BOTTLED AT THE SOURCE:
The Design and Implementation of Classroom Projects for Learning Mathematics via Primary Historical Sources

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ABSTRACT

As mathematics instructors, it is not unnatural for us to be tempted to provide students with clear and precise presentations, both in our teaching and in the written materials which we provide to them. But just as a water filtration process intended to remove impurities can also remove healthy minerals and their interesting tastes, efforts to remove potential impediments to student learning can inadvertently strip a subject down to a set of facts and formulas lacking in context, motivation and direction. Beyond this, teaching something very distilled is unlikely to help students see how they can develop and reason with ideas on their own.

Going back to the source from which a mathematical subject originally sprang is one means of restoring these vital ingredients to student learning. In this paper, I describe my own experiences with a particular approach to using primary historical sources in mathematics to promote student learning, and share some of the challenges and rewards that I have personally found the most exciting in using classroom projects based on original sources with my students.

Keywords: Primary Historical Sources, Original Sources, Classroom Project Design, Pedagogy, Elementary Set Theory, Boolean Algebra, Group Theory

1 Introduction

The audience of this HPM conference and readers of its *Proceedings* are likely to be familiar with various ideas for bringing a historical dimension to bear on students' learning of mathematics; many of you may also have used primary source materials in your teaching in some way. This paper describes my own experiences with a particular approach to using primary source material in mathematics to promote student learning. The specific classroom projects I use to illustrate the challenges and opportunities afforded by this approach are part of a larger compendium of such projects that have been developed and tested since 2008, with support from the US National Science Foundation (NSF), by an interdisciplinary team of mathematicians and computer scientists at New Mexico State University, Old Dominion University and Colorado State University-Pueblo. The full collection of these projects is located on our web resource [6]; additional projects developed with prior NSF support can be found both in print [7] and on a separate web site [5].

As the title of this paper suggests, the pedagogical approach our team has adopted is not simply to bring primary source material to the students and ask them to "drink" that material in. Instead, original source material is "bottled" for student consumption as part of a classroom project which includes a discussion of the historical context and mathematical significance of each primary source selection, as well as a series of tasks designed to illuminate the source material and prompt students to develop their own understanding of the underlying concepts and theory. By guiding students through the reading of an original source in this way, our classroom projects fall short of Fried's call to adopt a 'radical accommodation' strategy of (directly) studying (original) mathematical texts as a means of avoiding the danger of trivializing history when using it as a teaching tool [12].[1] We never-theless embrace the commitment to humanizing mathematics which Fried contends requires one to take history seriously by "look[ing] at [mathematics] through the eyes and works of its practitioners, with all their idiosyncrasies" and "as far as possible, [to] read *their* texts as *they* wrote them" [12, p. 401].

Returning to the water processing metaphor of the title, one of the major challenges of our approach is thus to design projects which avoid (or at least minimize) the amount of filtering which occurs as a result of the "bottling" process. Of course, as mathematics instructors, it is not unnatural for us to be tempted to provide students with clear and precise presentations. But just as a water filtration process intended to remove impurities can also remove healthy minerals and their interesting tastes, efforts to remove potential impediments to student learning can inadvertently strip a subject down to a set of facts and formulas lacking in context, motivation and direction. Beyond this, teaching something very distilled is unlikely to help students see how they can develop and reason with ideas on their own, or to help them develop mathematical competencies as well as mathematical techniques. As argued by Kjeldsen & Blomhøj [15], Jahnke [13] and others, going back to the source from which a mathematical subject originally sprang is one means of restoring these vital ingredients to student learning.

At the same time, our ultimate pedagogical goal is to develop projects for learning core material from the curriculum of contemporary courses in discrete mathematics and computer science.[2] Adopting Jankvist's terminology in [14], our primary focus is thus on 'history-as-a-tool,' rather than on 'history-as-a-goal.' Within the 'history-as-a-tool' category, my own emphasis has been on the role of history as a cognitive tool to support the learning of mathematics (versus its motivational potential). While pursuing that goal, I am repeatedly amazed by the ways in which an original source reading selected to address some specific content objective can both stimulate and quench a student's thirst for deeper understanding that reaches beyond that objective.

