VISUALIZATION OF THE MECHANICAL DEMONSTRATION TO FIND THE VOLUME OF THE SPHERE USING DYNAMIC GEOMETRY

María del Carmen BONILLA

AIPEM: Asociación Peruana de Investigación en Educación Matemática, Lima, Peru
mc_bonilla@hotmail.com, mbonilla@pucp.edu.pe

ABSTRACT

This research examines the mechanical demonstration of the mathematical process developed by Archimedes to determine the volume and surface area of the sphere. The process was divided into two parts: a mechanical demonstration and a mathematical proof (Archimedes, 2005). From an epistemological perspective we can see how mathematical knowledge arose from other sciences, specifically physics. For Archimedes the most important thing was to first understand, and then demonstrate. Motivated by the desire of enjoying the delight of the logical process, it was discovered that dynamic geometry can be used, Cabri 3D, to visualize the construction of the mechanical demonstration. These actions are guided by a didactic perspective developed in this historical study. Research does show that it is possible to combine the history of mathematics and computer sciences for the study and better understanding of the concepts that emerged throughout history.

Keywords: Archimedes, Volume of the sphere, Mechanical Method, dynamic geometry.

Theoretic framework

Philosophy of Mathematics

In the last three decades new types of evidence and argument have been created in the development of the mathematical practice, changing the rules in the area (Hanna, and Pulte Jahnke, 2006). Changes have been produced by the use of computers (as a heuristic device or as a means of verification), due to a new relationship of mathematics with empirical sciences and technology, and a strong unconsciousness on the social nature of the processes that guide the acceptance of a test. These changes have been reflected in new trends in the philosophy of mathematics. For years, philosophers have tried to define the nature of mathematics taking into account its logical foundations and its formal structure. In the past 40 years the search has drifted. The first one to highlight these changes was Imre Lakatos in the late sixties of last century, his work remains highly relevant to the philosophy of mathematics and mathematics education.

Imre Lakatos and the heuristic style

The philosophical approach that guides this work is that proposed by Imre Lakatos (1978), who emphasized the historic nature of mathematics, human activity that is not discovered but constructed.

Lakatos takes a quasi-empirical philosophical conception of mathematics, which gives it a more practical nature. He performed a comparison of two styles in the development of a mathematical proof. The first one, the deductivist, is developed by the Euclidean method. At first it mentions often artificial and mystically complex axioms, lemmas, and/or definitions whose origin is not explained, and must be accepted. Following is stated the theorem, full of harsh conditions, and finally the test. The expectation is that the student has some mathematical maturity, i.e. that he/she is endowed with the ability to accept the Euclidean arguments with no interest in the underlying problem and the heuristic aspect of the argument. Here mathematics is seen as a set of eternal and immutable truths, where counterexample, refutation and criticism are not accepted, amid an authoritarian environment. The deductivist style hides the struggle and covers the adventure in an environment where history fades, as well as the successive tentative formulations of the theorem throughout the test, the outcome being sacred.

In contrast, the heuristic style highlights the definitions generated by the test, the definitions generated by the previous tests and also the counterexamples that have led their discovery. This style emphasizes the problematic situation in the logic that has given birth to the new concept. The heuristic order would be given by stating conjecture, show proof and counterexamples to the theorem, and finally the description generated by the test.

Explanation vs. Justification

Work product of Lakatos and others led to conceptions of mathematics in general and the particular test, based on studies of mathematical practice, often combined with epistemological views and cognitive approaches. In this context, there has been a reconnaissance of central importance to mathematical comprehension, the way it is expressed, and what is considered as mathematical explanation. With these two changes in the focus attention has shifted from the supporting role to the explanatory role of proof.

> "The focus is not only why and how a test validates a proposition, but how it contributes to a proper understanding of the proposition in question, and what role it plays in this process by factors beyond logic" (Hanna et al., 2006).

