HISTORICAL ALGORITHMS IN THE CLASSROOM AND IN TEACHER-TRAINING

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ABSTRACT

A new national curriculum in French high-schools highlights the importance of algorithms in mathematics, and explicitly requires that elementary but varied work on algorithms be carried out in the classroom. To help teachers face this new demand, we decided to give our teacher-training sessions - set up on the use of historical mathematical sources- a more algorithmic flavor, in order to show them that *understanding*, *writing*, and *justifying* algorithms had been tasks of prime importance throughout the history of mathematics; tasks which could be carried out in the classroom on the basis of historical sources, and not only with a computer. We give in this paper some examples and commentaries.

The *history of mathematics* groups of the IREM (Institute for Research on Mathematics Education) have been involved in teaching (at the secondary level) and teacher-training since the 1980s, with an emphasis on the use of original sources. In the 1990s, the national IREM network published a large selection of historical texts presenting algorithms ([3], [4]), from the most elementary (reckoning) to the pretty sophisticated (approximation of solution of ODEs, inversion of matrices).

A new national curriculum in French high-schools, which is currently being implemented, highlights the importance of algorithms in mathematics, and explicitly requires that elementary but varied work on algorithms be carried out in the classroom. To help teachers face this new demand, we decided to give our teacher-training sessions a more algorithmic flavor, in order to show them that *understanding*, *writing*, and *justifying* algorithms had been tasks of prime importance throughout the history of mathematics; tasks which could be carried out in the classroom on the basis of historical sources, and not only with a computer.

Our involvement in these teacher-training sessions, along with work from professional historians, helped us deepen our understanding of what is at stake when working with algorithms in the math class.

1 The demands of a new curriculum

The new curriculum (here for the 5th Year of Secondary School) emphasizes the role of algorithms in all of the mathematics: “The algorithmic process is, since the beginning of time, an essential part of mathematical activity. In the first years of secondary education, pupils met algorithms (Algorithms of

Elementary Arithmetic Operations, Euclid's Algorithm, Algorithms in Geometrical Constructions). What is proposed in the curriculum is formalization in natural language.[1]"

Although some *notions* have to become familiar to students (such as loops, conditional branching, etc.), the emphasis is on *tasks*: understanding, writing, modifying algorithms. Both computer languages and "natural" language are considered to be relevant semiotic environments. Hence, the general spirit of this new curriculum is not to introduce computer science in the math class, but to promote an *algorithmic form of mathematical thinking*, in addition to (and not in place of) more traditional forms of mathematical activity (algebraic problem-solving, Euclidean-style geometry, calculus etc.).

This take on algorithms is highly consistent with the work on historical sources, since most of them up until the 17th century are of a more or less algorithmic nature, and these ones, and many others, will be improved and formalized in the following centuries.

2 Historical origin of the word and concept of "algorithm"

The first outbreak of the word I know comes from the *Carmen de algorismo , by* Alexandre de Villedieu (circa 1220) : "haec algorismus ars praesens dicitur, in qua talibus Indorum fruimur bis quinque figures : 0. 9. 8.7. 6. 5. 4. 3. 2. 1."[2]

Here "algorismus" refers to the "art" of Algus (or Argus, or Aldus), latinized name of Al Kwharizmi, whose "*The Book of Addition and Subtraction According to the Hindu Calculation*" survived only in its Latin translation . The first words of the manuscript (untitled) are Dixit *aldorizmi* :"So said al-Khwārizmī".

During the course of time, the meaning extended from routine arithmetic procedures "to mean, in general, the method and notation of all types of calculation. In this sense we say the algorithm of integral calculus, the algorithm of exponential calculus, the algorithm of sinus, etc." as wrote D'Alembert, in the article "Algorithme" of his *Encyclopédie*.

Now, here is the definition in our curriculum[3] : "An algorithm is defined as an operational method allowing one to solve, with a number of clearly specified steps, all the instances of a given problem. This method can be carried out by a machine or a person."

Today, the idea of finiteness has entered into the meaning of algorithm as an essential element. This concern arose in a more general context in Hilbert's 10th Problem (1900): given a polynomial equation with arbitrary rational coefficients, "a method is sought by which it can be determined, in a finite number of equations, whether the equation is solvable in rational numbers"[4]. This defines an "effective procedure" (which effectively achieves a result in a finite time).

