Mathematics of the 19th century engineers: methods and instruments

Dominique TOURNÈS

Laboratory of Computer Science and Mathematics (LIM, EA 2525), University of La Reunion,
Parc technologique universitaire, 2 rue Joseph-Wetzell, F-97490 Sainte-Clotilde, France.
Email: dominique.tournes@univ-reunion.fr

ABSTRACT

Traditionally, history of mathematics was more interested in the production of abstract concepts and major theories than in the one of methods and tools of calculation. As a result, it has often neglected to consider the mathematical practices of engineers. I want to sketch here an overview of these practices, at least as they can be identified in the 19th century, at the time when the professional community of engineers structures itself and deepens its mathematical culture[1].

1 Different mathematical practices

The mathematical needs of engineers seem very different from those of mathematicians. To illustrate this with a significant example, consider the problem of the numerical solution of equations, a pervasive problem in all areas of mathematics intervention.

In the early 19th century, the Joseph-Louis Lagrange's *Traité de la résolution des équations numériques de tous les degrés* is authoritative. This treatise, which went through three editions in 1798, 1808 and 1826, can be seen as a gigantic algorithm, fully established from the theoretical point of view, to detect all the roots of a polynomial equation and to calculate each of them numerically by using a always converging process. Given a polynomial equation, Lagrange's method consists in calculating a lower bound $\triangle$ of absolute values of differences between the distinct real roots. For this, one constructs an auxiliary equation whose roots are the differences between all ordered pairs of distinct roots. Then, using $\triangle$ and transforming possibly the equation by a change of scale, one determines a set of integers p such that each interval $[p, p+1]$ contains a single real root and each real root is contained in one of these intervals. Finally, on each interval $[p, p+1]$, one uses a continued fraction expansion which converges always to the root and provides an estimate of the error, contrary to what happens in some cases with Newton's method.

[1]For this text, I borrowed heavily in the work of three researchers with whom I had the pleasure of collaborating for several years: Konstantinos Chatzis, Marie-José Durand-Richard, and Joachim Fischer. I thank them for all they have brought me.

This algorithm, justified with the utmost rigor, is undoubtedly perfect for the algebraist who does not solve equations actually, but proves simply that it is possible to solve them numerically with arbitrary precision. That is why Lagrange's method quickly raised criticisms by some mathematicians like Joseph Fourier and Charles Sturm. These names are not trivial: indeed, they are scholars for whom mathematics is not an end in itself but a tool for natural philosophy, that is to say, the understanding of the physical world. It is precisely to study the propagation of heat that Fourier imagined how to model the evolution of temperature by using the trigonometric series that now bear his name. Similarly, Sturm focused on various problems of mechanics and physics, such as the compressibility of liquids or, following Fourier's tradition, the heat, which led him to his famous results on the "Sturm-Liouville problem". Sturm and Fourier are interested primarily in mathematical problems they encounter in the context of natural philosophy, and do not hesitate to use all the resources of calculus to deal with problems in the theory of equations which some other geometers of their time should have addressed by purely algebraic methods. On this occasion, they do not refrain from criticizing Lagrange.

In the *Analyse des équations déterminées*, published posthumously in 1831, Fourier writes: "Lagrange and Waring have offered to find the smallest difference of the roots of the equation, or a lesser amount than the smallest difference. Considered theoretically, the solution is correct [...]. But it is easy to judge that we cannot accept such a solution method. 1° Indeed the calculation that would furnish the value of the limit $\triangle$ is impracticable for equations of somewhat high degree [...]"[2] A few years later, in his "Mémoire sur la résolution des équations numériques" published in 1835, Sturm is no less severe: "This method, considered from a purely theoretical point of view, leaves nothing to be desired on the side of rigor. But, in the application, the length of the calculations necessary to form the equation for the squared differences, and the multitude of substitutions that one may have to perform, make it almost impracticable, and although Lagrange made there some simplifications, it still requires very painful calculations, so we have tried other solutions"[3].

These mathematicians-physicists, knowing well by nature what is actually the practice of mathematics within applications, are sensitive to the effectiveness of the calculation methods they study. But Sturm and Fourier are nevertheless academicians and professors at the École polytechnique; they are not practitioners. Looking at the side of engineers, the situation is radically different. Léon-Louis Lalanne, a French civil engineer who, throughout his career, sought to develop practical methods for solving equations, wrote what follows as a summary when he became director of the École des ponts et chaussées: "The applications have been, until now, the stumbling block of all the methods devised for solving numerical equations, not that, nor the rigor of these processes, nor the beauty of the considerations on which they are based, could have been challenged, but finally it must be recognized that, while continuing to earn the admiration of geometers, the discoveries of Lagrange, Cauchy, Fourier, Sturm, Hermite, etc., did not always provide easily practicable means for the deter-

[2] Joseph Fourier, *Analyse des équations indéterminées, première partie*, Paris: Didot, 1831, p. 116-117.

