarithmic spiral.
1638	René Descartes: Study on the logarithmic spiral which Descartes called the "*equiangular spiral*."
1753	Robert Simson: Study on the relation between the Golden Ratio (ϕ) and Fibonacci numbers[3].
1835	Martin Ohm: "*Die Reine Elementar-Mathematik*" (The pure elementary mathematics), in which the term of "Golden" Section was first used.
1830s	Alexander Braun, Auguste Bravais & his brother Louis Bravais: Discussion on the Fibonacci sequence seen in diverse botanic Phenomena (seeds of a sunflower, a pinecone, a kind of leaf arrangement, etc).
1843	J. P. M. Binet: Introduction (reintroduction) of the formula to gain any Fibonacci number; the formula is given by using ϕ.
1914	Thomas Cook: "*The Curve of Life*." Discussion many types of spirals in Nature, including the spiral of Nautilus shell.
1942	D'Arcy Thompson: "*On Growth and Form*" (first published in 1917). Introduction of the term "homogeneity" (self-similarity), which concerns with the Fibonacci numbers and the Golden Ratio.

Brief History of the Golden Ratio

Golden Ratio ($\phi = 1.6180\ldots$). In addition, as to all the segments seen in a pentagram, the ratio of every two segments taken in order of length is equal to this ratio. And, when the five diagonals are drawn in a regular pentagon, they will form another regular pentagon in the center; this operation can be continued without end. Therefore, the Golden Ratio can be represented with only the unit "1" (in the form of a continued fraction, $\phi = \frac{1}{1+\frac{1}{1+\frac{1}{\ddots}}}$). These properties of this ratio indicate that it possesses a kind of harmony in mathematics. According to this fact, the Golden Ratio also creates us some kind of mystic atmosphere.

(b) Harmony and mystery in human cultural activity In fact, the Golden Ratio has affected human mind. It stimulated the various artists to pursue their ideal harmony and beauty. Such was the case especially in the Renaissance period. For example, lots of studies on Leonardo da Vinci' drawings have been done from a viewpoint of this notion. Considering Leonardo's *dessin* of human proportion (*Vitruvian human proportion*), it is often said that the position of navel might indicate the Golden Section of the human height (between the top of a head and the sole of a foot). This idea has been still discussed even in the 19^{th} century, by the German psychologist Zeising[4]. We can find other example of this kind.

(c) Harmony and mystery seen in Nature The Golden Ratio often appears with various forms in Nature. For example, by investigating the features of the spiral that the *Nautilus* shell creates during its growth, we can find that the spiral is almost logarithmic and that the coefficient of the growth is related to the Golden Ratio. Another example is seen in the arrangement of sunflower seeds. The seeds of the sunflower spiral spread outward in both clockwise and counterclockwise directions from the center of the flower. The numbers of clockwise and counterclockwise spirals are considered to be two consecutive numbers in the Fibonacci sequence. Nowadays, we can find many cases related to the notion of Golden Ratio in the morphologic study on the nature. This fact impresses us the harmony and the beauty of the nature.

3 The name of "Divina Proportione" written by Luca Pacioli

Although the notion of the Golden Ratio is quite fecund and popular for many teachers and students, it is possible that an essential question will arise in human mind; it concerns the name of the notion. From a viewpoint of harmony and mystery, "Golden Ratio" should be just the right name, i.e. ideal and even perfect for naming. Then, who named it first? It is not so easy to clarify, because we should be confronted at least with the three different names called to each period in its history: "*extreme and mean ratio*," "*divine proportion*" and "*golden ratio*."

It is perhaps Euclid (c. B.C. 300) who was the first to define explicitly the notion. In the Book VI of his "*Elements*," he gives the definition, as follows:

Definition 3 A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less[5].

[4] About Zeising's discussion, for example see below: Adolf Zeising, *Neue Lehre von den Proportionen des menschlichen Körpers, aus einem bisher unerkannt gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetze*, Leipzig, Rudolph Weigel, 1854

And in Proposition 30, he also shows the geometrical construction of this ratio[6]. This definition means the golden section. It is considered as a quite mathematical definition which formally depends on geometry. In addition, in the Book XIII, he discusses the five regular polyhedrons, especially a regular dodecahedron with the notion.

On the other hand, the name of "Golden Ratio" appeared much later, in the 19^{th} century. Johannes Tropfke, a German mathematician, suggested this matter in his book "*Geschichte der Elementar-Mathematik*" (History of elementary mathematics). According to Tropfke, it was the German mathematician Martin Ohm who first adopted the name of "Golden Section" (not precisely "Golden Ratio") in his second edition of "*Die Reine Elementar-Mathematik*" (The pure elementary mathematics) published in 1835. Ohm said that "one also customarily calls this division of an arbitrary line in two such parts the golden section[7]." Tropfke also noted there were other mathematicians who used this name at that time. So it might be certain that the name of "Golden Ratio" was used and established around in the 1830s and in the 1840s.