But this does not give a formal definition of the notion of an algorithm.
In the 17th century Leibniz dreamed of a universal language that would allow reducing mathematical proofs to simple computations. Then, during the 19th, logicians such as Babbage, Boole, Frege, and Peano tried an "algebrization" of logic, but only the definition of a recursive function (Gödel and Church, between 1931 and 1936) gave a satisfactory formal definition to an algorithm.([2] [3] [4]).

[1] *Ressources pour la classe de seconde* : Algorithmique online
http://media.eduscol.education.fr/file/Programmes/17/8/Doc_ress_algo_v25_109178.pdf

[2] This new art is called the algorismus, in which we derive such benefit out of these twice five figures of the Indians : 0 9 8 7 6 5 4 3 2 1,

[3] 7th year, students majoring in Informatique et Sciences du Numérique

[4] The answer is no (Matijasevic, 1970)

3 A brief theoretical analysis

Teachers and students are used to formulas for solving problems. A formula is, from the Latin etymology, a "small form" condensing the relations between different objects. According to the Online Etymology Dictionary, modern sense is colored by Carlyle's use (1837) of the word for "rule slavishly followed without understanding"…but of course an algorithm too can be slavishly followed without understanding!

Our work with historical texts presenting algorithms prompted us to *describe* the texts, and *characterize* the various mathematical tasks involved in dealing with them. So, a rough sketch would go as follows: at the core of each text lies a mathematical procedure, which calls for:

I. Expression/transmission(in a given semiotic and instrumental context.)
I.a "Algebraic type": a mathematical relation between different elements (formula) a-i) given in natural language (rhetorical) a-ii) given in an algebraic form with symbols (symbolic)
I.b "Algorithmic type" : a list of instructions (algorithm) b-i) in natural language, given on an example (generic example) b-ii) in natural language, given on undetermined data b-iii) in a programming language

II. Justification (in a given epistemic and social context)
II.a) Simply checking the algorithm with a few examples
II.b) Justifying the mathematical procedure
II.c) Establishing properties of the algorithm itself (seen as a syntactic object): proving that it terminates, proving that it does what it is supposed to do, estimating its size (for comparison purposes), its rapidity of convergence (in case of approximation), etc. (But only finiteness is asked in our curriculum.)

Although this descriptive framework may seem sketchy – and calls for refinement – it turned out to be useful in teacher training. It is useful when it fits, and very useful when it doesn't fit the actual historical sources, since it helps raise interesting questions, of both historical and pedagogical natures.

4 Some Examples of Algorithms (to be discussed during the workshop).

4.1 Chinese and Indian iterative algorithms for the extraction of square roots

As told in the curriculum, we can look at "Algorithms of Elementary Arithmetic Operations". These operations have been performed in a variety of ways (tables for Babylonian and Egyptian, pebbles, knots on strings, bead on an abacus frame, tokens on counting board, marks in the dust…). Our pupils have learnt them by using paper and pencil, and the decimal system - like the method named "algorism" in Medieval Books. The procedures are not far from Indian and Chinese ones, but they were only taught on examples. That is not the case for Indian and Chinese ones ([5]).

In both cases, the algorithms indicate the flow of operations to be performed on the computing surface (dirt for instance).The results of operations, receive specific names (dividend, divisor, quotient, given by parallelism with the algorithm of division) and are put in assigned places. In each position, the numbers may vary, but are subjected to the same transformation. This allows an iterative algorithm. And that is what we do in a modern algorithm by assignations of the variables. Of course, we sometimes have to interpret the denominations by using the analogy with the division.

4.2 Heron of Alexandria gives a Method of Successive Approximations for square root on a generic example (circa 50 AD)

Heron gives first an algorithm, on a generic example, to find the area of a triangle, knowing the length of its 3 sides:

> "For instance, let the sides of the triangle be of 7, 8, 9 units.
>
> Compose the 7 and the 8 and the 9: the result is 24;
>
> from this take the half: the result is 12;
>
> subtract the 7: 5 remaining.
>
> Again from the 12, sub tract the 8: 4 remaining;
>
> and again the 9: 3 remaining.
>
> Make the 12 by the 5: the result is 60;
>
> these by the 4 : the result is 240;
>
> these by the 3 : the result is 720; from these take a side and it will be the area of the triangle."