Based on analysis of different studies on the problem of teaching of the proof, Hanna (1989) states that there is a growing trend away from formal proofs in the curriculum, and give more emphasis to the social criteria for acceptance of a proof. In this context she draws a distinction between mathematical proofs that prove and mathematical proofs that explain, both being legitimate, because they meet the requirements of mathematical proofs, although there are differences of opinion on the degree of rigor. The mathematics proofs that explain highlight in the convincing argument, in the role of the proof as a means of communicating mathematical ideas, generating new ways of teaching of the proofs. The proof that proves shows only that a theorem is true, a proof that explain shows that it is true and also shows why it is true, then highlights the characteristics mathematical properties that imply the theorem to be proved, ie the result depends on properties. It is therefore necessary to replace the non explanatory proofs by another equally legitimate as the first, but with an explanatory power provided by the mathematical properties on which they are based and by the mathematical message in the theorem. Mathematics educators have the task to make students understand mathematics and, therefore it is important to give a more prominent place in the mathematics curriculum to proofs that explain.

Visualization and proof from the Philosophy of Mathematics

For over two decades researches on the use of visual representation and their potential contribution to mathematical proofs are more numerous and important, mainly because computers have increased the possibilities of visualization (Hanna & Sidoli, 2007). Besides the traditional role of visual representations as evidence or source of inspiration for a mathematical statement, it analyze to what extent the visual representations can also be used in its justification, as substitutes for the traditional proof.

Researchers in mathematics have studied this issue and the different positions in this respect range from the traditional view that regards as useful complement to the proof, which can show the way for a rigorous proof but is unable to replace the rigor in the verification of knowledge gained in this way. In an intermediate term are located researchers that believe visual representations can play an essential role, though limited in the proofs, noting that in some cases visual proofs may constitute proofs if they meet certain requirements. At the other extreme, some researchers say that the visual representations may constitute evidence itself. The information can be presented in a linguistic and non linguistic way. Propositional logic is only linguistic representation. The task proposed is to extract the information implicit in a visual representation so as to obtain a valid proof. Some researchers do not believe that visual and propositional reasoning are mutually exclusive, and have developed the concept of "heterogeneous proof", where it facilitates reasoning with visual objects. It draws attention to the content of proof, it teaches logical reasoning and the construction of the proof by visual manipulation and propositional information in an integrated manner.

In summary we can see that there is no consensus on the role of visualization in mathematics, specifically in their roles of explanation and justification of the proof, although its role as an aid to mathematical understanding if accepted by all. In mathematics education there are research showing that the dynamic geometry is successful in improving students' ability to notice the details, conjecture, reflect and interpret relationships and provide provisional explanations and proofs, research that support the mathematics literature on the visualization and its help in mathematics understanding.

Prove and visualization from the Mathematics Education

In the classroom the key role of the test is to promote mathematical understanding (Hanna, 2000). One of the most effective ways to reach it is the use of dynamic geometry, helps students develop mathematical thinking, produces valid evidence and improves understanding of mathematics. In relation to the proof, even for experienced mathematicians, the rigorous nature of it, despite being defined, is secondary to understanding. It is compelling and legitimate if leads to understanding. Among the various functions of the proof, verification, explanation, systematization, discovery, communication, construction, exploration and incorporation, the first two are fundamental, but in education which comes first is the explanation, so is valued more proof that best explains.

There is a consensus among mathematicians on intuition, speculation and heuristics are useful in the preliminary stages of obtaining mathematical results, and that intuitive reasoning without proof is not a speculative branch, separate from mathematics. The dynamic geometry software has given a boost to intuition, speculation and heuristics, thanks to the exploration of mathematical objects by dragging. It encourages exploration and testing as it is easy to pose and prove conjectures. However, these potentials should not base an entirely experimental approach of mathematical justification, nor despise the focus of the proof in theory and mathematical practice. What you should do is use both exploration and prove because they are complementary and mutually reinforcing. Exploration leads to discovery and proof is to confirm the discovery. The use of dynamic geometry software on heuristics, exploration and visualization promote understanding of the test.

