THE FLATTENING OF THE EARTH: Its Effect on Eighteenth Century Mathematics

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ABSTRACT

In the sixteenth and seventeenth centuries, as long sea voyages became common, travelers began to report that pendulum clocks consistently ran slow near the equator. Isaac Newton mentioned this phenomenon in the Principia and suggested (Book III, Proposition 18) that the earth was not quite spherical; it bulged slightly near the equator and thus the force of gravity was less there. The theory was confirmed in the 1730's by measurements taken by Maupertius in Lapland and by La Condamine in what is now Ecuador.

We discuss how Euler and Lagrange incorporated the knowledge that the earth was not a perfect sphere in the development of the theory of conformal mapping, and suggest ways in which this case study of the interplay of technology (navigation and geodesy) with mathematical theory (conformal mapping) might be used with students.

Keywords: geodesy, cartography, history of mathematics

1 Flattening the Earth

The sixteenth-century round-the-world voyages of Fernão de Magalhães (Ferdinand Magellan), Francis Drake, Ignacio de Loyola, and others put an end to any lingering doubts about the correctness of Aristote's theory that the earth was round. However, two observations in the late seventeenth century led to a suspicion that the round shape was not a perfect sphere.

The first observation was by Giovanni Cassini in 1665, taking advantage of improvements in telescope making to make very precise measurements of the disc of Jupiter, discovering that the planet was in fact an oblate spheroid, with the equatorial radius about 1/15 larger that the polar radius. Ironically, Cassini and his descendants were to later lead a faction that believed the earth, unlike Jupiter, was a prolate spheroid with a larger polar than equatorial radius.

The second observation did not seem at first to have any relationship to the shape of the earth. In 1672-3, the Académie Royale sent Jean Richer on an expedition to Cayenne, "pour la perfection & et l'avancement de l'Astronomie;" while there, Richer conducted a number of astronomical and physical experiments. Curiously, he found that a pendulum clock which had been precisely calibrated in Paris, lost more than two minutes per day in the tropics. Perhaps it was Christian Huygens, then resident at the Paris Observatory, who suggested that the force of gravity was in fact less at the equator.

But why would gravity be diminished at the equator? Huygens thought it might be centrifugal force, caused by the earth's rotation. Isaac Newton published the *Principia* in 1687. In Book III, Proposition XVIII of that work, he proposed a different explanation: that the earth was, like Jupiter, an oblate spheroid. Objects at the equator were further from the center of the earth than those at the poles and, therefore, by the inverse-square law, the effect of gravity was less. Newton said: “If our earth was not higher about the equator than at the poles, the seas would subside about the poles, and rising towards the equator, would lay all things there under water.”[13, p.426] In Proposition XIX, Newton, assuming that “the matter of the earth is uniform, and without movement” uses geometrical results on the inverse- square law from Book I to compute the ratio of the equatorial to the polar axes, and arrives at a value of 230:229. In Proposition XX, the theory is extended to planets other than earth.

Newton’s work did not convince everyone. Especially in France, adherents of the Cartesian camp distrusted the notion of action at a distance. Jacques Cassini (son of Giovanni) measured the distance of one degree of latitude in the south of France, and found (erroneously) this was longer than the degree measured near Paris in an earlier survey. Thus, they argued for a prolate, rather than an oblate, spheroid. To resolve the dispute, the Académie obtained royal permission, and funding, to send expeditions to Lapland and to Spanish Peru (in a region which is now part of Ecuador).

The Lapland expedition, including Maupertuis, Clairaut, and Anders Celsius from Sweden, returned in 1638. Their results showed that a degree of latitude was significantly longer than in France. Maupertuis was acclaimed by Voltaire as the “flattener of the earth” [19]. Voltaire wrote the following epithet:

> This poorly known world which he knew how to measure
> Becomes a monument from which he derives his glory,
> His destiny is to describe the world,
> To please and enlighten it.[1]

The story of the two expeditions is told elsewhere ([9],[8], [1]); and Todhunter ([18]) still appears to be the authority on more refined measurements, and on further improvements to Newton’s theory, especially by Stirling and Clairaut. The rest of this paper will treat the specific problem of how the model of earth as an oblate spheroid was adopted for use in cartography. But one last incident needs to be be mentioned.

A survey to measure the length of a degree of latitude in the the Paris-Amiens region had been undertaken by Jean Picard in 1669.[2] Then a 1720 survey in the south of France reported that the length of a degree of latitude was slighty longer than Picard’s figure. This was one reason the Cassinis held out so long for the prolate earth theory. But, in 1740, Cassini de Thury (grandson of Giovanni) admitted to the Academy that a resurvey had shown significant errors with Picard’s original measurements. This was interpreted as a concession by the Cartesian party that the earth might indeed be flattened at the poles.

