AN ESSAY ON AN EXPERIMENT IN MATHEMATICS CLASSROOM

—the golden ratio related in the form of the Nautilus shell—

Yoichi HIRANO* & Yoshihiro GOTO**

* Department of European Civilization, Faculty of Letters,
Tokai University
** Postgraduate, Course of Mathematics and Mathematical Sciences,
School of Science, Graduate School of Tokai University
4-1-1, Kitakaname, Hiratsuka-shi, 259-1292 Japan,
yhirano@tsc.u-tokai.ac.jp

ABSTRACT

What do children learn in mathematics classroom? Of course, it might be mathematical concepts as well as mathematical theories that they should learn. But usually the concepts and theories are quite abstract, and they cannot understand them so easily. In addition, they should also understand that they learn mathematics for the activities in their everyday life. For the purpose, they should also see concrete features of mathematics. Then, how can teachers give them the concrete subjects of mathematics? In this paper, we discuss an experiment in mathematics classroom, as an example of concrete subject.

Here we show the experiment about Nautilus shell. It is often said that the Nautilus shell has a logarithmic spiral whose growing rate is related to the golden ratio. From a viewpoint of biological investigation, Theodore Cook and D'Arcy Thompson published each of their works on the morphology of the nature at the beginning of the 20th century. In their works, they discussed the spiral of Nautilus shell. Recently, the subject of this kind is treated in many books on mathematics for the specialists and even for the general public.

From this experiment, we can find that the growth curve of the Nautilus shell is almost exactly a logarithmic spiral. Through this discussion, we also show that Nautilus shell is one of most applicable examples, because it can be examines according to children's level (both their school curriculum and their ability). Moreover, we can get the same results repeating the procedure with another type of Nautilus shell called *Nautilus macrompharus* (native to New Caledonia), etc.

Finally, we discuss the benefit of the experiment in mathematics classroom, as follows:

(1) Children can understand the feature (the form) of living things mathematically,

(2) Children can understand concrete features of mathematical subjects,

(3) Children can understand mathematics through various concrete features of mathematics

(4) Teachers can encourage students to interest themselves to mathematics

*First Author

In addition, the experiment presented here had been already conducted in the high school mathematics classroom, as well as even in the teacher-training course, and this trial also included the analysis applied with Excel software.

1 Introduction

1.1 Background

Nowadays, mathematics is considered to be universal. The universality of mathematics seems to be argued from the fact that mathematics has developed, especially after Descartes in the 17th century, with the aim of forming a conceptual system in spite of various aspects of its development process. We are now sharing almost the same mathematics all over the world and its globalization is very important to develop our scientific and technological civilizations at present. Thinking about the historical development, universality is not always true; on the contrary, it is sometimes a prejudiced perspective under eurocentrism[1]. After the Scientific Revolution of the 17th century, the framework of human thinking has kept a certain kind of universality. It has been based on the ”new scientific thinking” of the 17th century, and we can find the features of our modern civilization in that extension realized through the 18th century. Mathematics is considered to have the same evolutionary process.

With this in mind, we can see the reason why mathematics is considered to be a universal discipline at present. Mathematics today has, in a sense, developed under eurocentrism. Although various kinds of mathematics were developed in each civilization, we perceive mathematics to be a conceptual discipline which has been formed by cutting off many concrete human-cultural parts and by rearranging the remaining conceptual parts into a logical and concise system. This kind of mathematics is surely convenient for mathematicians, but not so comprehensible to the public as well as mathematics education; for, universal mathematics formed under eurocentrism loses the features that represent the original character related to human life and culture.

All that human beings have built up should be considered to be a part of civilization, and therefore, mathematics, which is a product of human wisdom, must also be a kind of civilization. When we consider mathematics as a key element to understanding our civilization, we should see a society or a community including mathematics as an integrated system of multi-cultured dimensions just as human cultures, human life-styles, science and technology, etc. Here, we could find some elements for mathematics classroom; we could find concrete subjects which are related closely to children’s direct understanding, for example in everyday life, in the nature, etc.

Then, how can teachers give such concrete subjects to children? When they choose a material, they can show its mathematical feature theoretically; this might be an explanation from a deductive point of view. But they can also show it in another way. Especially concerning materials seen in the nature, we can analyze them through an experiment from a viewpoint of mathematical theory.

[1]For example, let us discuss the historical process of the abstraction of mathematics. Generally speaking, there exist three typical periods of abstraction: ancient Greece, the 17th century and the 19th century. This gives us a composition that is most easy to understand when we look at mathematics from a macroscopic perspective. It is also from a viewpoint of eurocentrism that mathematics has been developed around Europe.

1.2 Problématique

In this paper, we discuss the possibility of "mathematical experiments". Here, by taking the case of Nautilus shell, we try to show how to investigate it from a mathematical viewpoint.