[3] Charles Sturm, "Mémoire sur la résolution des équations numériques", *Mémoires présentés par divers savans à l'Académie royale des sciences de l'Institut de France, section sciences mathématiques et physiques*, 6 (1835), p. 274.

mination of the roots"[4].

Lalanne says that as politely as possible, but his conclusion is clear: the methods advocated by mathematicians, that they be "pure" or "applied" (provided this distinction be meaningful in the 19th century), are not satisfactory. These methods are complicated to understand, long to implement and sometimes totally impracticable for ground engineers, foremen and technicians, who, moreover, did not always receive a high-level mathematical training.
Given such a situation, 19th century engineers were often forced to imagine by themselves the operational methods and the calculation tools that mathematicians could not provide them. The objectives of the engineer are not the same as those of the mathematician, the physicist or the astronomer: the engineer rarely needs high accuracy in his calculations, he is rather sensible to the speed and simplicity of their implementation, especially since he has often to perform numerous and repetitive operations. He needs also methods adapted for use on the ground, and not just for use at the office. Finally, priority is given to methods that avoid performing calculations by oneself, methods that provide directly the desired result through a simple reading of a number on a numerical or graphical table, on a diagram, on a curve or on the dial of a mechanical instrument.

2 A growing need for calculation

The 19th century is the moment of the first industrial revolution, which spreads throughout the Western world at different rates in different countries (Great Britain: 1780-1850, France: 1815-1850, Germany: 1835-1870, United States: 1860-1885, Japan: 1870-1900, Russia: 1880-1905). Industrialization causes profound transformations of society. In this process, the engineering world acquires a new identity, marked by its implications in the economic development of industrial states and the structuration of new professional relationships that transcend national boundaries. The engineer then changes its status: it was formerly a practitioner of the arts in service of princely courts; it is now becoming a professional working for the civil society that develops simultaneously with the appearance of the nation-states. The constitution of a specific milieu of engineers, resulting among other things in the creation of numerous professional associations and many specialized journals, is closely related to major public works and ambitious industrial projects that accompany the political changes implemented in Europe, the United States and in the emerging colonial empires.

In this context, the engineer is faced with ever more numerous calculations, longer and more complex, requiring an increasing mathematization as well as a collective work organization more effective. This leads him to create by himself mathematical tools adapted to the new problems and the new implementation constraints that face him. Let us mention here briefly some of these problems and tools.

During the years 1830-1860, the sector of public works experiences a boom in France and more generally in Europe. The territories of the different countries are covered progressively by vast networks of roads, canals, and, after 1842, of railways. These achievements require many tedious calculations of

[4]Léon-Louis Lalanne, "Exposé d'une nouvelle méthode pour la résolution des équations numériques de tous les degrés (troisième partie)", *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 82 (1876), p. 1487.

surfaces of "cut and fill"[5] on cross-sections of the ground. Civil engineers try then different methods of calculation more or less expeditious. Some, like Gaspard-Gustave Coriolis, calculate numerical tables giving the surfaces directly based on a number of features of the road and its environment, such as roadway width or slope. Other engineers, especially in Germany and Switzerland, design and build mechanical devices, planimeters and integraphs, used to quickly calculate all kinds of surfaces on a plane. These instruments are the perfect mechanical translation of the calculus principles (continuous summation of infinitesimal surfaces for the planimeters, continuous drawing of a curve whose slope is equal at every moment to the ordinate of a given curve for the integraphs), and for this reason they will have significant applications in many other scientific fields. Still others, like Lalanne, imagine replacing numerical tables by graphical tables, cheaper and easier to use. It is within this framework that a new mathematical discipline, called "nomography", develops itself and will be deepened throughout the second half of the 19th century and beyond.