In both philosophy of mathematics and in mathematics education there is a mutual awareness of new types of proofs and explanation, mainly due to visualization and dynamic geometry, which is necessary to strengthen, and also it is imperative to develop a converging theoretic framework based on recent developments in both fields, in the light of the strong and realistic empiricists trends now shared and worked on by philosophers mathematicians and mathematics educators in different institutions and different research programs.

Miguel de Guzman and the History of Mathematics

Guzman (1993) considers that the History of Mathematics provides a truly humane vision of science and mathematics, not deified, sometimes crawling and sometimes painfully fallible, but also able to correct their mistakes. This view is very different from that perceived by the student when the theorems are presented as truths emerging from the darkness and lead to nowhere. The theorems acquire perfect sense in theory, after having studied it further, including its biographical and historical context. The History of Mathematics provides a temporal and spatial framework to great ideas, along with its justification and precedents; points out open problems of every age, their evolution, the situation in which they are now; and serves to point out the historical connections of mathematics with other sciences, in whose interactions traditionally have emerged many important ideas. The study of history of mathematics allows us to appreciate how the logical order of the theory mismatches the historical order, as seen in this study, and the didactic order mismatches the other two.

Knowing the history of mathematics will allow the professor to better understand the difficulties of generic man, of humanity, in the development of mathematical ideas, and through it those of their own students. History can be used to understand and to make a difficult idea be understood in a better way. Mathematical thinking has gone a long way before giving the rigorously formalized notion of the concepts. If those who gave birth to the concepts did wonderful things with them even though they did not reach the rigor, how can we intend to introduce these concepts to students with unnatural and hard to swallow structures that only after several centuries of work could eventually reached formalization.

Greek mathematics

The Greeks were interested in drawing tangents to curves, in defining and calculating lengths, areas, volumes and centers of gravity (Thiele, 2003). Instead of using algebra, operations with letters, they used the ratios and magnitudes instead of variables, and the role of the field of real numbers was played by the Eudoxus' s Theory of Proportions.

The dilemma analysis vs. synthesis is present in the Greek thought, understanding analysis as the splitting up of a given problem using accurate and logic steps until ending up in something that is already known to be true or a contradiction. With the synthesis completes the proof inverting the process of analysis, deducing the thesis that has been found true in the analysis. They have opposite meanings. Both are the two essential parts of the scheme of a proof in Greek mathematics developed in the field of geometric constructions. Thiele asserts that the Greeks make a distinction between the discovery or invention of mathematical concepts, *inventio* in Latin, and the proof that a given fact is true, verificatio in Latin. One could argue that there is a correspondence between analysis / synthesis, and inventio / verificatio. In relation to the interpretation of Greek mathematical texts, a basic problem is given by its geometric-verbal character.

With regards to concepts of number and magnitude, for the Greeks the number is not only the meaning of thought but the object of thought itself. They conceived three different kinds of numbers: the natural numbers and two ratios, the ratio of natural numbers (positive fractions), and the ratio of magnitudes (positive real numbers). The magnitude is characterized by the property of being able to increase and decrease. The magnitude has a dual interpretation: as a mathematical object and, as an object that can be measured, and the result of such a measurement is a magnitude.

The Greeks distinguish between numerical magnitudes or natural numbers and geometric magnitudes or continuous quantities. The first ones cannot be split, but the second ones (lines, surfaces, solids, angles) can be divided indefinitely. The Eudoxus' s Theory of Proportions, the problem of quadrature, the method of exhaustion, the method of compression, and others are contributions of the Greek mathematics, all of them of extreme importance to the subsequent development of mathematics.

Archimedes

Considered the greatest mathematician of antiquity, and one of the greatest in the history of mathematics, is well known not only for his work as a geometrist but for his inventiveness in the field of engineering, thus in the field of Physics and Technology. He combined theory with practice.

Extant works of Archimedes are all theoretical, both on geometry and mathematical physics (Archimedes, 2005). They have two characteristics: depth and originality, due to the topics treated as well as the methods employed. His works are not compilations of previous mathematical discoveries but actual scientific essays, whose purpose was to communicate his discoveries to the scientific community. His findings derived from his work in research lines proposed by the Greek mathematics, as in the case of finding the solution to the measurement of the circle, through triangulation and the compression method; the Quadrature of the parable, seeking equivalence between plane figures and areas of curvilinear figures, problem known as application areas; and, in the field of solid the seeking of equivalences between them (*On the sphere and cylinder, On conical and spheroids*).