Voltaire now addressed Maupertuis as “flattener of the earth, and of the Cassinis” [17].

2 Euler Looks at the Data

The return of Maupertuis and the Lapland expedition, and their findings confirming Newton, were a major event in eighteenth century science. Leonard Euler, at the Academy in St. Petersburg, would

[1] “Le Globe mal connu qu’ il a sçu mesurer, Devient un Monument où sa gloire se fonde; Son sort est de fixer la figure du Monde, De lui plaire, et de l’ éclairer.

[2] Newton mentions this survey in the *Principia*.

certainly have heard of it. Moreover, Euler would have had a interest in the cartographic applications of the expedition's findings, since a large part of his duties then involved work at the Academy's Department of Geography, working to produce a map of the Russian empire. In 1740, Euler produced a short paper[2], which used Maupertuis' figures to compute the length of one degree of latitude and one degree of longitude at several different places on the globe. Euler's method was apparently used by C.N. de Winsham[3] in the construction of a book of tables[18, p. 132].

In 1754, while at the Berlin Academy, Euler presented another paper[3] on the calculation of distances on the surface of the globe. This paper extends the methods of a prior paper on spherical trigonometry to an ellispoidal earth. The paper begins with an analysis of the problem, where general formulae are derived, followed by an estimate of the size and shape of the earth, using data from four surveys at different latitudes, and then a solution of a some specific arithmetical problems.[4]

The first section, interesting for its general treatment of elliptical geometry, uses only simple calculus to derive an expression for the radius of curvature, which is needed to compute the surface distance along a meridian. An approximation to surface distance as a function of the equatorial radius and eccentricity is found, again by simple calculus.

He shows that, at latitude ϕ, the distance on the surface of the ellipsoid corresponding to a small displacement in latitude, $d\phi$, is equal to

$$\frac{2\sqrt{2}\,a^2b^2\,d\phi}{(a^2+b^2)^{3/2}}\left(1-\frac{3}{2}\,\delta\cos(2\phi)\right),\quad \text{or}\quad A\cdot\left(1-\frac{3}{2}\,\delta\cos(2\phi)\right),$$

where a and b are the major and minor semi-axes, and δ is a constant related to the eccentricity.[5] In this case, we are measuring the length of one degree, so $d\phi = \pi/180$.

In the second section, Euler estimates the equatorial radius and eccentricity, He uses data from four different surveys from four different latitdudes:

- the La Condamine/Bougeur expedition to South America;
- the Maupertuis expedition to Lapland;
- a new measurement by Nicolas-Louis de La Caille, at the Cape of Good Hope in South Africa;
- A survey of France from near Paris to near Amiens (this appears to be the 1740 survey mentioned previously).

Here are the data Euler uses:

[3] or Winsheim

[4] Example: Knowing the meridian at a point L, determine the latitude by observing the stars, then walk in a straight line keeping a fixed angle with the meridian, to another point M, and determine the distance LM after a new observation of the stars.

[5] Euler used e for the equatorial radius(semi-major axis) and a for the polar radius (semi-minor axis). To make it easier for the modern reader, and to conform to current usage, we changed these to a for the semi-major and b for the semi-minor axis.

Location	Latitude	Length[6] of One Degree
South America	$-0°30'$	56753
Cape of Good Hope	$-33°18'$	57037
France	$49°23'$	57074
Lapland	$66°20'$	57438

Thus, the four approximate equations to resolve are

$$\begin{aligned} A\cdot\left(1-\frac{3}{2}\,\delta\cos\,1°\right) &= A\,(1-1.4997715\,\delta) = 56753+p \\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,66°36'\right) &= A\,(1-0.5957219\,\delta) = 57037+q \\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,98°46'\right) &= A\,(1-0.2286183\,\delta) = 57074+r \\ A\cdot\left(1-\frac{3}{2}\,\delta\cos\,132°40'\right) &= A\,(1-1.0165980\,\delta) = 57438+s \end{aligned}$$

where the quantities p, q, r, s are unknown errors in the measurements. Euler subtracts the first equation from the other three, than divides the last two equations by the last two, to obtain

$$\frac{321+r-p}{284+q-r}=\frac{65}{34}, \quad \text{and} \quad \frac{685+s-p}{284+q-p}=\frac{437}{157},$$

or

$$31p-65q+34r=7546 \quad \text{and} \quad 280p-437q+157s=16563$$

or, finally, eliminating p,

$$-150q+307r-157s=51594.$$

So far, so good. But now a subjective judgment enters. Euler says "if one wanted to suppose these three errors equal, each would be 84 toises, which would be too much, given the great exactness with which the second and fourth of the measurements were made." Somewhat arbitrarily, it seems, he takes r equal to 125 toises, and assumes that q and s are equal, so that $q=s=-43$, and so $p=15$.