In the rest of this paper, I share some of the challenges and rewards that I have personally found the most exciting in my work with original source projects, focusing primarily on the project "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce." In the next section, I use excerpts from that project to illustrate how primary source selections and student tasks can be combined to address a specific curricular topic in ways that go beyond simple mastery of that topic. In the final section of this paper, I more briefly describe design issues surrounding the project "Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley," relating those issues back to the water processing metaphor of my title.

[1] 'Radical accommodation' is one of two alternatives that Fried identifies as a means to avoid this danger; the second alternative is that of 'radical separation' in which the study of the history of mathematics is placed on an entirely different track from the regular course of study.

[2] More detailed discussion of our pedagogical design goals is found in [8]; an overview of how our projects can be used as replacement units for specific topics within a discrete mathematics course or to teach such a course in its entirely is found in [9].

2 Unexpectedly refreshing! "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce"

The project "Origins of Boolean Algebra in the Logic of Classes" [1] was initially designed for use in a first course in discrete mathematics which includes both elementary set theory and boolean algebra as part of its curriculum; in its current incarnation, it can be used simply as a unit on elementary set theory. Arranged in five sections, the project begins with a historical introduction describing the general context of Boole's work. The second section of the project employs extensive excerpts from Boole's 1854 *An Investigation of the Laws of Thought* [10] as a means to introduce students to the operations of logical addition (i.e., set union), logical multiplication (i.e., set intersection) and logical difference (i.e., set difference). Current terminology and notation for set operations, however, is intentionally *not* employed. Instead, students use Boole's deliberately algebraic notation — an integral part of his effort to develop a symbolic algebra for logic — to complete project tasks which explore the basic laws governing this algebra (e.g., commutativity, idempotency) and Boole's justification for those laws. Certain restrictions imposed by Boole on the use of the operation symbols (now long since lifted) are also explored in ways that raise important mathematical themes that I have found difficult to pursue in conjunction with a standard textbook treatment of elementary set theory.

Boole's justifications for these restrictions rely in part on his definitions of the operations, and in part on the analogy of his symbols with those of 'standard algebra.' The following excerpt from the project illustrates how Boole's own writing is woven together with project tasks that prompt students to examine and evaluate Boole's arguments:[3]

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PROPOSITION I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.[4]

...

6. Now, as it has been defined that a sign is an arbitrary mark, it is permissible to replace all signs of the species above described by letters. Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x.

[3] To set them apart in this paper, project excerpts are indented. Within project excerpts, primary source selection are set in sans serif font and bracketed at their beginning and end by the following symbol : ∞∞∞∞∞∞∞∞

[4] Emphasis added.

⋯By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms "nothing" and "universe," which as "classes" should be understood to comprise respectively "no beings," and "all beings." ⋯Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep;" and in like manner, if z stand for "horned things," and x and y retain their previous interpretations, let zxy represent "horned white sheep," i. e. that collection of things to which the name "sheep," and the descriptions "white" and "horned" are together applicable. Let us now consider the laws to which the symbols x , y , &c., used in the above sense, are subject.

⋯

10. We pass now to the consideration of another class of the signs of speech, and of the laws connected with their use.

11. *Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts.*

We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, applicable to each individual of the group under consideration, but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. **In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical.**[5] Thus the expression "men and women" is, conventional meanings set aside, equivalent with the expression "women and men." Let x represent "men," y, "women;" and let $+$ stand for "and" and "or," then we have

$$x + y = y + x, \tag{3}$$

an equation which would equally hold true if x and y represented numbers, and $+$ were the sign of arithmetical addition.

Task 4

Note that Boole imposed a restriction on the use of the addition symbol "+" in his system by asserting that "the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another." In this task, we consider the question of whether this restriction reflects standard language usage, as Boole claimed.

[5] Emphasis added.