He also ventured into new fields, studying different curves besides the circumference in *On Spiral Lines*; he applied the rigor of geometrical methods to experiments in the field of statics and hydrostatics (Mathematical Physics) as read *On the Balance of the Figures* and *On Floating Bodies*; in *The Sandreckoner* he addresses the problem of expression and large numbers notation; and *the Problem of the Oxen*, he presents a solution that in terms of current mathematics uses a diophantine equation of the Pell-Fermat type.

The Method

Archimedes stated in Method (1986) that the order of the demonstration is not the order of discovery. In the analysis of his works a methodological dualism in mathematical research is clearly differentiated, represented by the opposition between ars inveniendi, the path of discovery, and ars disserendi, the path of the demonstration, as a supported presentation of what is already known for sure, being both complementary. In the specific case of the volume of the sphere, Archimedes, in the path of discovery, performs a mechanical demonstration, using a heuristic logic in explaining the methodology used in the treatment of geometric issues with the help of mechanical notions, such as lever, center of gravity. Archimedes said that this mechanical method is not less useful with regards to the proof of the theorems themselves.

The Method is the only case in the Greek mathematics literature in which the heuristic system is exposed by an author. In the Letter to Erastosthenes, Archimedes states that he wanted to publish The Method because he believes that it could make a not-little-benefit contribution to mathematical research, while others using the method described will be able to come up with other theorems he never thought about (Archimedes, 1986). This assertion implies an educational perspective, communicating his work so it can be learned and used by others.

Fifteen examples are developed in The Method, being the most significant for Archimedes the demonstration that the cylinder circumscribed to the sphere has a volume one and a half times the volume of the sphere. These demonstrations are of special interest due to the fact that for the first time it was possible to find equivalence between figures made up of flat surfaces and figures made up of curved and flat surfaces.

According to Thiele (2003), the basic idea of the mechanical method is based on atomistic concepts evaluated mechanically. In the process, Archimedes sectioned the solids of revolution into "slices" infinitesimal. Each of these circular regions is made of physical matter, with a weight proportional to its volume, which allows the application of the law of the lever.

The Mechanical Demonstration

Previous assumptions

1. A1. If a magnitude is removed from another magnitude and the center of gravity of both the full magnitude and the magnitude removed is the same point, this same point is the center of gravity of the remaining magnitude.
2. A2. If a magnitude is removed from another magnitude with the center of gravity of the full magnitude and the removed magnitude not being the same point, the center of gravity of the remaining magnitude is in the prolongation of the line joining the centers of gravity of the full magnitude and the removed magnitude, placed at a distance whose ratio with the straight line between the centers of gravity is the same ratio between the weight of the magnitude that is removed and the weight of the remaining magnitude (*On the Equilibrium of the planes*, I, 8).
3. A3. If the centers of gravity of whatever number of magnitudes are on the same line (line segment, in this context), the center of gravity of the magnitude made up of all these magnitudes will be also found on the same line (Ibid., I, 4 and 5, II, 2 and 5).

4. A4. The center of gravity of any line is the point dividing the line into two equal parts (Ibid., I, 4).
5. A5. The center of gravity of any triangle is the point where straight lines drawn from the angles of the triangle to the midpoints of the sides intersect (Ibid., I, 14).
6. A6. The center of gravity of any parallelogram is the point where the diagonals converge (Ibid., I, 10).
7. A7. The center of gravity of the circle is the very center of the circle.
8. A8. The center of gravity of any cylinder is the point which divides the axis into two equal parts.
9. A9. The center of gravity of any prism is the point which divides the axis into two equal parts.
10. A10. The center of gravity of any cone is on the axis at a point which divides it so that the portion situated towards the vertex triples the remaining part.