Another large field of problems is that of construction. The study of conditions of stability and resistance of structures (beams, bridges, roofs, arches, vaults, retaining walls, etc.) also gives rise to a new discipline, called "graphic statics", which uses systematically the concepts of polygon of forces and funicular polygon, and whose goal is to replace analytical methods by graphical methods exploiting the achievements of projective geometry, a branch of mathematics that was booming at this time. Graphic statics takes immediately on an international character, since it is developed mainly by engineers from France (Jean-Victor Poncelet, Gabriel Lamé, Paul-Émile Clapeyron), Germany (Carl Culmann, Otto Mohr, Wilhelm Ritter), Great Britain (William Rankine James Clerk Maxwell), Italy (Luigi Cremona, Antonio Favaro) and United States (Henry Turner Eddy). Graphic statics arrives timely, when metal structures of all types are multiplying: suspension bridges, railway stations, exhibition halls, etc. These metal structures are composed of a multitude of individual elements. Going through the algebraic calculation to estimate the internal forces working inside this type of structure is a tedious process, which may simultaneously give rise to errors because of the large number of pieces that form these structures. This type of construction culminated with the spectacular achievements of Gustave Eiffel, who, in collaboration with the engineer Maurice Koechlin, a student of Culmann, realized notably the Garabit Viaduct (1878-1923) and the Eiffel Tower (1887-1889). The famous Eiffel Tower has been calculated completely by using the techniques of graphic statics: "To give an idea of the importance of the labor of studies, it suffices to say that the principal drawing office, directed by Maurice Koechlin, made for the backbone of the Tour alone, not including elevators and ancillary works such as basements, stairways, tanks, restaurants, etc., more than 1,700 general drawings; and that the auxiliary office headed by Mr. Pluot established 3,629 drawings for execution. The total surface of these 5,300 drawings exceeds 4,000 m2. The number of different pieces that are detailed is 18,038. This considerable work has required hard work of thirty draughtsmen for eighteen months"[6].

If we look now on the military side, the engineers from the artillery schools face difficulties in constructing firing tables from the fundamental equation of ballistics. The increasing ranges and speeds of projectiles make it impossible to keep the old assumption that air resistance is proportional to

[5] "Cut and fill" is the process of earthmoving needed to construct a road, a canal or a railway: to minimize the construction labor, the amount of material from cuts should roughly match the amount of fills.

[6] Gustave Eiffel, *La Tour de trois cent mètres*, Paris: Société des imprimeries Lemercier, 1900, p. 101.

the square of the velocity. The determination of ballistic trajectories is a place of tension between mathematicians and engineers. Here as elsewhere, the theoretical solutions are not satisfactory for the artilleryman on the battlefield, who must determine very quickly the firing angle and the initial speed to give to his projectile to reach a given target, and needs for this firing tables accurate and easy to use. In 1892, Francesco Siacci, a major figure in Italian ballistics, writes: “Our intention is not to present a treatise of pure science, but a book of immediate usefulness. Few years ago ballistics was still considered by the artillerymen and not without reason as a luxury science, reserved for the theoreticians. We tried to make it practical, adapted to solve fast the firing questions, as exactly as possible, with economy of time and money”[7].

To calculate their firing tables, military engineers conceive numerous numerical and graphical methods for integrating by approximation the ballistic differential equation. In 1834, Jacob Christian Friedrich Otto uses integration by successive arcs to compute firing tables that will experience a great success and will be in use until the early 20th century. In France, an original approach is due to Alexander-Magnus d’Obenheim in 1818. His idea was to replace the numerical tables by a set of curves carefully constructed by points calculated with great precision. These curves are drawn on a portable instrument called the “gunner board”. The quadrature method used to construct these curves is highly developed. Obenheim employs a method of Newton-Cotes type with a division of each interval into 24 parts. In 1848, Isidore Didion, following Poncelet’s ideas, constructs ballistic curves that are not a simple graphic representation of numerical tables, but are obtained directly from the differential equation by a true graphical calculation: he obtains the curve by successive arcs of circles, using at each step a geometric construction of the center of curvature. During the last part of the 19th century, there is a parallelism between the increasing speeds of bullets and cannonballs, and the appearance of new instruments to measure these speeds. Ballisticians are then conducted to propose new air resistance laws for certain intervals of speeds, and they used almost 40 different empirical laws to calculate tables.

It would be easy to multiply examples, speaking also, among other subjects, of hydraulics and, more generally, of fluid mechanics. This domain forced engineers to imagine many new approaches to problems that are modeled by partial differential equations. But the key is to have highlighted the enormous computational requirements which appeared during the 19th century in all areas of engineering sciences and which caused an increasing mathematization of these sciences. This leads naturally to the question of engineering education: how were engineers prepared to use high-level mathematics in their daily work and, if necessary, to create by themselves new mathematical tools?

3 Mathematical education of the engineers in the 19th century

The French model of engineering education in the early 19th century is that of the École polytechnique, founded in 1794 through the main impetus given by Gaspard Monge. Although it had initially the ambition to be comprehensive and practice-oriented, this school promoted quickly a high-level teaching dominated by mathematical analysis. This teaching only theoretical was then completed,

[7] Francesco Siacci, *Balistique extérieure*, trad. fr. par P. Laurent, Paris & Nancy: Berger-Levrault, 1892, p. x.

from the professional point of view, by two years in application schools with civil and military purposes: École des ponts et chaussées, École des mines, École de l'artillerie et du génie de Metz, etc. This training model, which subordinates practice to theory, has produced a corporation of "ingénieurs savants" capable of using the theoretical resources acquired during their studies to achieve an unprecedented mathematization of the engineering art. In particular, the engineers of the Ponts et Chaussées played a leading role in the creation of new instruments and methods of calculation: indeed, in the years 1830-1860, they had to face, as we saw it before, the many technical and organizational problems posed by the massive achievement of modern infrastructure (roads, canals, railways) throughout the French territory.

