THE DIVISION OF THE TONE AND THE INTRODUCTION OF GEOMETRY IN THEORETICAL MUSIC IN THE RENAISSANCE: AN HISTORIC-DIDACTICAL APPROACH

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ABSTRACT

This presentation intends to show the role of the division of the tone and of mathematical conceptions underlying such a procedure in a substantial change undergone by the conception of western music throughout its history from Antiquity to the Renascence. It aims at showing therefore that such a change comprises also a significant extension in the spectrum of techniques used in theoretical music, which begin to include explicitly geometry, subsequently widely used, among the mathematical tools of solving problems. In a wider sense, during such a period, western music came from a cosmological-mathematical-speculative understanding, in which the main attention was placed on a rational activity of speculation and the purpose of the musical sound was to imitate a supramusical order and regularity to assume a mathematical-empirical concept, in which the main emphasis was set on the quality of the sound itself and music was examined by means of its laws and effects in people. The possibility of division of the tone, proscribed since the Pythagoreans not only into halves, but also into any number of parts, triggered new possibilities hindered by the authority and legitimacy of the Platonic-Pythagorean conception of music, according to which only whole numbers and ratios of integers should participate of the discourse concerning theoretical music, thereby promoting greater interaction between arithmetic and geometry in musical contexts. Such a possibility also unchained significant changes in the conceptions of theory of ratios underlying theoretical music, promoting the emergence of arithmetical conceptions of ratio in such contexts. This presentation also intends to raise the didactical potential of such changes approached in musical context inasmuch as such a context is fertile to differentiate ratios and proportion from structural analogous, but semantically different, ones, concepts sometimes approached indifferently in the dynamics of learning/teaching.

1 Division of the tone

The equal division of the tone played an important part in the historical process that led to the emergence of equal temperament. Mathematically, the equal division of the tone $8 : 9$ provides incommensurable ratios[1] underlying musical intervals. Attempts to divide the tone were already made

[1] The equal division of the tone $(8 : 9)$ means mathematically to find x so that $8 : x = x : 9$; that result, anachronistically speaking, in irrational numbers, inconceivable in the Pythagorean musical system.

in Antiquity, for instance by Aristoxenus (fourth century B.C.). In contrast with the Pythagoreans, who defended the position that musical intervals could properly be measured and expressed only as mathematical ratios, Aristoxenus rejected this position, asserting instead that the ear was the sole criterion of musical phenomena (Winnington-Ingram, 1995, 592). In preferring geometry to arithmetic in solving problems involving relations between musical pitches, Aristoxenus sustained, also against the Pythagoreans, the possibility of dividing the tone into two equal parts, conceiving musical intervals—and indirectly ratios—as one-dimensional and continuous magnitudes, making possible in this way their division. This idea provoked a large number of reactions expressed for instance in the *Sectio Canonis*[2] (Barbera 1991, 125), which was in Antiquity attributed to Euclid[3] and much later in the *De institutione musica*[4] (Bower and Palisca 1989, 88) of Boethius in the early Middle Ages, which gave birth to a strong Pythagorean tradition in theoretical music throughout the Middle Ages. Following the Platonic-Pythagorean tradition, a great part of medieval musical theorists sustained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

The division of the tone is directly linked to the arithmetization of theories of ratio in the context of mathematics and music, a process which developed throughout Middle Ages up to the Renaissance. During Latin, Byzantine and Arabic Middle Ages, such a process had already received meaningful contributions, culminating in the Renaissance in a strong confluence of such traditions, which brought about during this time an unprecedented acceleration of the arithmetization of theories of proportions.

Up to the Renaissance, the utilization of ratios and proportions did not occur in a well-defined structure. Sometimes, such a use had arithmetic features, other times, geometric-musical features or even occurred as a combination of such tendencies. To such different structures, which kept up with the concepts of ratio and proportion since Antiquity, corresponded theories on these concepts underlying music and mathematic treatises up to the Renaissance.