Euler plugs his estimates into the above equations, and concludes that $\delta=0.00436055$. Since

$$\delta=\frac{a^2-b^2}{a^2+b^2},$$

this means that the ratio of major axis to minor axis is

$$\frac{a}{b}=\sqrt{\frac{1+\delta}{1-\delta}}=1.00437,$$

or about 230 to 229, which is precisely what Newton predicted. One gets the impression that Euler has set out to prove Newton right and the Cassinis wrong.

It is tempting look at these errors with the benefit of current knowledge. The current figures[7] for the major and minor axes of the earth are 6378136.6 and 6356751.6 meters, or 3272470 and 3261498

[7] The current standard for the figure of the earth, known as WGS84, is essentially a compromise ellipsoid based on many thousands of satellite measurements. The data used here is taken from [8], p. 264, who in turn cites [7].

toises. These figures agree well with Euler's estimates, although the earth appears somewhat less flattened than was assumed in Euler's time. However, it is ineteresting to look at errors in the four surveys in the light of current knowldege.

	Latitude	Euler	WGS84
South America	$-0°30'$	15	-20
Cape of Good Hope	$-33°18'$	-43	-132
France	$49°23'$	125	-11
Lapland	$66°20'$	-43	-224

Euler thought that the most accurate surveys were the second and the fourth; these were in fact the least accurate. The French survey, which Euler assumed was the least accurate, turns out to have the smallest error.

Of course, it is unfair to criticize Euler for lack of access to data which did not yet exist. The French survey had some notoriety, as suggested above, and it is likely that its accuracy was widely questioned. On the other hand, the very large errors in the Lapland survey were not known until the 1920s.

Euler's data analysis appears somewhat ad-hoc. But in his defense, the method of least squares had not yet been invented. Yielding to temptation again, let us see what a least-squares analysis might tell us. We start with the four equations above, in linear form as

$$A - A\delta x_i = y_i + \epsilon_i,$$

with A and δ as above, the x_i are equal to $-3/2$ times twice the cosines of each latitude, and the y_i the estimated survey measurements.

A standard least-squares calculation[8] shows

$$A = 57019.50896, \qquad A\delta = 191.222296, \quad \text{so that} \quad \delta = 0.00335363.$$

Imitating Euler's calcualtions, the result are (units in toises)

	Euler's estimates	Least Squares estimates	WGS84
Major axis	3281168	3280417	3272470
Minor axis	3266892	3266134	3261498
Flattening Ratio	230	229.66	298.25

Compared to Euler's predictions, the method of least squares shows the earth slightly smaller, but with the same shape (flattening ratio). But again, what is interesting is the least-square residuals at each data point, compared with Euler's estimates:

	Latitude	Euler	Least Squares
South America	$-0°30'$	15	2
Cape of Good Hope	$-33°18'$	-43	-57
France	$49°23'$	125	111
Lapland	$66°20'$	-43	-56

[8]Computation was performed using the `lm()` module in the open-source statistical package *R*. For information about *R*, see[14].

The least-squares residuals show the Africa and Lapland surveys about equal in goodness of fit, while the French survey is much worse than the others. This is precisely the same pattern that Euler achieved in his manipulations. Perhaps what may have seemed subjective judgment was really an instinctive feel for how the data fit together.

In summary, later data, unavailable at the time, would eventially prove Euler wrong in his assessments of errors. But he did impressively well, based on the data he had to work with.

3 Practical Cartography

Euler published, in 1775, three treatises[4, 5, 6] which constitute his principal contribution to mathematical cartography. However, all of these papers assume a spherical, rather than an ellipsoidal earth. Euler's interest was in the abstract properties of a mapping from one surface to another, and not so much in the tools of day-to-day cartography. He closes the first treatise[4] with these word: "⋯it is not easy to derive methods of practical use from our formulae. Nor, indeed was the intention of the present work to dwell on practical uses, especially since, for the usual projections, these matters have been explained at length by others." [9]

Both Lambert[11] and Lagrange[10] published papers in the 1770s which did attempt to incorporate knowledge of an ellipsoidal earth into their work on angle-preserving (conformal) cartographic maps. Their treatments are similar. We shall describe Lagrange's work here since his treatment is more systematic.