Consider, for instance, the following expressions:

(I) infants and teenagers	(III) lying or confused
(II) dancers and singers	(IV) conqueror of Gaul or first emperor of Rome

For which of these is the conjunction (*or* , *and*) used in the exclusive sense specified by Boole? That is, is there an implication in standard language usage that a particular individual under discussion will belong to at most one of the named classes, but not both classes simultaneously?

Task 5

In his section 11 above, Boole referred to the analogy of symbolic logic with arithmetical algebra as further justification for restricting the use of "+" to disjoint classes:

> In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical.

Recall that the standard definition of addition for whole numbers m, n defines $m + n$ to be the total number of elements in the union (or aggregate) of a set containing m objects with a set containing n objects. Provide one or more specific examples to illustrate why it is important to use disjoint sets in this definition.

In my own classroom implementation of this project, student responses to the two tasks above have been as intriguing as their responses to Boole's claims themselves! One exciting features of this project which is exemplified by their responses is the way in which Boole's attention to issues related to language use and set operations (e.g., inclusive versus exclusive 'or') ensures that the inherent subtleties of these issues are made explicit. In contrast, contemporary textbook authors often downplay (or ignore) these subtleties in a way that can exacerbate their difficulty for students. Beyond the reassurance which an acknowledgement that the matter is not altogether unproblematic provides for some, all students are explicitly required to wrestle with these subtleties in completing Task 4 and other related project tasks. Because Boole's own writing both recognizes that there are decisions to be made about how symbols are used *and* offers criteria for making these decisions, the question of what constitutes sufficient evidence for a claim also naturally arises during class discussion of these and later tasks. In this way, students' introduction to elementary set theory is elevated above the level of simply applying basic set operations and their laws.

Of course, the context of Boole's own motivation for studying the laws of thought was an influential factor in the criteria he applied for making decisions about symbol usage.[6] In particular, his work in logic was strongly influenced by the general state of nineteenth century British mathematics, and especially by the concept of a 'symbolical algebra.' Although today's students acquire their mathematical ideas in a quite different context, Boole's development of the *algebraic* aspects of of symbolic

[6]The importance of considering context when reading original source material is discussed at come length in Jahnke [13].

logic is not completely foreign to their experience. Their own experience with algebra and their work on Tasks 7 and 8 below, for example, make them especially sympathetic to the restriction which Boole imposes on the usage of '$-$' in the following passage:

11. ⋯The above are the laws which govern the use of the sign $+$, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from the whole. This operation we express in common language by the sign *except*, as, "All men except Asiatics," "All states except those which are monarchical." **Here it is implied that the things excepted form a part of the things from which they are excepted.**[7] As we have expressed the operation of aggregation by the sign $+$, so we may express the negative operation above described by $-$ minus. Thus if x be taken to represent men, and y, Asiatics, i. e. Asiatic men, then the conception of "All men except Asiatics" will be expressed by $x - y$. And if we represent by x, "states," and by y the descriptive property "having a monarchical form," then the conception of "All states except those which are monarchical" will be expressed by $x - xy$.

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13. ⋯Let us take the Proposition, "The stars are the suns and the planets," and let us represent stars by x, suns by y, and planets by z; we have then

$$x = y + z. \tag{7}$$

Now if it be true that the stars are the suns and the planets, it will follow that the stars, except the planets, are suns. This would give the equation

$$x - z = y, \tag{8}$$

which must therefore be a deduction from (7). Thus a term z has been removed from one side of an equation to the other by changing its sign. This is in accordance with the algebraic rule of transposition.

Task 7

Note Boole's comment (in his section 11) that the operation *minus* will require "that the things excepted form a part of the things from which they are excepted." For example, if x represents men and y represents women, then the expression $x - y$ is meaningless in Boole's system since the class of women does not form part of the class of men.

(a) Suppose we wanted to drop Boole's restriction in the particular example just described. What class would the expression $x - y$ denote? Based on your reading of Boole up to this point, could this class be represented symbolically in some other way within his system? Explain.