Throughout history of the controversy involving arithmetization of ratios, diverse theoreticians contributed to shaping the theories mentioned above. In the 4^{th} century b.C., Aristoxenus used musical intervals—and indirectly ratios—as uni-dimensional continuous magnitudes, making possible with this the division of musical intervals in any number of parts. In the 4^{th} century a.C., Theon of Alexandria inserted interpolations in the book VI of "The Elements" of Euclides, modifying with this the original sense of compounding ratios. In the 11^{th} century, Psellus proposed the geometric division of the tone. Starting from theoretic-musical contexts, such a conception implied in the interpretation of ratios as continuous magnitudes. In the translation of book V of "The Elements" of Euclid in the 13^{th} century, Campanus of Novara conferred to the definition 5 an arithmetic interpretation, in which he introduced for instance the terminology "denominatio", although such a concept was not present in the original text. In the end of the 15^{th} century, Erasmus Horicius published for the first time in musical context a treatise, which made use of ratios as a continuous quantity. In the 16^{th} century, the

[2]Neither one nor many mean numbers will fall proportionately between a superparticular interval. Superparticular intervals are those produced by an epimoric ratio, i.e. a ratio of the type $m + 1 : m$.

[3]For discussions supporting the attribution of the *Sectio Canonis* to Euclid, see Menge, 1916, pp. 39–40.

[4]Demonstration against Aristoxenus that a superparticular ratio cannot be divided into equal parts, and for that reason neither can the tone.

process of arithmetization accelerated so that in the 17^{th} century the arithmetic theory of ratio and proportion became predominant.

Goldman suggests that Nicholas Cusanus (1401–1464) was the first to assert in *Idiota de Mente* that the musical half-tone is derived by *geometric division* of the whole-tone, and hence would be defined by an irrational number (Goldman, 1989, 308). As a consequence, Cusanus would be the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the High Renaissance music theorists Faber Stapulensis (1455–1537) and Franchino Gafurius (1451–1524), published half a century later (Goldman, 1989, 308). Nevertheless, one can find in the Byzantine tradition Michael Psellus (1018–1078), who suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia* (Psellus, 1556) a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchetus of Padua (1274 ? –?) proposed, in his *L*ucidarium in Arte Musice Planae written in 1317/1318, the division of the tone into five equal parts (Herlinger, 1981, 193), an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach. At the end of the fifteenth century and the beginning of the sixteenth century, Erasmus Horicius, one of the German humanists gifted in musical matters, wrote his *Musica* (Erasmus Horicius, 1500?, fo. 66v), where he suggested a geometric division of the whole tone (Palisca, 1994, 160). Erasmus stated that any part of any superparticular ratio can be obtained, in particular the half of $8 : 9$, which corresponds to divide equally the whole tone (Palisca, 1994, 159). Theoretically based on many geometrical propositions and, unusually, modeled on Euclidean style, his *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as a truly geometric tradition in the treatment of ratios in theoretical music contexts during the sixteenth century. Such a change from an arithmetical to a geometrical basis in the theory of music represents a meaningful structural transformation in the basis of theoretical music, strongly tied to the change in the conception of western music mentioned at the beginning of this article.

2 The introduction of geometry in theoretical music in the Renaissance

The period from the end of the fifteenth century to the end of sixteenth century witnessed more intense structural changes in the conceptions underlying ratios and proportions in the contexts of theoretical music. With the need of equal temperament which brings together the need of the division of the whole tone and consequently structural changes in the conceptions of ratios, treatments with such concepts in theoretical music ceased to be a subject exclusively of arithmetic and became a subject of geometry.

In this context, Erasmus Horicius contributed immensely to the introduction of geometry as an instrument for solving structural problems in theoretical music. Notwithstanding the announcements of the need for geometry in theoretical music by previous authors, Erasmus could be considered the first in the Renaissance to apply Euclidean geometry extensively in his *Musica* (Erasmus Horicius, 1500?) for the resolution of structural problems in theoretical music. Relying mainly on books V and VI of Euclid, Horicius used geometry in different ways to solve musical problems, applying it to intervals, in contradiction to the Boethian arithmetical tradition. He used in his *Musica* the *denominatio* terminology taken from Campanus's Latin translation of the Elements, a procedure which contributed to the emergence of an arithmetical theory of ratio in the context of theoretical music. Making use of geometrical resources hitherto unusual in musical contexts, Erasmus showed that the intervals of the fifth ($3 : 2$) and the whole tone could be divided through a proportional mean, namely by finding a magnitude b between a and c so that $a : b$ is proportional to $b : c$ considering the whole tone mathematically expressed by $a : b$, although such resources involved potentially irrational numbers. Procedures like those in musical contexts intensified the conflicts associated with the Pythagorean tradition concerning theoretical music, according to which only whole numbers and ratios of whole numbers could serve as the basis for theoretical music, whether through a stiff distinction between consonance and dissonance defined by the first four numbers or through the search for a perfect system of intonation based on commensurable ratios.