Archimedes uses this theorem as well [established in the previous post *On Conical*]:

T1. If whatever number of magnitudes and other magnitudes in equal number relate to one another, taking in pairs the ones ordered in a similar way, in the same rate; if, in addition, all or some of the first magnitudes have any rates with other magnitudes, and the second ones have the same rates with other magnitudes taken in the same order, the set of first magnitudes is to the set of magnitudes placed in connection with them as the set of second magnitudes is to the set of the related to them.

The Volume of the sphere

Archimedes takes as a basis the construction of a sphere in which he plots the great circle ABCD and two perpendicular diameters AC and BD (Gonzalez, 2006). From the diameter BD he constructs a circle perpendicular to the circle ABCD and, based on the circle BD, he constructs a cone with its vertex on point A (Fig. 1). The ABD cone surface is extended to where it cuts a plane parallel to the base of the cone passing through point C. The intersection of the cone with the plane is a circumference perpendicular to AC with diameter EZ (Fig. 2).

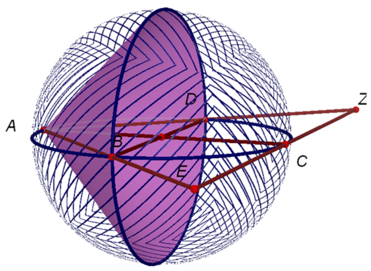

Fig. 1

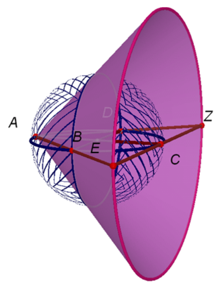

Fig. 2

A cylinder is constructed from EZ diameter having as axis AC and as generatrixes the segments EL and ZH (Figs. 3 and 4).

CA diameter is extended to construct AT so that AT=AC, considering CT as a lever, with A as midpoint. A line MN is drawn parallel to the diameter BD, so that it intersects the circumference ABCD in Q and O, which cuts the diameter AC in S, the segment AE in P and AZ in R (Fig. 5). A plane is drawn through MN, perpendicular to AC, which intersects the cylinder at the circumference with diameter MN, that intersects the sphere ABCD at the circumference with diameter QO and that intersects the cone AEZ at the circumference with diameter PR (Fig. 6).

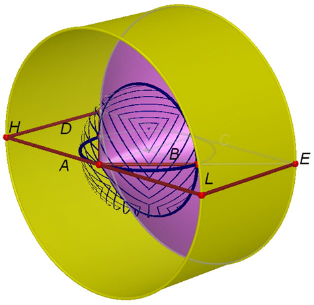

Fig. 3

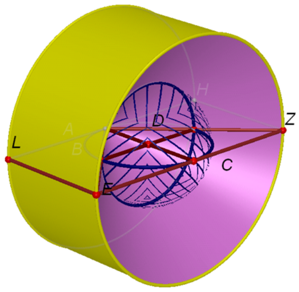

Fig. 4

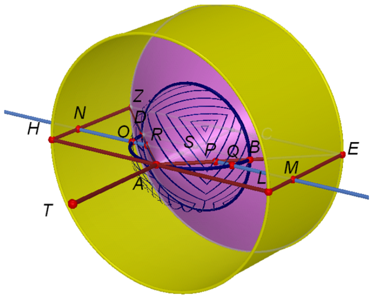

Fig. 5

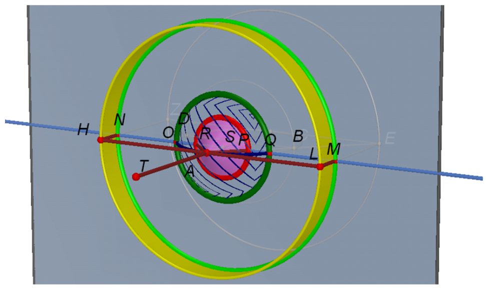

Fig. 6

The intersection of the plane and the cylinder becomes a circumference that delimits a circular region. The same applies to the intersections of the plane with the cone and the sphere. Taking point A as the midpoint of the lever, the circular regions formed by the intersection of the plane with the cone and the sphere are moved to the point T. Multiplying the segment AT by the sum of the areas of the circular regions which are in T, the result will be the same as the multiplication of segment AS by area of the circular region formed by the intersection of the plane with the cylinder (Fig. 7).