This model is considered to have influenced the creation of many polytechnical institutes throughout Europe and to the United States. However, unlike France, the teaching of the theory and practice are held together in these new schools, and the training they offer is far from monolithic. In particular the teaching of mathematics, while occupying an important place, is less centered on mathematical analysis than in France. In the first half of the century, polytechnical institutes are created on the model of elite military school. They train mostly State military engineers, who may also be in charge of certain public works. Later, polytechnical institutes better meet the growing needs of industry and train mainly civil engineers in the strict sense. They are more similar to the École Centrale des Arts et Manufactures in Paris and are integrated, sometimes into the higher technical schools, sometimes into the universities. These institutes offer courses directly adapted to the future professional practice of engineering students, such as courses of descriptive geometry, graphic statics, or graphical computation.

Unlike continental Europe, there are no polytechnical institutes in England for the training of civil engineers; the latter is organized by employers inside an apprenticeship system. This is done so first and foremost in the business, before being integrated later, towards the end of the century, into the new universities of industrial cities: the "red-brick universities" or "civic universities". The result is a much less intense standardization of the engineering profession than on the Continent, and a less formal training. Thus, in reports made in 1889, 1892 and 1893 for the British Association for the Advancement of Science, Henry Selby Hele Shaw noted the delay of the British engineers in the operation of new calculation methods, including graphic statics, and calls for reform of engineering education, including the introduction of specialized courses in high schools incorporated into new universities. He notes in particular that the higher geometry is taught widely on the Continent: "This kind of geometry, which is called, under the various names of 'modern geometry', 'higher geometry', 'projective geometry', or, best of all, 'geometry of position', has advanced enormously in importance in recent years, and has become a subject of instruction chiefly on the Continent, in the polytechnical schools"[8], and then he regrets that this is not the case in Great Britain: "The fact is, that very few engineers in this country understand even the nature of projective, modern, higher geometry, or geometry of position, by all of which names it is variously called"[9]. The lack of specialized engineering schools and

[8]Henry Selby Hele Shaw, "Second report on the development of graphic methods in mechanical science", *Report of the Sixty-Second Meeting of the Bristish Association for the Advancement of Science Held at Edinburgh in August* 1892, London: Murray, 1893, p. 385.

[9]*Ibid*., p. 423.

the lack of high-level mathematical courses offered to engineering students in the universities then explain the fact that the English engineers, at least for a certain period, have contributed less than their continental counterparts to develop new mathematical methods.

4 An example of a mathematical discipline created by engineers: nomography

Unable to deal with everything in the limited framework of this conference, I will devote myself to nomography. Indeed, this is the paradigmatic example of a corpus of mathematical tools, constituting an autonomous discipline, which was created from scratch by engineers themselves to meet their needs. Moreover, the theoretical foundations of this discipline are almost entirely due to French engineers who came out the École polytechnique and then worked in civil engineering: Léon-Louis Lalanne, Charles Lallemand, Maurice d'Ocagne, Rudolph Soreau, etc. We must add to this list the Belgian engineer Junius Massau, an ancient student and then professor at the school of civil engineering of the University of Ghent, where training was comparable to that of the École polytechnique, with high-level courses of mathematics and mechanics.

The main purpose of nomography is to construct graphical tables to represent any relationship between three variables, and, more generally, relationships between any number of variables. Isolated examples of graphical translation of double-entry tables are found already in the first half of the 19th century, mainly in the scope of artillery, but this is especially Lalanne, engineer of the Ponts et Chaussées, who gave a decisive impetus to the theory of graphical tables. In 1843 was published in Paris, in a French translation, the *Cours complet de meteorologie* by Ludwig Friedrich Kämtz, professor of physics at the University of Halle. In an Appendix to this course, Lalanne provides consistent evidence that any law linking three variables can be graphed in the same manner as a topographic surface using its marked level lines. In this same year 1843, Lalanne presents at the Académie des sciences de Paris a memoir in which he outlines the idea of using non-regular scales on the x-axis and the y-axis: by replacing the primitive variables by auxiliary functions of those ones, suitably chosen, it is possible in some cases, to reduce to straight lines the marked level lines. For example, for multiplication, after remarking that the relationship $\gamma = \alpha\beta$ can also be written $\log\gamma = \log\alpha + \log\beta$, just graduate the axis with the new variables $x = \log\alpha$ and $y = \log\beta$, and then the bundle of hyperbolas $\gamma = \alpha\beta$ becomes a bundle of straight lines with equations $x + y = log\gamma$. By analogy with an optical phenomenon, Lalanne names "anamorphosis" this transformation. He also introduces the word "abacus" in this context to designate the new graphical tables by inscribing them into the historical continuity of the art of computation (previously an abacus was a calculating table on which jetons could be displayed and moved inside columns)[10].