Then, what about the name of "divine proportion?" It is known that this naming appeared in the Renaissance period. It was perhaps Luca Pacioli, an Italian friar and mathematician, the first one to use it, when he published a book with the name of "divine proportion" as a title. In fact, he wrote "*De Divina Proportione*" (Divine Proportion)[8] in 1498, and presented the manuscript to the grand Duke of Milan. And in 1509, he published the book of the same title in the form of the letterpress printing.

In his "*De Divina Proportione*", Pacioli treats various kinds of polyhedrons, but the discussion centers round on the five regular polyhedrons. In addition, the figures of various kinds of polyhedron are attached at the end of this book, and Leonardo da Vinci drew them. We should notice, here, that a regular dodecahedron is formed with twelve regular pentagons as its faces and that the Golden Ratio can work as an important factor in the relation between the regular polyhedrons: for example the relation between an octahedron and an icosahedron[9].

At the starting point, Pacioli, by himself, adopted Plato's idea on the five regular polyhedrons. In "*Timaeus*"[10] Plato had considered that the five regular polyhedrons correspond to the four elements of the world and the fifth essential element; Earth corresponds to a cube, Water to an icosahedron, Air to an octahedron, Fire to a tetrahedron, and finally, the fifth essential element, i.e. the sacred Cosmos, to a dodecahedron. On the basis of this idea, Pacioli developed his argument that the ratio appearing in a pentagon was considered as an important an even noble factor to discuss polyhedrons. Therefore, for Pacioli, such a ratio should be called "divine proportion".

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[5]Thomas L. Heath, "*The Thirteen Books of Euclid's Elements*," vol. 2, Dover, 1956, p. 188.

[6]Ibid., pp. 267–268. "To cut a given finite straight line in extreme and mean ratio."

[7]Johannes Tropfke, *Geschichte der Elementar Mathematik in systematischer Darstellung*, Berlin und Leipzig, 1923, Band 4, p. 187.

[8]Luca Pacioli, *De Divina Proportione*, Introduzione di Augusto Marinoni, Silvana Editoriale, Milano, 1986. See also French translation, Divine proportion, (Traduction de G. Duchesne et M. Giraud avec la collaboration de M.-T. Sarrade), Librairie du Compagnonnage, 1988.

[9]For example, see H. S. M. Coxeter, *Introduction to Geometry*, Second Edition, John Wiley & Sons, Inc., 1989.

Coxeter writes: "The faces surrounding a vertex of the icosahedron belong to a pyramid whose base is a regular pentagon. Any two opposite edges of the icosahedron belong to a rectangle whose longer sides are diagonals of such pentagons. . . . . this rectangle is a golden rectangle. . .". Therefore, "the twelve vertices of the icosahedron are the twelve vertices of three golden rectangles in mutually perpendicular planes". Considering that a golden rectangle can be inscribed in a square so that each vertex of the rectangle divides a side of the square in the golden ratio, an octahedron can be also constructed with these three rectangle mentioned above (pp. 162–163).

[10]For example, see Plato, *Timeaeus and Critias*, Penguin Books, 1977, pp. 73–77.

4 Concerning the name of "Divine Proportion"

As mentioned above, Pacioli's "*De Divina Proportione*" seems to be the first book with this name as a title. But Pacioli seems not to be the first to call "divine proportion" to the ratio in question. To investigate the problem, we should look around the situation at that time.

At first, we should pay attention to the fact that Pacioli's "*De Divina Proportione*" is said to be a kind of plagiarism. Pacioli seemed to plagiarize Piero della Francesca's writing on the same subject, "*De quinque corporibus regularibus*" (On the Five Regular Bodies)[11]. But in the latter book, although Piero treated the five regular polyhedrons, he did not use the term "Divine Proportion". Secondly, we should also take notice of Pacioli's book "*Summa*"[12] written in 1494. In this book, Pacioli discusses geometry by presenting certain of the books of Euclid's "*Elements*." But in the part correspondent to the Book VI of the "*Elements*," Pacioli did not use the term "divine proportion," but the same term as Euclid. Consequently, it is suitable to think that Pacioli and Piero have not used yet the term "divine proportion" before 1498.

On the other hand, we can find the term "divina proportione" (divine proportion) in the writing of Leonardo da Vinci, precisely in the "Paragone" of his "*Libro di Pittura*" (On painting)[13]. This is Leonardo's posthumous work rearranged from the manuscripts by his disciple Francesco Melzi[14]. But, since Pacioli mentioned Leonardo's work on painting in his book "*De Divina Proportione*," Leonardo had already written some parts of "Libro" in 1509. In it, the term "divina proportione" appears three times, and all of them correspond with harmony and beauty which are kept within the sublime deity. Some studies suggest that Leonardo's term "divina proportione" signifies, in itself, the mathematical meaning of the Golden Ratio[15], and however, this insistence might be still disputable. Anyway, it is certain that Leonardo used the term.