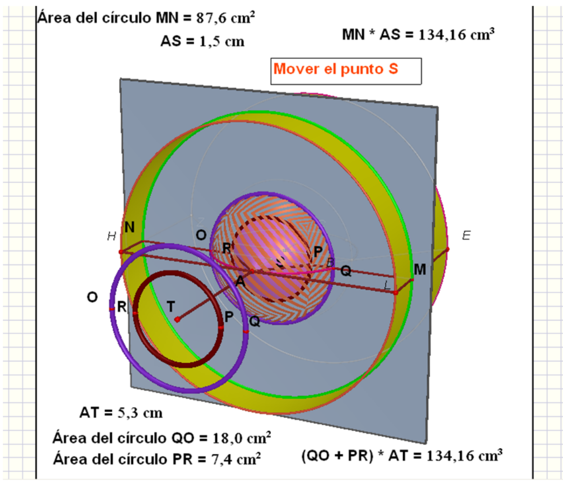

Fig. 7

The sphere and the cone are filled as the circles are moved, both of which, keeping the cylinder in the same place, are balanced about point A, being the center of gravity of the cone and the sphere at point T, and the center of gravity of the cylinder at point K (Fig. 8). From this process we can say that,

(Volume of sphere + volume of cone) × AT = Volume of cylinder × AK

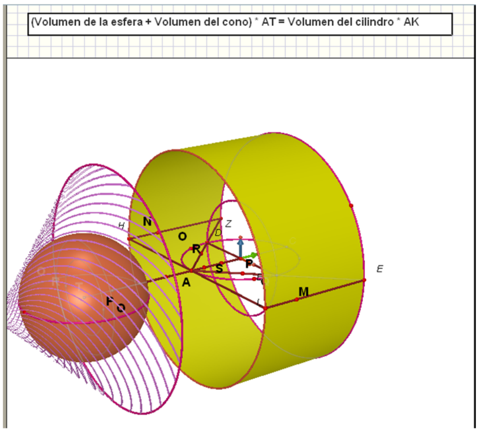

Fig. 8

Resulting that the volume of the cylinder is to the volume of the sphere and the volume of the cone together, as TA is to AK. But TA is twice AK, therefore the volume of the cylinder is double the volume of the cone and the sphere, and in turn the volume of the cylinder is three times the volume of the cone (Euc., XII, 10). Then the volume of the cone AEZ equals twice the volume of the sphere. But the volume of the cone AEZ equals to 8 times the volume of the cone ABD because EZ is twice BD (Euc., XII, 12). Then, 8 times the volume of the cone ABD equals twice the volume of the sphere; therefore, the volume of the sphere ABCD is equal to the volume of 4 cones ABD (Fig. 9)

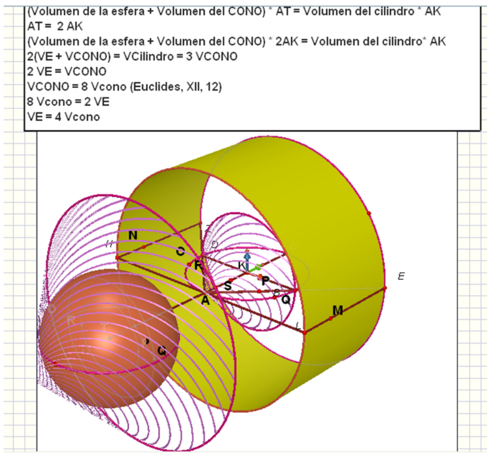

Fig. 9

Segments are drawn through B and D parallel to AC generating a cylinder that circumscribes the sphere. The volume of this cylinder is double the volume of the cylinder circumscribing the cone ABD. In turn, the latter cylinder volume is three times the volume of the cone ABD. Therefore, the volume of the cylinder circumscribing the sphere is six times the volume of the cone ABD. But as was shown earlier, the volume of the sphere is equal to four times the volume of the cone ABD. Therefore, the cylinder volume is one and a half times the volume of the sphere, or, the volume of the sphere is two thirds of the volume of the cylinder (Fig. 10).