Lalanne obtained a massive spread of graphical tables in the sector of public works, where his ideas came to a favorable moment. Indeed, the Act of June 11, 1842 had decided to establish a network of major railway lines arranged in a star from Paris. To run the decision quickly, one felt the need for

[10]This new sense of the word $\ll abacus \gg$ was not adopted in England and the United States, where this kind of graphical table was called $\ll contourlinechart \gg$.

new ways of evaluating the considerable earthworks to be carried out. In 1843, the French government sent to all engineers involved in this task a set of graphical tables for calculating the areas of cut and fill on the profile of railways and roads.

After Lalanne, the graphical tables resting on the principle of concurrent lines spread rapidly, until becoming, in the third quarter of the 19th century, very common tools in the world of French engineers. The Belgian engineer Junius Massau succeeded Lalanne to enrich the method and its scope of application. Professor at the school of civil engineering of the University of Ghent, Massau distinguished himself in the field of engineering sciences by contributing in a creative manner to rational mechanics, to nomography and especially to graphical integration, a discipline of which he is considered as the creator. Essentially, his contributions to the theory of graphical tables are contained in two large memoirs published in 1884 and 1887.

After recalling the work of Lalanne that inspired him, Massau introduces a notion of generalized anamorphosis, seeking what are the functions that can be represented using three pencils of lines, without requiring the first two pencils to be parallel to the coordinate axis. In this case, the lines of α, β and γ shall have respective equations of the form

$$f_1(\alpha)x + g_1\alpha)y + h_1(\alpha) = 0,$$
$$f_2(\beta)x + g_2\beta)y + h_2(\beta) = 0,$$
$$f_3(\gamma)x + g_3\gamma)y + h_3(\gamma) = 0,$$

and the concurrency condition of these lines, coming from the elimination of x and y, is

$$\begin{vmatrix} f_1(\alpha)\, g_1(\alpha)\, h_1(\alpha) \\ f_2(\beta)\, g_2(\beta)\, h_2(\beta) \\ f_3(\gamma)\, g_3(\gamma)\, h_3(\gamma) \end{vmatrix} = 0.$$

When we can put in this form a relationship $F(\alpha, \beta, \gamma) = 0$, this relationship is precisely representable by an abacus with concurrent straight lines (called in English a "straight line chart"). Determinants of the above type, called "Massau determinants", played an important role in the subsequent history of nomography; they are encountered in research until today.

A variant of straight line charts was designed by Charles Lallemand, an engineer of Mines. The type of graphical table he invented, called "abaque hexagonal" in French and "hexagonal chart" in English, prolonged undoubtedly the interest in the charts with concurrent lines. We are at a time when a broad program of public works is elaborated in France. The execution of this program required more precise knowledge of the relief of the ground, hence it was decided to undertake what geodesists call the leveling of the whole country: complementary to the triangulation which fixes the position of points on the ground in horizontal projection, the geodetic leveling consists in determining their elevations above mean sea-level. From 1880, Lallemand was responsible for creating a Service du nivellement general de la France, which officially began in 1884. It is within this context that he invented the hexagonal charts, designed as a graphical method to automate the long and tedious calculations necessary for the operation of numerous measurements on the ground. Lallemand's efforts allowed tripling the accuracy of previous results, while significantly reducing costs. Hexagonal charts operate a method of graphical addition based on the fact that the sum of the projections of a straight line segment on two axis forming between them an angle of 120°, is equal to the projection of the same segment on the internal bisector of these axis. By graduating the three axis with non-regular scales $x = f(\alpha)$, $y = g(\beta)$ and $z = h(\gamma)$, one can represent by such a chart any equation with three variables of the form $f(\alpha) + g(\beta) = h(\gamma)$.

With Massau's and Lallemand's publications, the theory of contour lines charts was entering into a mature phase, but in the same time a new character intervened to orient this theory towards a new direction. Philibert Maurice d'Ocagne entered the École Polytechnique in 1880, and then made his entire career in the corps of Ponts et Chaussées. In particular, he was called from 1891 to 1901 at the Service du nivellement, to help Lallemand. Meanwhile, he taught tirelessly for 45 years at the École polytechnique, at the École des ponts et chaussées and at the Sorbonne (University of Paris). Closely linked to this dual activity as an engineer and a teacher, Ocagne continued during all his life an important research that resulted in over 400 publications. Within this burgeoning work that touches on many themes, it is essentially the part about graphical charts that won him fame.