It is easy to recognize the corresponding formula, precisely known as "Heron's formula. Then, always on the same example, Heron gives an algorithm in order to find the "side" of 720:

> "Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square root nearest to 720 is 729, having a root of 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$; Take half of this; the result is $26\frac{2}{3} + \frac{1}{3} = 26\frac{5}{6}$; therefore the square root of 720 will be very nearly $26\frac{5}{6}$.
> For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $\frac{1}{36}$. Il we wish to make the difference less than $\frac{1}{36}$, instead of 729, we shall take the number now found, $720\frac{1}{36}$, and by the same method, we shall find an approximation differing by much less than $\frac{1}{36}$.[5]

There is no justification for the algorithm. Note that Heron names it a "method", and tells it is a "synthesis".

We find another way of presenting this algorithm by Theon of Alexandria (circa 370 BC).The procedure is described following a geometrical figure in the style of Euclid's Elements (II.4)

This algorithm (along with the following one, from Al Khwarizmi) is a favorite of teachers. It is easy to program with a recurrent sequence, and offers interesting ways of justification, and of error evaluation, in the frame of the curriculum.

[5]Heron , *Metrica*, in Ivor Thomas, *Greek mathematical Works*, vol.II, p.471, and [2], p. 202

4.3 Al-Kwharizimi : *al-Kitab al-mukhtasar fi hisab al-jabr w'al-muqabala or The Compendious Book on Calculation by Completion [or Restoring] and Balancing.*

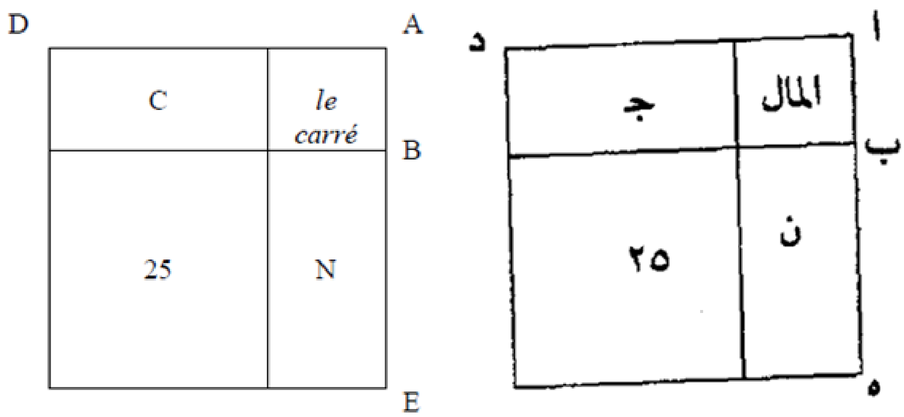
"*Squares and numbers are equal to roots*: for instance, 'one square and ten roots of the same, amount to thirty-nine dirham'; that is to say, what must be the square which, when increased by ten of its own roots, amount to thirty-nine.

The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by himself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine. [⋯]"

We have said enough so far as numbers are concerned [⋯] Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers "

Here is a first geometrical proof; Al Kwharizmi gives a second one:

There is another figure which leads to this. Let the area AB, which is the *square*. We are seeking to add to it ten of its roots. We take half of ten: we'll get five.,of which we make two area applied to each side of AB ; let it be the two areas C et N. The length of each of these areas will be five, which is the half of the ten roots., and its breadth is equal to the side of AB. It remains a square from an angle of AB, Which is five times five, and five is the half of the ten roots that we added to each side of the first area. We know then that the first area is the *square*, that the two areas added on each side are ten roots, and the whole is thirty-nine, and that it remains to complement the grater area the square of five by five – which is twenty-five, that we add to thirty-nine to complement the grater area, which is DE. We get from this sixty-four ; we take its root, which is eight, and which is one side of the greater area ; if we subtract from this an equal quantity to the one we have added, which is five, it remains three, which is the side of the area AB – which is the *square* – and which is its root ; the *square* is nine. Here is the figure:



4.4 : Diophantus (III^{rd} century AD) : "formula" vs" program"

We chose this text, because Diophantus gives three presentations for the same mathematical property: one with a "rhetorical formula", and one with an algorithm, then the inverse algorithm.[1]

In the short Treatise *About polygonal numbers (De polygonis numeris*), with the first four propositions and a subtle additional argument, Diophantus proves the validity of the following defining relation of each type of polygonal numbers:

> "Every polygonal, multiplied by the octuple of the number less by a dyad than the multiplicity of the angles, and taking in addition the square on the number less by a tetrad than the multiplicity of the angles, makes a square "

In "symbolic formula" language, a polygonal number P with v angles satisfies the relation $8P(v-2)+(v-4)^2$ =*square*. Prop. 4 makes it explicit what is the side of the *square*; its expression contains the side l of the polygonal number P: the full-fledged relation is: $8P(v-2)+(v-4)^2 = [2+(v-2)(2l-1])^2$ amounting to a definition of a *specific* polygonal number, that is, to an identification of any number greater than 2 as a specific polygonal one.