[7] Emphasis added.

(b) Now consider the expression $d-p$ in the case where d represents drummers and p represents pianists. Explain why Boole would consider the expression $d-p$ to be meaningless, then describe the class which the expression $d-p$ would denote if Boole's restriction were removed. Based on your reading of Boole so far, how could this class be represented symbolically within his system? Explain.

(c) Based on these examples, do you agree or disagree with Boole's restriction, and why? What argument, if any, does he give for adopting this restriction in his section 11?

Task 8

Now consider Boole's equations (7) and (8) in his section 13 above, and recall his earlier restriction that "+" can only be applied to classes which share no members. Suppose we were to drop this restriction, and let y represent men, z represent doctors, and $x = y + z$ as in Boole's equation (7). Can we still deduce Boole's equation (8) in this case? That is, does $x - z = y$? Explain.

As with Boole's restriction on '$+$' to classes which are disjoint, his restriction on the use of '$-$' is not part of today's set theory. One concern about using an original source in which the author's conception differs in this way from that currently in use is that students may be unduly confused when they are eventually required to adopt current usage. But in truth, many beginning students of set theory unconsciously adopt Boole's 'common language' argument for restricting the use of '$x - y$' to cases where y is a part of x; after all, how can one remove something that is not already there?? Thus, a potential difficulty to understanding our current definition of set difference which exists independently of instructional materials is again made explicit through reading of the primary source. Boole's use of the notation '$x - xy$' to represent 'All states except those which are monarchical' in the preceding excerpt also anticipates a way of sidestepping the restriction on subtraction which my students have picked up on in responding to Task 7(b). In their responses to Task 8, many students also identify "$x = y + z \Rightarrow y = x - z + yz$" or "$x = y + z \Rightarrow y = x - z(1 - y)$" as legitimate 'subtraction rules' in the case where y and z are not disjoint.[8] Later in the project, students read excerpts from Boole's *Laws of Thought* in which he justifies the use of the symbol '1' to represent the 'Universe' and begins to use expression '$x(1 - y)$' as an (algebraic) equivalent to '$x - xy$' which allows the aggregate of non-disjoint sets x, y to be 'legally' expressed as '$y + x(1 - y)$.' Long before that point, however, many of my students have anticipated this algebraic strategy!

In short, Boole's explicitly algebraic notation meshes with students' experiences in algebra in ways that allow them to quite naturally anticipate the current definition of set difference as $A - B = A \cap \overline{B}$. In the third section of the project, which examines refinements to Boole's system made by John Venn in his 1894 edition of *Symbolic Logic* [17], students are thus unsurprised by the following pronouncement by Venn:

[8] Task 8 also begins an exploration of a second compelling reason for adopting Boole's restricted interpretation of '$-$'; namely, an unrestricted use of '$+$' would not allow for a well-defined inverse operation. Later project tasks based on original source excerpts from Boole, Venn and Peirce further explore the concept of inverse operations for logical operations.

It will now be seen how it is that the process of subtraction, with the corresponding symbolic sign, can be dispensed with. If y be excepted from x it must be a part of x, and may therefore be written xy $\cdots$. But this, as we have just explained, is the same as to 'multiply' x by $1 - y$, or by not-y. In other words, the legitimate exception of y from x is the same thing, in respect of the result, as taking the common part of x and not-y. 'The clergy, except the teetotalers', means, when the exception is duly interpreted, the class common to those of 'the clergy' and 'the non-teetotalers'.