In 1884, when he was only 22 years old, Ocagne observes that most of the equations encountered in practice can be represented by an abacus with three systems of straight lines and that three of these lines, each taken in one system, correspond when they meet into a point. His basic idea is then to construct by duality, by substituting the use of tangential coordinates to that of punctual coordinates, a figure in correlation with the previous one: each line of the initial chart is thus transformed into a point, and three concurrent lines are transformed into three aligned points. The three systems of marked straight lines become three marked curves. To clarify this, consider three arbitrary curves defined by parametric equations

$$x = \frac{f_i(t)}{h_i(t)}, \quad y = \frac{g_i(t)}{h_i(t)} \quad (i = 1,\ 2,\ 3).$$

Three points marked with $t = \alpha$, $t = \beta$ and $t = \gamma$, taken on these three curves respectively, are aligned when

$$\begin{vmatrix} f_1(\alpha)\, g_1(\alpha)\, h_1(\alpha) \\ f_2(\beta)\, g_2(\beta)\, h_2(\beta) \\ f_3(\gamma)\, g_3(\gamma)\, h_3(\gamma) \end{vmatrix} = 0.$$

A given relationship between three variables is then representable by an "alignment chart" if and only if, it can be put into the form of a determinant of the above type. One recognizes unsurprisingly a Massau determinant, because it is clear that the problem of the concurrency of three straight lines and the problem of the alignment of three points, dual to each other, are mathematically equivalent. Using an alignment chart is particularly simple: in practice, to avoid damaging the chart, one does not draw actually the auxiliary straight line on the paper: one uses either a transparency marked with a straight thin line or a thin string tightened between the points to join.

After this first achievement in 1891, Ocagne deepened the theory and applications of the alignment charts until the publication of a large treatise in 1899, the famous *Traité de nomographie. Théorie des abaques. Applications pratiques*, which became for a long time the reference book of the new discipline.

In this treatise, Ocagne still employs the accepted term of "abacus" to refer to any graphical table; however, a little later, he introduced the generic term "nomogram" to replace "abacus", and the science of graphical tables became "nomography". From there, alignment charts were quickly adopted by many engineers for the benefit of the most diverse applications. At the turn of the 20th century, nomography is already an autonomous discipline well established in the landscape of applied sciences.

5 Engineering mathematics as a source of new theoretical developments

The mathematical practices of engineers are often identified only as "applications", which is equivalent to consider them as independent from the development of mathematical knowledge in itself. In this perspective, the engineer is not supposed to develop a truly mathematical activity. We want to show, through some examples, that this representation is somewhat erroneous: it is easy to realize that the engineer is sometimes a creator of new mathematics, and, in addition, that some of the problems which he arises can in turn irrigate the theoretical research of mathematicians.

To return to nomography, the problem of general anamorphosis, that is to say, of characterizing the relationships between three variables admitting a representation by straight lines charts (or, in an equivalent formulation, relationships that may be put into the form of a Massau determinant), has inspired many theoretical research to mathematicians and engineers: Augustin-Louis Cauchy, Paul de Saint-Robert, Junius Massau, Leon Lecornu, and Ernest Duporcq have brought partial responses to this problem before that in 1912 the Swedish mathematician Thomas Hakon Gronwall gives a complete solution resulting in the existence of a common integral to two very complicated partial differential equations.

Beyond the central problem of nomographic representation of relationships between three variables, which define implicit functions of two variables, there is the more general problem of the representation of functions of three or more variables. Engineers have explored various ways in this direction, the first consisting in decomposing the functions of any number of variables into a finite sequence of functions of two variables, which results in the combined use of several charts with three variables, each connected to the next by means of a common variable. Such a practical concern was echoed unexpectedly in the formulation of the Hilbert's 13th problem, one of the famous 23 problems that were presented at the International Congress of Mathematicians in 1900. The issue, entitled "Impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments" is based on the initial observation that up to the sixth degree, algebraic equations are nomographiable. Indeed, up to the fourth degree, the solutions are expressed by a finite combination of additions, subtractions, multiplications, divisions, square roots extractions and cube roots extractions, *i. e.* by functions of one or two variables. For the degrees 5 and 6, the classical Tschirnhaus transformations lead to the reduced equations $f^5 + xf + 1 = 0$ and $f^6 + xf^2 + yf + 1 = 0$, whose solutions depend again on one or two parameters only. The seventh degree is then the first actual problem, as Hilbert remarks: "Now it is probable that the root of the equation of the seventh degree is a function of its coefficients which does not belong to this class of functions capable of nomographic construction, i. e., that it cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary *that the equation of the seventh degree* $f^7 + xf^3 + yf^2 + zf + 1 = 0$ *is not solvable with the help of any continuous functions of only two arguments*"[11].