It is often said that the Nautilus shell has a logarithmic spiral whose growing rate is related to the golden ratio. The logarithmic curve is thought to have been developed by Rene Descartes, French mathematician. At that time, Jan Swammerdam (1637-1680), Dutch biologist, and Christopher Wren (1632-1728), English architect, studied the spirals of snail shells. From a biological viewpoint, Theodore Cook and D'Arcy Thompson published each of their works on this subject at the beginning of the 20th century[2]. In their works, they discussed the spiral of Nautilus shell. Recently, the subject of this kind is treated in many books on mathematics for the specialists and even for the general public. For example, Ian Stewart as well as other authors discussed the formation of the spiral of Nautilus shell[3].

We can investigate the features of the spiral that the Nautilus shell creates during its growth. By tracing the spiral curve that appears along the cross section of the cut Nautilus shell, and by drawing a series of the tangents of the curve which intersect with each other perpendicularly, we can see a series of the length of the tangents and even the radii. Then by calculating the rate of increase between each successive length, we can find that this sequence is a geometrical progression and that the lengths and the numbers of order (the value of the angles) are related to each other as an exponential function. And on further investigation we can also find that the coefficient of the growth is slightly larger than the golden ratio.

This is an example of the practicable experiments in the mathematics classroom. This experiment had been already conducted in the high school mathematics classroom, as well as even in the teacher-training course. This trial also included the analysis applied with Excel software.

In this paper, we also discuss how we can realize the experiment and how to examine the results according to children's level (children's school curriculum or even mathematical ability). As mentioned in the section 3, the result of this experiment can discuss in various manner. Therefore, we could handle the result depending on the subjects that children learn at school (i.e. the problems related to a geometrical progression, an exponential function, etc.). In addition, since the example is quite concrete, children could be convinced of the result even depending on each of their own abilities.

In conclusion, we discuss the benefit of the experiment in mathematics classroom.

2 Method of the Experiment on Nautilus shell

2.1 Preparation

(a) Material
- Nautilus pompilius shell (a species native to the Philippines), which is cut right down the middle into the two similar parts.

[2] Cook, Theodore Andrea, *The corves of Life*, Dover Publication, 1979 (originally published in 1914). Thompson, D'Arcy, On the Gowth and Form, Dover Publication, 1992 (originally publishd 1917).

[3] Stewart, Ian, *Nature's Numbers*, Basic Books, 1995

(b) Tool
- a flat vat, absorbent cotton, a clear glass plate (a clear acrylic plate),
a tracing paper (squared graph paper), a ruler (a caliper)

2.2 Experiment

The experiment shoud be carried out in the following procedures.
[1] Trace the spiral curve of the Nautilus shell on the tracing paper:

(1-1) cover the bottom of the vat with plenty of absorbent cotton
(1-2) put the cut Nautilus shell onto the absorbent cotton (the cross section upward) and cover it with a clear glass plate to fix the shell firmly on the absorbent cotton
(1-3) put a tracing paper on the glass plate and trace by hand the spiral curve of the shell on the paper (handwriting)

[2] Investigate the properties of the spiral curve:

(2-1) draw a series of the tangents of the curve which intersect with each other perpendicularly (see Fig. 1)
(2-2) measure the length of each tangent in ascending order from the shortest one (P_1P_2, P_2P_3, P_3P_4, $\cdots$, in the Fig. 1)
(2-3) draw a segment joining the points of contact on the horizontal tangents (T_9T_{11}) and another segment joining the points of contact on the vertical tangents (T_8T_{10})
(2-4) find the center of the spiral as the intersection of these segments (O)
(2-5) measure the length of each radius in ascending order from the shortest radius at the right angle (here radius means the length between the center and each point of contact, OT_1, OT_2, OT_3, $\cdots$)
(2-6) calculate the ratio between the lengths of two consecutive tangents (between horizontal tangents, between vertical tangents, etc.)

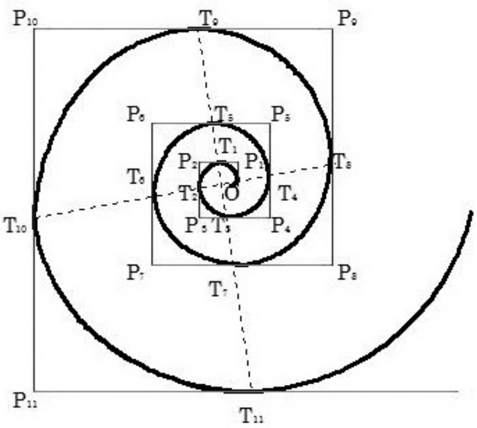

Fig.1 The tangents drawn of the spiral

We show the picture just about a specific sample (see Fig. 2).