Nor are students particularly surprised by Venn's removal of the restriction on '$+$' as applicable to disjoint sets only, a modification to Boole's system which Venn justifies as follows:

Boole, as is well known, adopted the $\cdots$plan [of] making all his alternatives mutually exclusive, and in the first edition of this work I followed his plan. **I shall now adopt the other, or non-exclusive notation:– partly, I must admit, because the voting has gone this way, and in a matter of procedure there are reasons for not standing out against such a verdict**;[9] but more from a fuller recognition of the practical advantages of such a notation. $\cdots$as a rule and without intimation of the contrary, I shall express 'X or Y,' in its ordinary sense, by $X + Y$. I regard it as a somewhat looser mode of statement, but as possessing, amongst other advantages, that of very great economy. [10]

By this point in the project, students have completed various project tasks calling for reflection on Boole's arguments in favor of an exclusive interpretation for '$+$', and analyzed specific examples which illustrate the 'very great economy' to be gained by a non-exclusive interpretation of '$+$'. What is surprising to many students, however, is the 'non-mathematical' reason that Venn gives for having changed his mind about this issue between the first and second edition of his *Symbolic Logic*; namely, by popular demand! This example of the ways in which non-technical factors can (and do) play a decisive role in the development of mathematics affords students a superb view of mathematics as a human endeavor.

One last design comment before leaving discussion of this particular project. Because I personally use this project to lay the ground work for a more abstract treatment of boolean algebra as a discrete axiomatized structure later in the course, the project concludes with primary source material from a C. S. Peirce, who took a more formal approach to the algebra of logic.[11] In fact, Peirce was one of those mathematicians who voted for a non-exclusive interpretation of '$+$' prior to Venn's 1894 adoption of

[9]Emphasis added.

[10]Interestingly, Venn does not remove Boole's restriction on '$-$', nor does he dispense with the process of subtraction altogether.

[11]This section of the project could be omitted by an instructor seeking only to introduce elementary set theory.

that interpretation. Although chronological earlier than Venn's work, Peirce's notation and general approach in his 1867 *On an Improvement in Boole's Calculus of Logic* [16] is more abstract than that of Boole or Venn, as illustrated by the following excerpt:

Logical addition and logical multiplication are doubly distributive, so that

(8) $$(a +\!, b),c \mathrel{=\!\!,} a,c +\!, b,c$$

and

(9) $$(a,b) +\!, c \mathrel{=\!\!,} (a +\!, c),(b +\!, c).$$

Proof. Let

$$\begin{aligned} a &\mathrel{=\!\!,} a' + x + y + o \\ b &\mathrel{=\!\!,} b' + x + z + o \\ c &\mathrel{=\!\!,} c' + y + z + o \end{aligned}$$

where any of these letters may vanish. These forumulæ comprehend every possible relation of a , b and c; and it follows from them that

$$a +\!, b \mathrel{=\!\!,} a' + b' + x + y + z + o \qquad (a +\!, b),c \mathrel{=\!\!,} y + z + o$$

But

$$a,c \mathrel{=\!\!,} y + o \qquad b,c \mathrel{=\!\!,} z + o \qquad a,c +\!, b,c \mathrel{=\!\!,} y + z + o \qquad \therefore (8)$$

In this excerpt, '$+$' indicates the two sets being joined are disjoint, while '$+\!,$' is used for the union of non-disjoint sets. In contrast to the expository writing styles of Boole and Venn, Peirce offers no rationale for the use of two separate symbols and simply states the basic dual algebraic laws (i.e., commutativity, associativity, idempotency), asserting that they are "evident." His proof (above) of the first of the dual distributivity law is also more formal than anything in either Boole or Venn. Yet students' grounding in the more concrete arguments of Boole and Venn allows them to make sense of Peirce's formalism, while Peirce's use of somewhat different notation in turn prepares students to shift to the standard set theoretical notation. The decision to place excerpts from Peirce in non-chronological order relative to the work of Venn was thus a deliberate design choice based on pedagogical, rather than historical, considerations. In the subsequent closing section of the project, the standard terminology and notation for set operations in use today is then (finally!) introduced and the algebraic properties of those operations are compared with the standard axioms for an abstract boolean algebra.[12]

[12] A companion project entitled "Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization" [2] goes on to explore the early axiomatization of boolean algebra as a fully abstract structure and introduces students to the use of a model to establish the independence and consistency of an axiomatic system, while a second companion project entitled "Applications of Boolean Algebra: Claude Shannon and Circuit Design" [3] explores the application of boolean algebras to the problem of circuit design. All three projects can be completed independently of each other.