In 1901, Ocagne had found a way to represent the equation of the seventh degree by a nomogram involving an alignment of three points, two being carried by simple scales and the third by a double scale. Hilbert rejected this solution because it involved a mobile element. Without going into details, we will retain that there has been an interesting dialogue between an engineer and a mathematician reasoning in two different perspectives. In the terms formulated by Hilbert, it was only in 1957 that the 13th problem is solved negatively by Vladimir Arnold, who proved to everyone's surprise that every continuous function of three variables could be decomposed into continuous functions of two variables only.

I will take another example in ballistics. As we saw it before, in the second half of the 19th century, ballisticians were conducted to propose new air resistance laws for certain intervals of speeds. The fact that some functions determined by artillerymen from experimental measurements fell within the scope of integrable forms has reinforced the idea that it might be useful to continue the search for such forms. It is within this context that Francesco Siacci resumed the theoretical search for integrable forms of the law of resistance. In two papers published in 1901, he multiplies the differential equation by various multipliers and seeks conditions for these multipliers are integrant factors. He discovers several integrable equations, including one new integrable Riccati equation. This study leads to eight families of air resistance laws, some of which depend on four parameters. In his second article, he adds two more families to his list. The question of integrability by quadratures of the ballistic equation is finally resolved in 1920 by Jules Drach, a brilliant mathematician who has contributed much in Galois theory of differential equations in the tradition of Picard, Lie, and Vessiot. Drach puts the ballistic equation into a new form that allows him to apply a theory developed in 1914 for a certain class of differential equations, which he found all cases of reduction. Drach exhausts therefore the problem of theoretical point of view, by finding again all integrability cases previously identified.

From a more general point of view, experimental, numerical and theoretical research on the ballistic equation has nevertheless played the role of a laboratory where the modern numerical analysis was able to develop. Mathematicians have indeed been able to test on this recalcitrant equation all possible approaches to calculate the solution of a differential equation. There is no doubt that these tests have helped to organize the domain into a separate discipline at the beginning the 20th century.

I could also highlight fruitful interactions between the development of graphic statics and the one of projective geometry. I could cite Karl Pearson who used some ideas from graphic statics and nomography to give a new impetus to mathematical statistics. I could also mention Massau's research on graphical integration that was the source of important theoretical developments in the field of partial differential equations, but it is unnecessary to multiply further examples to be convinced that knowledge and practices of 19th century engineers were in constant interaction, immediate or delayed, with

[11] David Hilbert, "Mathematical problems", translated by Mary Winston Newson, *Bulletin of the American Mathematical Society* 8 (1902), p. 462.

those of pure and applied mathematicians.

Conclusion

Recent research by historians shows more clearly that mathematical knowledge and mathematical representations are part of various social groups in interaction, in which they find various legitimacies. Within this new framework, history of mathematics should enrich itself by taking greater account of the engineering community, within which specific mathematical practices, original and fruitful, did exist. Moreover, as these old practices are often based on numerical, graphical and instrumental methods translating in a simple and concrete manner the key concepts of mathematics, these practices should constitute a fruitful source of inspiration for creating relevant activities to be exploited nowadays in mathematics education[12].

REFERENCES

– Belhoste (Bruno) & Chatzis (Konstantinos), "From technical corps to technocratic power: French state engineers and their professional and cultural universe in the first half of the 19th century", *History and Technology*, 23 (2007), 209-225.

– Chatzis (Konstantinos), "La réception de la statique graphique en France durant le dernier tiers du XIXe siècle", *Revue d'histoire des mathématiques*, 10 (2004), 7-43.

– Chatzis (Konstantinos), "Les ingénieurs français au XIXe siècle (1789-1914): émergence et construction d'une spécificité nationale", in Gabriel Lamé, les péregrinations d'un ingénieur au XIXe siècle, numéro spécial de la revue *SABIX*, 44 (2009), 53-63.

– Chatzis (Konstantinos), "Theory and practice in the education of French engineers from the middle of the 18th century to the present", *Archives internationales d'histoire des sciences*, 60 (2010), n° 164, 43-78.

– Chatzis (Konstantinos), "La statique graphique: heurs et malheurs d'une science d'application", in D. Tournès (éd.), *Histoire du calcul graphique*, Paris: Cassini, to appear.

– Chatzis (Konstantinos) & Ribeill (Georges), "Des périodiques techniques par et pour les ingénieurs. Un panorama suggestif, 1800-1914", in P. Bret, K. Chatzis & L. Hilaire-Pérez (dir.), *La presse et les périodiques techniques en Europe, 1750-1950*, Paris: L'Harmattan, 2008, 115-157.