Diophante explains ensuite how to find, for a polygone number from a given type (ie that the multiplicity of angles v is fixed), a polygone number P whose side l is given and the reverse. The description is formulated by a mix of grammatical participle and future.

> 'Thus, taking the side of the polygone [number], always doubling it, we'll subtract one unity, and multiplying the remainder by the number lesser by a dyad than the multiplicity of angles, we'll add a dyad to the product; and taking the square on the result, we'll subtract from it the square of the number lesser by a tetrad than the multiplicity of angles, and dividing the remainder by eight times the number lesser by a dyad than the multiplicity of angles, we'll find the required number.
>
> Again, given the polygone itself, we'll find the side like this: Multiplying it by eight times the number lesser by a dyad than the multiplicity of angles , and adding to the product the square on the number lesser by a tetrad than the multiplicity of angles, we'll thus find a square (assuming that the given number is a polygone), and subtracting always a dyad to the side of this square ,we'll divide the remainder by the number lesser by a dyad than the multiplicity of angles, and adding to the result one unit, and halving the result, we'll get the side required.

Here, the "rhetorical formula" of the beginning justifies the algorithms.

Moreover, we can analyze how Diophantus writes the "inverse" algorithm, by inverting exactly the order and the operations (add/subtract, etc.).

We are used to the formal manipulation of formulae, but, we can analyze here a formal manipulation of algorithms. There are other documents: in the *Nine chapters*, for instance, we can find that two algorithms are proved to be equivalent because some steps cancel in pairs.

The text of Al-Khwarizmi is quite often used in the classroom to express an algorithmic solution to one that is summarized in a formula. Teachers are used to going from the algorithm to the formula, as formula is more familiar to them (and to current students). We see here how Diophantus extracts an algorithm from a formula. This is a commonplace task in elementary mathematics.

4.5 Jordanus de Nemore (circa 1220)

"(*a*) *If a given number is separate in two parts whose difference is known, then , each of the parts can be found.*

Since the lesser part and the difference equal the larger, the lesser with another equal to itself together with the difference make the given number.
Subtracting therefore the difference from the total, remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part.
For example, separate 10 in two parts whose difference is 2. If that I subtracts from 10, 8 remains, whose half is 4."

This is not as easy as algebra to follow, but it gives both method and justification. But, for the Problem (b) it is more difficult! So Jordanus introduces letters for intermediary results.

"(*b*) *If a given number is separated in two parts such that the product of the parts is known, then each of the parts can be found.*
Let the given number a be separated into x and y so that the product of x and y is given as b. Moreover, let the square of $x+y$ be e, and the quadruple of b be f. Subtract this from e to get g, which will then be the square of the difference of x and y. Take the square root of g, and call it h. h is also the difference of x and y. Since h is known, then x and y can be found.
The mechanics of this is easily done thus; For example, separate 10 into two numbers whose product is 21. The quadruple of this is 84, which subtracted from the square of 10, namely from 100, yields 16. 4 is the root of this and also the difference of the two parts. Subtracting this from 10 to get 6, which halved yields 3, the lesser part; and the greater is 7."[6]

Here, there is no more justification. We can notice that this algorithm comes back to the previous, using it as a procedure .

4.6 Euler, (1774) an algorithm to find the square root; on a generic example and with letters for undetermined data.

Elements of algebra, Chap XVI, On the Resolution of Equations by Approximation

784. When the roots of an equation are not rational, whether they may be expressed by radical quantities, or even have not that resource, as in the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at least the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

785. The first method which we shall mention, supposes that we have already determined, with tolerable exactness, the value of one root; that we know, for example, that such as 4, and that it is less than 5. In this case, if we suppose this value $= 4 + p$, we are certain that p expresses a fraction. Now, as p is a fraction, and consequently less than unity, the square of p, its cube, and in general, all the higher powers of p, will be much less with regard to the unity ; and, for this reason since we require only an approximation, they may be neglected in the calculation; When we have, therefore, nearly determined the fraction p, we shall know more exactly the root $4 + p$; from that we proceed to determine a new value still more exact, and continue the same process till we come as near the truth as we desire.