3 Preservatives Added? "Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley"

The goal of the Boolean Algebra project discussed in the preceding section is to develop an understanding of the modern paradigm of elementary set theory as a specific example of a boolean algebra. The design of that project and its two companion projects (footnote 12) also provides an opportunity for students to witness how the process of developing and refining a mathematical system plays out, the ways in which mathematicians make and explain their choices along the way, and how standards of rigor in these regards have changed over time. To achieve these goals, all three projects rely on student reading of primary sources and completion of associated project tasks, with little commentary provided on that source material apart from historical information.[13] The central design issue was thus to ensure a selection (filtration) of original source excerpts sufficient to build a modern conception of set theoretical operations and the construction of project tasks to frame (bottle) those sources while remaining faithful to their original meanings.

In this concluding section, the challenges of achieving a proper balance between "filtration" and "bottling" are considered in the context of the project "Abstract awakenings in algebra" [4]. Designed for use in a first course in abstract algebra, my original intention was to base this project on just one original source, Cayley's 1854 paper *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$ [11]. In that paper, Cayley explicitly recognizes the common features of various (apparently) disparate mathematical developments of the early nineteenth century and defines a 'group' to be any (finite) system of symbols subject to certain algebraic laws. Asserting (without proof) that the concept of a group corresponds to the "system of roots of [the] symbolic equation [$\theta^n = 1$]," Cayley states (without proof) several important group theorems and proceeds to classify all groups up to order seven. While focusing on the classification of arbitrary (finite) groups and their properties, Cayley does not neglect to motivate this abstraction through references to specific nineteenth century appearances of the group concept, including "the system of roots of the ordinary equation $x^n - 1 = 0$" and the theory of elliptic functions. He also remarks that "The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations."

The fact that Cayley's paper provides such a powerful lens on the process and power of mathematical abstraction makes it simultaneously attractive and difficult to use in a classroom project aimed at developing an understanding of elementary group theory. Unlike Boole's introduction of symbolic algebra as a tool for studying logic, Cayley's paper does not make a radical departure from previous work in an existing field of study that requires no particular knowledge of previous treatments of that field. Instead, Cayley's insight into the common features of a variety of existing mathematical objects was dependent on his familiarity with those objects. Even at the time, his insight was premature and his paper attracted little attention from other mathematicians until late in the nineteenth century. Thus, simply reading Cayley's paper as one's first introduction to group theory seems unlikely to lead to a robust understanding of that theory.

One obvious alternative to using Cayley's paper as students' introduction to group theory is for students to study basic group theory using a contemporary textbook prior to reading Cayley's paper

[13] The interested reader should consult the projects at [6] to ascertain the extent to which commentary on the source material is employed.

in an effort to glean additional insights into that theory from his treatment of it. My own experiences with trying this approach many years ago, however, suggested that the few insights which students ultimately gained from reading the paper justified neither the time spent doing so nor the frustration experienced along the way. As beautiful as the paper is, students were coming to it without a proper historical context for Cayley's ideas, and the modern treatment to which they had already been exposed seemed to interfere more than it assisted with understanding those ideas.

With respect to the pedagogical goals of our NSF grant, another disadvantage of my first attempts to use Cayley's paper in teaching abstract algebra was the way in which it became simply an add-on to the course, rather than the primary vehicle by which students acquired an understanding of group theory. My experiences with the design and implementation of the Boolean Algebra and other primary source projects further suggested that something far more powerful could result from reading Cayley within a properly framed historical context. This led me to the question "What did Cayley himself read?" as a means of identifying prior source material that could lead up to a successful reading of Cayley's own paper. Among the mathematical works on which Cayley was building, that of two authors emerged as especially important precursors to Cayley's definition of abstract group: Lagrange's writing on algebraic solvability and Cauchy's writing on permutation theory.