– Durand-Richard (Marie-José), "Planimeters and integraphs in the 19th century. Before the differential analyser", *Nuncius*, 25 (2010), 101-124.

– Durand-Richard (Marie-José), "Mathématiques entre science et industrie: Grande-Bretagne 1850-1950", in *Histoire, logique et mathématiques : Hommages à Charles Morazé*, Marc Ferro & Marc Barbut (éds), Paris: Association pour la Recherche à l'École des Hautes Études en Sciences Sociales (AREHESS), 2007, 63-82.

– Durand-Richard (Marie-José), "Historiographie du calcul graphique: heurs et malheurs d'une science d'application", in D. Tournès (éd.), *Histoire du calcul graphique*, Paris: Cassini, to appear.

– Fischer (Joachim), "Instrumente zur Mechanischen Integration. Ein Zwischenbericht", in *Brückenschläge. 25 Jahre Lehrstuhl für Geschichte der exakten Wissenschaften und der Technik an der Technischen Universität Berlin, 1969-1994*, H.-W. Schütt & B. Weiss (eds), Berlin: Verlag für Wissenschafts- und Regionalgeschichte Dr. Michael Engel, 1995, 111-156.

[12]I began to conceive and experiment such activities in classrooms. In the bibliography, see my papers dated from 2005, 2007, 2010, and 2012.

– Fischer (Joachim), "Instrumente zur Mechanischen Integration II. Ein (weiterer) Zwischenbericht", in *Chemie - Kultur - Geschichte. Festschrift für Hans-Werner Schütt anlässlich seines 65. Geburtstages*, A. Schürmann & B. Weiss (eds), Berlin-Diepholz: Verlag für Geschichte der Naturwissenschaften und der Technik, 2002, 143-155.

– Fischer (Joachim), "Zur Rolle von Heinz Adler zwischen Ludwig Albert Ott und Alwin Oswald Walther", in *Rechnende Maschinen im Wandel: Mathematik, Technik, Gesellschaft. Festschrift für Hartmut Petzold zum 65*. Geburtstag, U. Hashagen & H. D. Hellige (Hg.), München: Deutsches Museum, 2011, 33-110.

– Fischer (Joachim), "Instruments graphomécaniques d'intégration", in D. Tournès (éd.), *Histoire du calcul graphique*, Paris: Cassini, to appear.

– Tournès (Dominique), "Pour une histoire du calcul graphique", *Revue d'histoire des mathématiques*, 6 (2000), 127-161.

– Tournès (Dominique), "Du compas aux intégraphes: les instruments du calcul graphique", *Repères-IREM*, 50 (2003), 63-84.

– Tournès (Dominique), "L'intégration graphique des équations différentielles ordinaires", *Historia mathematica*, 30 (2003), 457-493.

– Tournès (Dominique), "Junius Massau et l'intégration graphique", *Revue d'histoire des mathématiques*, 9 (2003), 181-252.

– Tournès (Dominique), "Constructions d'équations algébriques", *Repères-IREM*, 59 (2005), 69-82.

– Tournès (Dominique), "Les méthodes graphiques dans l'histoire et dans l'enseignement", in *Histoire et enseignement des mathématiques. Rigueurs, erreurs, raisonnements*, Évelyne Barbin & Dominique Bénard (éds), Paris: INRP & Clermont-Ferrand: université Blaise-Pascal, 2007, 263-285.

– Tournès (Dominique), "Une approche graphique de la méthode d'Euler", in *De grands défis mathématiques d'Euclide à Condorcet*, Évelyne Barbin (éd.), Paris: Vuibert, 2010, 139-155.

– Tournès (Dominique), "Une discipline à la croisée de savoirs et d'intérêts multiples: la nomographie", in *Circulation Transmission Héritage, Actes du XVIIIe colloque inter-IREM Histoire et épistémologie des mathématiques (Université de Caen Basse Normandie, 28-29 mai 2010)*, Pierre Ageron & Évelyne Barbin (éds), Caen: Université de Caen Basse-Normandie, 2011, 415-448.

– Tournès (Dominique), "Calculer avec des hyperboles et des paraboles", in *Des mathématiques éclairées par l'histoire. Des arpenteurs aux ingénieurs*, Évelyne Barbin (éd.), Paris: Vuibert, 2012, 131-148.

– Tournès (Dominique), "Abaques et nomogrammes", in D. Tournès (éd.), *Histoire du calcul graphique*, Paris: Cassini, to appear.

– Tournès (Dominique), "Résolution graphique des équations différentielles", in D. Tournès (éd.), *Histoire du calcul graphique*, Paris: Cassini, to appear.