$$V_c = \Pi r^2 * h$$
$$h = 2r$$
$$V_e = \frac{2}{3} * \Pi r^2 * 2r$$
$$V_e = \frac{4\Pi r^3}{3}$$

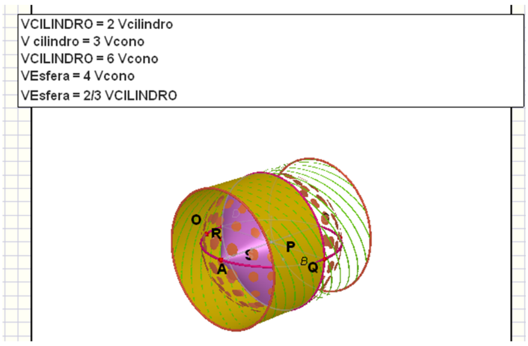

Fig. 10

Surface area of the sphere

To calculate the surface area of the sphere Archimedes established an analogy. As the area of every circle is equal to the area of a triangle whose base is the circle's circumference and a height equal to the radius of the circle (Fig. 11), he assumed that the volume of any sphere is equal to the volume of the cone whose base is the surface of the sphere and whose height is the radius of the sphere (Fig. 12). As the volume of the sphere and the length of the radius are known, it can be deduced therefrom that the area of the surface of the sphere is 4 times the area of its great circle.

$$A_{\bigcirc} = A_{\triangle} = \frac{\text{circumferece} - \text{length} * \text{height}}{2}$$
$$A_{\triangle} = \frac{2\Pi r * r^2}{2}$$
$$A_{\triangle} = \Pi r^2$$

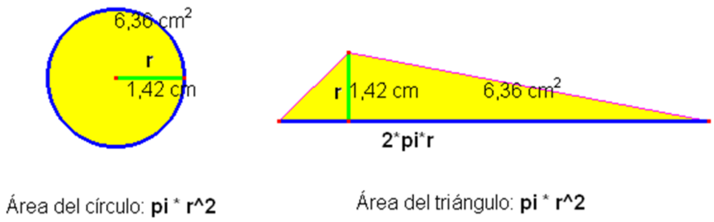

Fig. 11

$$V_e = V_c = \frac{\text{Area} - \text{surface} - \text{sphere} * \text{radio}}{3}$$
$$\frac{4\Pi r^3}{3} = \frac{\text{ASE} * r}{3}$$
$$\text{ASE} = 4\Pi r^2$$

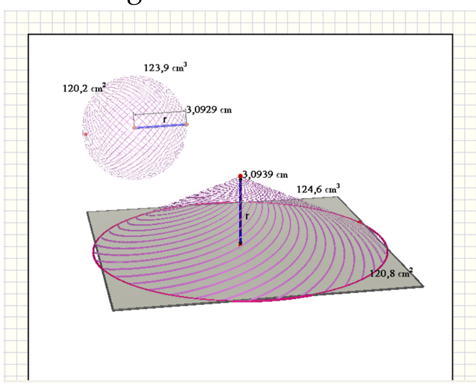

Fig. 12

Conclusions

- From the perspective of the philosophy of mathematics, regarding the dilemma deductivist style versus heuristic style, the mathematical practice proposed in this research work is strongly tinged with the heuristic style, as Archimedes explore, conjecture, experiment, deduced and build knowledge when using the mechanical method to determine the volume of the sphere.

It is possible to visualize this process through the manipulation and dragging of mathematical objects, actions that are facilitated by the dynamic geometry, thus allowing optimize the comprehension of the process that led to the discovery of these mathematical notions. Imre Lakatos actually picks up, put back in place, the heuristic style typical of the construction of mathematical knowledge, which is appreciated in its fullness in The Method of Archimedes, and which was devalued by the formalist trends of Modern Mathematics.