Erasmus represents an intensification in the conceptual change undergone by theoretical music at this time, and his contribution is relevant to the research on mathematics and music at the end of the fifteenth century and beginning of the sixteenth century at the University of Paris, inasmuch as one can find the use of geometry in the solution of musical problems, for instance in the geometric division of superparticular intervals[5] presented in Faber Stapulensis's *Elementalia musica*, first published in 1494. This work had influence in the Spanish tradition of theoretical music in the sixteenth century, with authors like Pedro Ciruelo (1470–1548) and Juan Bermudo (1510–1565), who also presented respectively in the works *Cursus quatuor mathematicarum artium liberalium*, published in Alcalá de Henares in 1516 (Ciruelo, 1526) and *Declaración de Instrumentos*, published in 1555 in Osuna (Santiago Kastner, 1957) the same division of the tone with the geometrical mean presented by Faber Stapulensis. In the Iberian Peninsula, the tendency to use geometry occurred also in Salinas's *De Musica* published in Salamanca in 1577, which contains a geometrical systematization for the equal Temperament that makes extensive use of Euclid's *Elements*.

Such a tendency spread also to the German and Italian production in theoretical music. For instance, the German mathematician Heinrich Schreiber (1492-1525) published in the appendix *Arithmetica applicirt oder gezogen auff die edel kunst Musica* of his "Ayn new kunstlich Buech⋯" of 1521 (Bywater, 1980) a geometric division of the tone into two equal parts making, use of the Euclidean method for finding the geometric mean. He also operated with ratios with a very arithmetical structure, for instance, compounding them as one, anachronistically, multiplies fractions.

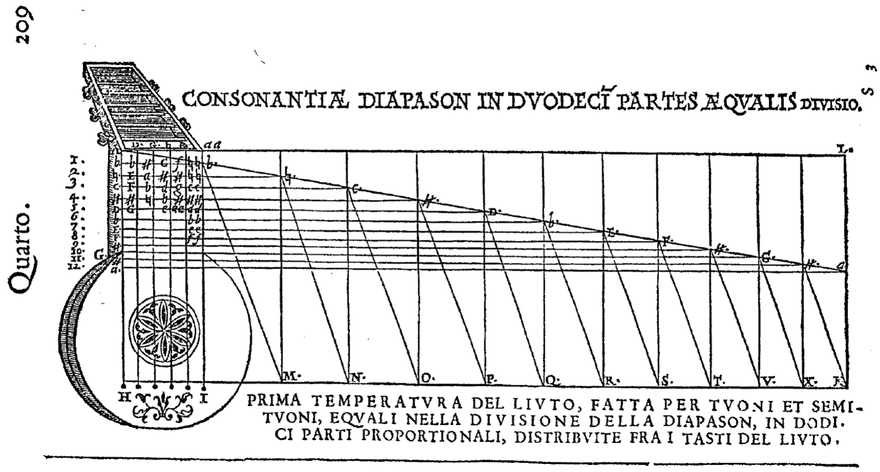
In the Italian tradition the tendency to use geometry was also strong. A representative example of such a tendency is Gioseffo Zarlino, a leading Italian theorist and composer in the sixteenth century. One of the most important works in the history of music theory, Zarlino's *Le istitutioni harmoniche* (1558), represents an important attempt to unite speculative theory with the practice of composition on the grounds that "music considered in its ultimate perfection contains these two parts so closely joined that one cannot be separated from the other (Palisca, 1995, 646). The tendencies for reconciling theory and practice also manifested themselves in this period in the context of structural problems underlying theoretical music. Such a reconciliation seemed to be incompatible with a Pythagorean perspective on theoretical music, in which there was no place for geometry, an essential tool for modeling a new language claimed by practical music.