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.

Here we perceive, that x is greater than 4 and less than 5 ; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2$; but as p^2 must be very small, we shall neglect it, in order to that we may have only the equation $16 + 8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p$; we are sure that p expresses a fraction much smaller than before, and that we may neglect p^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p = 20$, or $9p = -\frac{1}{4}$; and consequently $p = . - \frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.
And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p = 20$; so that $8\frac{34}{36}p = -\frac{1}{1296}$, $322p = -\frac{36}{1296} = -\frac{1}{36}$ and $p = -\frac{1}{36 \cdot 322} = -\frac{1}{11592}$, therefore, $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

787. Now, in order to generalize what we have here laid down, let us suppose the given equation $x^2 = a$, and that we previously know x to be greater than n, but less than $n + 1$. If we now make $x = n + p$, p must be a fraction, and p^2 may be neglected as a very small quantity, so what we shall have $x^2 = n^2 + 2np = a$; or $2np = a – n^2$; and $p = \frac{a-n^2}{2n}$ and consequently $x = n + \frac{a-n^2}{2n} = \frac{n^2+a}{2n}$. Now , if n approximated towards the true value, this new value will approximate much nearer ; and , by substituting it for n, we shall obtain a new value, which may again be substituted, in order to approach still nearer ; and the same operation may be continues as long as we please.”

5 Questions and Perspectives

. We shall mention two here, which we plan to study in some detail in the future:

A. Algorithms in geometry: to what extent can construction procedures be read as algorithms? For instance, in Euclid's elements, can the *ekthesis* be seen as an initializing phase? Can *diorisms*, (related to the numbers of solutions of a problem in function of the value of the "givens" of the problem) and *case distinctions* be seen as instances of conditional branching?

B. Formulae vs algorithms : Beyond issues of cognitive flexibility ("translation" tasks), one can investigate the following issues:

- Compare the two semiotic environments (formulae / algorithms) in terms of manipulation potential
- Finiteness issues: many iterative algorithms (for square root approximation for instance, or iterated Euclidean division to find a continued fraction expansion) lead to infinite formulae. Could we see the later as the counterpart of algorithms? This lead to a wealth of questions, beyond that of convergence.

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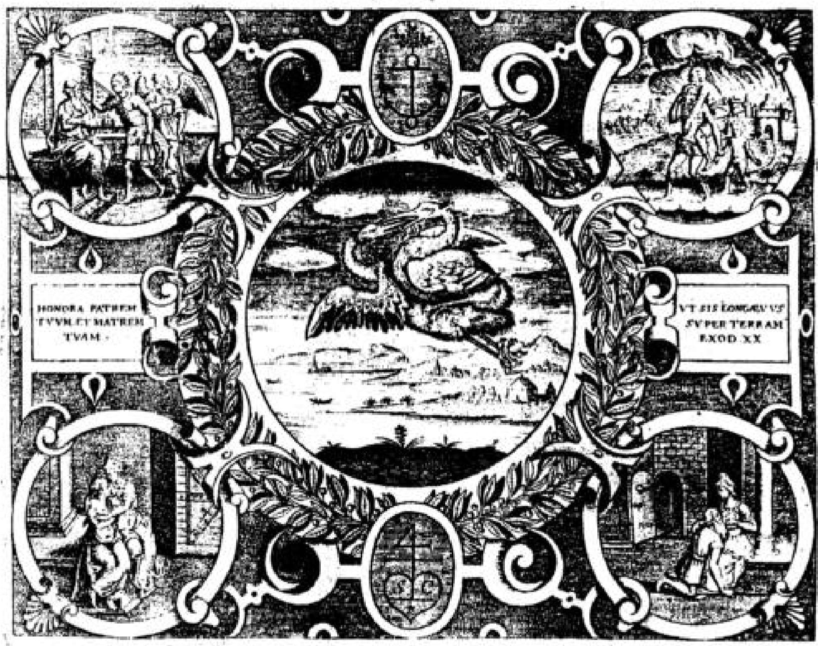

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