Moreover, we should also notice that Pacioli and Leonardo had a close acquaintance with each other. Since 1481, Leonardo has resided in Milan, where Pacioli was assigned in 1496. The two deepened their friendship and perhaps they might exchange their discipline. Therefore, it is possible to consider that Pacioli's book "*De Divina Proportione*" might be the result of their collaboration.

Considering such a situation, we could frame a hypothesis. It would be Leonardo who first introduced the term "divine proportion" in his "*Libro*" around in the 1490s. Although Leonardo's term would be based on his philosophical idea, it might not be appreciable whether it contains the mathematical meaning. Pacioli might be influenced by Leonardo's idea through their collaboration, and finally the former would adopt the term "divine proportion" including its philosophical meaning.

In consequence, Pacioli achieved his "*De Divina Proportione* as a mathematical treatise on the regular polyhedrons with the notion of "Golden Ratio" and it should be supported with Leonardo's idea and terminology (see Fig. 1).

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[11] Piero della Francesca, *Libellus de quinque corporibus regularibus*, Giunti Gruppo Editoriale, Firenze, 1995.

[12] Luca Pacioli, *Summa de arithmetica, geometria, proportioni et proportionalita*, Reprint (Originally published,Venice, 1494), Kyoto, Daigakudo Books, 1973.

[13] Leonardo da Vinci, *Libro di Pittura*, Giunti Gruppo Editoriale, Firenze, 1995.

[14] Concerning Melzi's effort to compile Leonardo's manuscripts on Painting, see for example: *Leonardo on Painting, edited by Martin Kemp*, Yale Nota Bene, 2001.

[15] For example, Soichi Mukogawa, a Japanese historian of Art, discusses this matter in his article as follows: Soichi Mukogawa, On the Meaning of Leonardo da Vinci's 'divina proportione' in His 'paragone', *Bijutushi*, 51(2), 2002, pp. 282–296 (in Japanese).

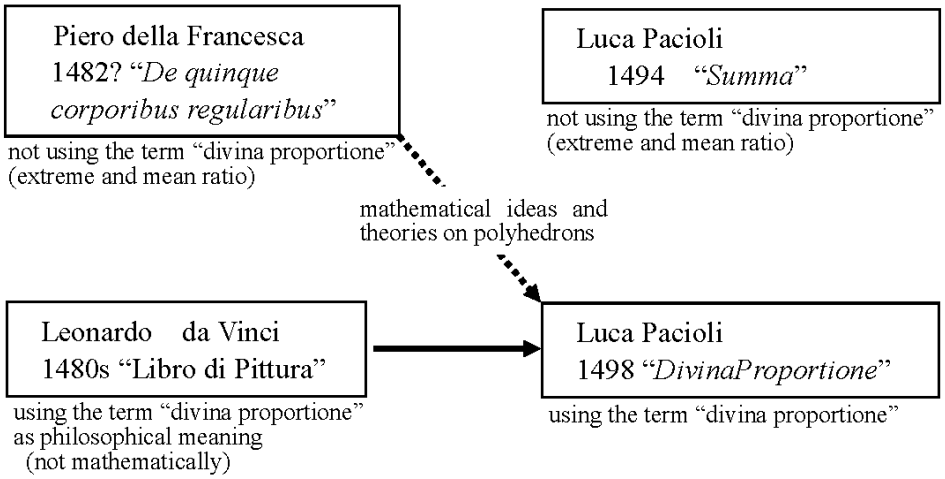

Fig. 1

5 Conclusion

The "Golden Ratio" is certainly a mathematical notion, but it has also influenced various aspects of human activities through its history; i.e. development of mathematics in itself, human artistic activities and human understanding of Nature. The name of this notion was fixed at least in the 19^{th} century after many twists and turns. And the name of "divine proportion" introduced in the Renaissance period seems to symbolize the significance of the notion; for it has provided us for some conceptual factors of harmony and mystery.

In this paper, we tried to suggest that it would be Leonardo da Vinci who first introduced the term "divine proportion." It might be the first hypothesis in the discussion on the development of the Golden Ratio. The fact seems also to symbolize the period, because in the Renaissance period, mathematics was considered to be much linked with human activities and to developed practically rather than theoretically. J. V. Field says that "the name Renaissance is really a label for a style not for a period"[16] In fact, the Renaissance mind pushed up the significance of the notion to the level of harmony, beauty, mystery and even divinity.

It is certain that historical topics are applicable for mathematics classroom, because they can give us lots of interesting and attractive contents. But it is also true that we should remember keep in our mind that the history of mathematics contains various kinds of factors, because it is also the history of human thoughts and activities.

[16]J. V. Field, *The invention of infinity*, Oxford University Press, 1997, p. 1.