Lagrange considers the question of a mapping from points on the globe with latitude s and longitude t to points on the plane (x, y). Specifically, "let us consider two points infinitely close, which are determined on the surface of the globe by the variables s, t, $s + ds$, $t + dt$, and on the map by the variables corresponding variables x,y, $x + dx$, $y + dy$, and look for the distances of these two points on the gobe and on the map. Evidently, the distance on the globe is expressed by $\sqrt{ds^2 + q^2 dt}$, where $q\,dt$ is the arc on the parallel [of latitude] between two meridians [of longitude], and the distance on the map is expressed by the usual formula $\sqrt{x^2 + y^2}$." Lagrange sets as a fundamental condition, that such a mapping should satisfy

$$\sqrt{ds^2 + q^2\,dt^2} : \sqrt{dx^2 + dy^2} = 1 : m,$$

or

$$dx^2 + dy^2 = m^2 ds^2 + q^2\,dt^2$$

where m is some arbitrary constant. Now Lagrange prepares for the possibility of mapping from an ellipsoidal shape. Observing that "the ordinate q of the curve of the meridian is given by the nature of the curve, so that $\dfrac{ds}{q}$ is integrable", he replaces the latitude s by the unspecified function $u(s)$. Setting $n = mq$, the previous condition becomes

$$dx^2 + dy^2 = n^2 du^2 + dt^2$$

The next fifteen pages are devoted to finding mappings from (u, t) to (x, y) which satisfy this condition under suitable restrictions. Finally Lagrange returns to consider the shape of the globe. If the earth is

[9] "⋯minus facile est methodos usu receptas ex formulis nostris generalibus elicere. Neque vero institutum praesens permittit, ut huic negotio immoremur, praecipue cum consuetae proiectiones ab aliis iam abunde sint explicatae"

in fact a sphere, then

$$q = \sin s, \quad \text{so that} \quad du = \frac{ds}{\sin s}, \quad \text{and} \quad u = \ln\left(k \tan\frac{s}{2}\right)$$

(for an arbitrary constant k), a value which can then be inserted in previously found expressions for $x(u,t)$ and $y(u,t)$. If, on the other hand, the meridians are elliptical, then taking z (rather than s) for the distance to the pole, and ε for the (linear) eccentricity, the expression for u becomes

$$u = \log\left(k \tan\frac{z}{2}\left(\frac{1+\varepsilon\cos z}{1-\varepsilon\cos z}\right)^{\frac{\varepsilon}{2}}\right).$$

Now simply let ζ be an angle which satisfies

$$\tan\frac{\zeta}{2} = \tan\frac{z}{2}\left(\frac{1+\varepsilon\cos z}{1-\varepsilon\cos z}\right)^{\frac{\varepsilon}{2}};$$

if the ellipse is near a circle, ζ can be approximated by a rapidly convergent series. Now the formulae previously found for the sphere can be used, with the slightly different angle ζ in place of z.

Lambert, writing seven years earlier, had come up with almost identical results to those of Lagrange, but his treatment is much briefer. (One speculates that Lagrange simply took Lambert's results and packaged them into a more readable format.) In an analysis of angle-preserving maps, Lambert comes up with an with an expression like Lagrange's(above) for the value of u. He defines a (small) auxiliary angle as a function of latitude and shows how to use this to make corrections to the latitude via a radidly convergent series.

Modern cartographers, when designing maps where the ellipsoidal shape of the earth is relevant, use the same technique pioneered by Lambert and Lagrange. For each given latitude, a slightly corrected "auxiliary latitude" is computed, and then the auxiliary is used in the projection computations. Different auxiliary latitudes are used, depending on the desired propoerites of the map. Detailed information on auxiliary latitudes can be found in [12] and [15].

4 Uses in Pedagogy

The Euler and Lagrange works discussed above show how to model the earth as an ellipsoid. Each author develops the geometry of the ellipse; in particular, the relationship between a the coordinates of a point on the curve and the angle of the tangent line at that point. The development requires only the fundamental equation and some simple calculus, but offers material not often seen in an undergraduate calculus class.

The Euler paper in the second section offers other intriguing opportunities for discussion, including:

- How do we incorporate subjectively known differences of error in an analysis?
- Alternatively, how do we read and assess the merits of a subjective data analysis?
- What are the advantages and potential drawbacks of the least-squares method?
- Are there alternatives to the least-squares method?

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