- In relation to the dilemma explanation versus justification, history of mathematics demonstrates the increased importance of understanding over justification, assertion proven in the work of Archimedes. To determine the volume of the sphere he develops the mechanical demonstration to understand the problem to be solved, and once the solution is found out he develops a mathematical proof and the axiomatic justification. Regarding the distinction between proof that prove and proof that explain, the mechanical demonstration is a proof that explains, because based on logical reasoning, exploration, experimentation and heuristics Archimedes was able to established mathematical properties and of other sciences, described in the process, which allowed to find the volume of the sphere. The mathematical demonstration that subsequently made is an example of proof that proves.
- Regarding the dilemma that analyzes how visual representations can also be used in justification, as substitutes for the traditional proof, mechanics demonstration of the volume of the sphere that has developed in this work using dynamic geometry software, is located at an intermediate point, as propositional and visual reasoning are used in an integrated manner. This demonstration is to constitute a *heterogeneous proof* proposed by Barwise and Etchemendy (Hanna, 2007).
- From an epistemological perspective the mechanical method shows that the emergence of many mathematical concepts has been given from an interdisciplinary field, in a context in which it has had to resort to other sciences, in this case, Physics. On the other hand, the ideas of atomism, one of the principles of chemistry as a science that arise in Greek thought, involve in the heuristic that seeks to solve the problem. Perhaps one could argue that three sciences converge in the mechanical method: mathematics, physics and chemistry. Only a man of genius as Archimedes could, over two thousand years ago, design and develop such a complex process. In particular, the infinitesimal employed by Archimedes show him as a precursor of analytical thinking.
- With regards to didactics, taking into account how Archimedes managed to make knowledge emerge, one must ask whether it would be more effective that students from different educational levels, work in an interdisciplinary context to achieve the construction of mathematical notions.
- In relation with the computer sciences, this study shows the usefulness of dynamic geometry, especially the Cabri 3D for visualization and a better understanding of complex processes that have led to mathematical notions, such as the case of volume and surface area of the sphere.

REFERENCES

– Arquímedes, 1986, *El Método*. (L. Vega & M.L. Puertas, Trans.), Madrid, España: Alianza Editorial, S.A.

– Arquímedes; Eutocio, 2005, *Tratados I. Comentarios*. (P. Ortiz, Trans.), Madrid, España: Editorial Gredos, S.A.

– González, P., 2006. “A un siglo del descubrimiento de “El Método” de Arquímedes por Heiberg” [Electronic version], *L*a Gaceta de la RSME, 9 (3), 715–144.

– Guzmán, M. de, 1993, “Enseñanza de las Ciencias y la Matemática. MATEMÁTICA”, *Biblioteca Digital de la OEI*, Recovered 15 Dec. 2011: http://www.oei.org.co/oeivirt/edumat.htm

– Hanna, G., 1989, “Proofs That Prove and Proofs That Explain”, In: G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the International Group for the Psychology of Mathematics Education*, Paris, Vol II, pp. 45–51.

– Hanna, G., 2000, “Proof, explanation and exploration: An overview, *Educational Studies in Mathematics*, Special issue on "Proof in Dynamic Geometry Environments", 44 (1–2), 5–23. Copyright 2001.

– Hanna, G., Jahnke, H. & Pulte, H., 2006, *Conference Program (preliminary) of The International Conference Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, Recovered 20 Dec. 2008: http://www.uni-due.de/imperia/md/content/zis/programm_tagung_011106.pdf

– Hanna, G., & Sidoli, N., 2007, “Visualization and proof: a brief survey of philosophical perspectives”, ZDM. *The International Journal on Mathematics Education*, 39(1–2), 73–78.

– Lakatos, I., 1978, *Pruebas y Refutaciones, La lógica del descubrimiento matemático*, Madrid: Alianza Editorial, S.A.

– Thiele, R., 2003, “Antiquity”, In H. N. Jahnke (ed.), *A History of Analysis, History of Mathematics, 24* (pp. 1–40). Providence, RI, USA: American Mathematical Society.