The 90-page project which resulted from this research develops a significant portion of the core elementary group theory topics from the standard curriculum of a first course in abstract algebra, and has been successfully tested at three institutions as a textbook replacement for this part of the curriculum.[14] Structured in four major sections containing a total of 84 project tasks, the project also includes an introduction that provides a broad overview of the historical roots of group theory in the theory of equations, a conclusion which states the theorem now known as 'Cayley's Theorem' in modern terminology and sketches its proof in a project task, and an appendix that provides current definitions of 'coset' and 'normal subgroups.' Although not explicitly discussed by Cayley, each of these "modern" concepts is implicit in Cayley's paper; Cayley's Theorem, for instance, is an almost trivial observation after working with the numerous Cayley tables in the paper.

In fact, one of the most exciting aspects of implementing this project with students has been the way in which so many theorems of elementary group theory stated by Cayley become obvious as a result of having read Lagrange and Cauchy in the first half of the project. Even the proof of "Lagrange's Theorem" — typically established in contemporary textbooks using cosets in a fashion that causes many students to struggle — becomes straightforward after working through Cauchy's proof of this same result for permutation groups.[15] This phenomenon was not wholly unexpected; after all, the rationale for including works from Lagrange and Cauchy in the project was to provide students with concrete examplars (roots of unity and permutations respectively) of the abstract group concept identified by Cayley. What was unexpected (but welcome!) was the extent to which those examplars prepared students for that shift in abstraction.

The use of this project as a replacement for such a substantial portion of a required course did, however, raise new tensions for me in terms of design decisions.[16] Ultimately, my students need to

[14]See http://www.cs.nmsu.edu/historical-projects/Blog/cayley-notestoinstructor.pdf on [6] for a detailed overview of the project contents.

[15]See [4, pp. 43 - 46] for Cauchy's proof of this theorem and related projects tasks.

[16]A few words about implementation seem appropriate at this point. Depending on the project(s) selected, instructors should generally allow one to several weeks to implement a project in class. (The Cayley project is an exception in that full implementation takes approximately 10 weeks.) While a project is being implemented, several strategies are possible.

know that they are leaving the course with a firm understanding of the current paradigm of group theory and an ability to read and write proofs that meet today's standard of rigor and formality. This concern led me, for example, to include more commentary on the original sources than is the case with the Boolean Algebra projects. This additional commentary includes "modern" proofs of certain propositions, as well as certain examples (e.g., infinite groups) and terminology (e.g., isomoporhism) that are not found in the original sources.

Besides fleshing out the historical context of Cayley's work by including original source material from his predecessors, I have also tried to exploit the ways in which my students' mathematical context is different from that of Cayley to promote their understanding of group theory in its current form. For example, although Cayley's pioneering work in matrix theory does not seem to have influenced his thinking about groups, my students have completed a pre-requisite course in linear algebra which makes invertible matrices a natural example of a non-commutative group for them.

In short, I have not only filtered the sources in this particular project in a variety of way, but have also mixed a few additives and preservatives into the bottle before offering it to students. Yet student learning and assessment results (as measured by homework and exam performance) in my classes have been so positive as a result of completing this project, that I now plan to extend it into a full-length abstract algebra textbook. As I continue to wrestle with the tension between remaining true to my sources while fulfilling the demands of the curriculum in this work, I remain convinced that doing so is merited not only by the pedagogical value which primary sources provide with respect to providing context, motivation, and direction for students' mathematical endeavors, but also by the deep and robust mathematical understanding which can flow from those sources as a result of those endeavors.

4 Acknowledgment

The development of the projects described in this paper has been partially supported by the US National Science Foundation's Course, Curriculum and Laboratory Improvement Program under grants DUE-0717752 and DUE-0715392. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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Students could work on the project entirely in class, either individually or in small groups, as the instructor monitors and assists their progress. Whole class discussions or brief lectures may also be appropriate at certain junctures. In preparation for class activities, many instructors assign select project tasks for students to complete based on their own reading. Some type of student writing or presentation is also recommended; again, instructors have considerable flexibility in how this is done. Our experience further suggests that each assigned project should count for a significant portion of the course grade.

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