In this context, it is worthwhile to mention Zarlino's *Sopplimenti musicali* (1588), in which the Italian theorist demonstrated much greater penetration into the ancient authors, particularly Aristoxenus and Ptolemy, than in *Le istitutioni harmoniche* (Palisca, 1995, 648). In spite of the still existing authority

[5]Superparticular intervals are those produced by an epimoric ratio, i.e. a ratio of the type $m + 1 : m$.

of Pythagoreanism in the context of theoretical music in the sixteenth century, Zarlino's *Sopplimenti musicali* already gave evidence of the tension between speculative theory and practice in the contexts of structural problems in theoretical music, inasmuch as it presented geometrical solutions for the equal temperament but was also based on Pythagorean foundations.

The figure below shows Zarlino's first proposal for a theoretical accomplishment of the equal temperament displayed on the lute, which is presented in chapter 30 of book 4 of the third volume of *Zarlino's Sopplimenti musicali* (Zarlino, 1588, 208). Entitled *Come si possa dirittamente diuidere la Diapason in Dodici parti ò Semituoni equali & proportionali*, this chapter presented the first theoretical possibility for the equal temperament as the *temperament of the lute, made by tones and semitones, equally made in the division of the diapason*[6], *in twelve proportional parts, distributed between the keys of the lute*.



First theoretical proposal for the accomplishment of the equal temperament, from Zarlino, Gioseffo, Sopplimenti musicali del rev. M. Gioseffo Zarlino da Chioggia. Venetia: Francesco de Franceschi, 1588, fo. 209.

3 Concluding remarks

The 16^{th} century saw a major revolution in the production of treatises on theoretical music. In contrast with the Pythagorean tradition, it witnessed the introduction of geometry as a tool to solve the problem of division of the tone and consequently to solve theoretical problems related to the systematization of the temperament, representing a considerable change in the base of theoretical music. Such facts are representative, in a wider sense, of a greater change undergone by the foundations of theoretical music in this period, which gradually ceased to be based on an arithmetical dogmatism to assume a geometric-physical approach, resulting in a weaken of the Platonic-Pythagorean tradition in musical contexts.

Nevertheless, it is worthwhile to mention that in spite of the new conceptions of theoretical music, Pythagorean ideas are still present in music contexts in another sense even in the 17th century, when such ideas seemed to be less present and less prominent. A representative example of such a presence can be found in book V of Kepler's *H*armonices mundi (1619), where the Platonic-Pythagorean cos-

[6]The diapason means the musical interval of the octave.

mos received a magnificent restatement, before being withdrawn (Werner, 1966). On the one hand, this procedure represents possibly a last evidence of the Platonic-Pythagorean speculative tradition in music rescuing the old doctrine of the music of the spheres; on the other hand, it also equips the old doctrine with a mathematical-empirical conception under which, in a wider sense, western music had been approached since the late Middle Ages in detriment to the persistent mathematical-cosmological-speculative conception of music predominant since Antiquity.

The resistance to the use of geometry for solving the problem of the division of the tone and consequently other structural problems in music can be put down in greater extent to the authority and legitimacy of the Platonic-Pythagorean tradition in musical contexts, since carrying out such a division would be impossible making use only of whole numbers and would demand to handle ratios as continuous quantities, and/or as a number. Such a practical need eventually demanded an arithmetization of ratios in music contexts, despite its incompatibility with the Platonic-Pythagorean tradition in music. It is worthwhile thus to mention the didactical potential of music contexts in dealing with such changes involving the concepts of ratio, proportion, number, equality as well as of continuous and discrete quantities, inasmuch as at the light of the historical changes mentioned above and also considering the corresponding musical meanings of such concepts, the semantic difference between such concepts stands out, despite the structural similarity, which brings about occasionally the overlooking by the didactic approach some inherent difference they have.

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