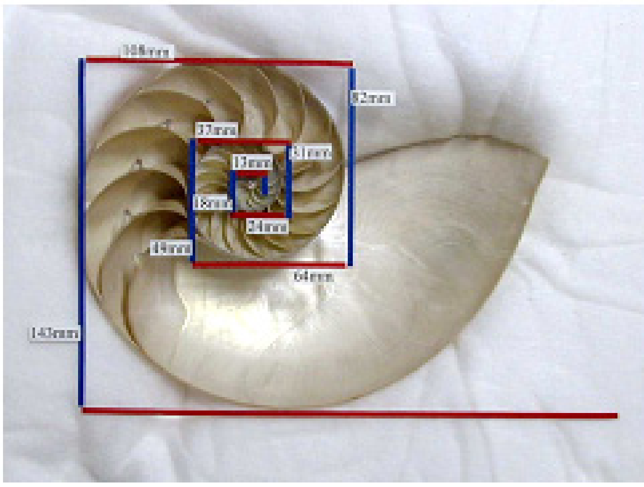

Fig.2 Nautilus shell and its tangents

3 Results and Discussion of the the Experiment

3.1 Result

According to the experiment mentioned above, we can get the result as follows (just about the specific sample shown in the Fig. 2):

Table 1 The length of tangents

Tangents	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5
Length(horizontal)	13		24		37		64		108	
Length(vertical)		18		31		49		82		143

(mm)

Table 2 The length of radii

Radii	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}
Length	6.9	9.6	12.7	16.4	19.6	26.0	34.3	43.5	57.3	75.9

(mm)

where a_i, b_j and r_k are defined as follows:

– a series of the length of the tangents
  horizontally: $a_1 = P_1P_2$, $a_2 = P_3P_4$, $a_3 = P_5P_6$, $\cdots$
  vertically : $b_1 = P_2P_3$, $b_2 = P_4P_5$, $a_3 = P_6P_7$, $\cdots$
– a series of the length of the radii
  $r_1 = OT_1$, $r_2 = OT_2$, $r_3 = OP_3$, $\cdots$

Then, the ratio of two consecutive horizontal tangents is as follows:

$$a_2/a_1 = 1.84\ldots,\ a_3/a_2 = 1.54\ldots,\ a_4/a_3 = 1.72\ldots,\ a_5/a_4 = 1.68\ldots$$

and concerning the vertical tangents,

$$b_2/b_1 = 1.72\ldots,\ b_3/b_2 = 1,58\ldots,\ b_4/b_3 = 1.67\ldots,\ b_5/b_4 = 1.74\ldots.$$

Moreover, the ratio of each two consecutive tangents is as follows:

$$b_1/a_1 = 1.38\ldots,\ a_2/b_1 = 1.33\ldots,\ b_2/a_2 = 1.29\ldots,\ a_3/b_2 = 1.19\ldots,\ b_3/a_3 = 1.32\ldots,$$
$$a_4/b_3 = 1.30\ldots,\ b_4/a_4 = 1.28\ldots,\ a_5/b_4 = 1.31\ldots,\ b_5/a_5 = 1.32\ldots.$$

In addition, concerning each two consecutive radii, we get the ratio between them as follows:

$$r_2/r_1 = 1.39\ldots,\ r_3/r_2 = 1.32\ldots,\ r_4/r_3 = 1.29\ldots,\ r_5/r_4 = 1.19\ldots,\ r_6/r_5 = 1.32\ldots,$$
$$r_7/r_6 = 1.31\ldots,\ r_8/r_7 = 1.26\ldots,\ r_9/r_8 = 1.31\ldots,\ r_{10}/r_9 = 1.32\ldots.$$

From these values, it is resulted that

(A) the ratio between two consecutive horizontal tangents seems to be nearly equal to the value 1.7,
(B) the ratio between two consecutive vertical taqgents seems to be nearly equal to the value 1.7,
(C) the ratio between two consecutive tangents seems to be nearly equal to the value 1.3.
(D) the ratio between two consecutive radii seems to be nearly equal to the value 1.3.

3.2 Discussion (1) – from a viewpoint of mathematics classroom

From the results of the experiment about the Nautilus shell, we can investigate various properties depending on the situation of children's understanding. For example, in the first year (or second year) of a high school, students study the theory of progression. Therefore, they can understand that the sequence of the length of the consecutive tangents (or the consecutive radii) appeared in the Nautilus shell is considered as a geometrical progression. Perhaps, as to the ratio between two consecutive tangents (or the consecutive radii), even junior high school students can understand the fact. But it is just high school students in a science and engineering course who can understand that the spiral possessed in Nautilus shell could be considered as a logarithmic spiral.

Thinking so, we can say that Nautilus shell is one of most applicable materials in mathematics classroom; for it includes various aspects that we show below.

(1) From a viewpoint of a geometrical progression
The sequence of the length of consecutive tangents is considered as a geometrical progression, with the value about 1.3 as the common ratio. In addition, the sequence of the length of consecutive horizontal (vertical) tangents is also considered to be a geometrical progression with the value about 1.7 as the common ratio. Here, it should be noted that the value 1.3 is nearly equal to the square root of 1.7. And the value 1.7 is considered to be nearly equal to the golden number $1.618\cdots$. Then, each tangent seems to increase at the rate of the square root of the golden ratio at each right angle, and therefore at the rate of the golden ratio at each 180 degrees.

(2) From a geometrical view point
In the experiment, the length of each radius is also measured. Then we can see the ratio between two consecutive radii is equal to the ratio between two consecutive tangents. This can be argued from a geometrical construction shown in the Fig. 3. Here, the triangles OPQ, OQR, ORS are similar to each other, then we can easily obtain the result as

$$PQ : QR : RS = OU : OV : OW.$$

According to this fact, it is also true that the ratio of two consecutive tangents is considered as the rate of the growth of Nautilus shell.

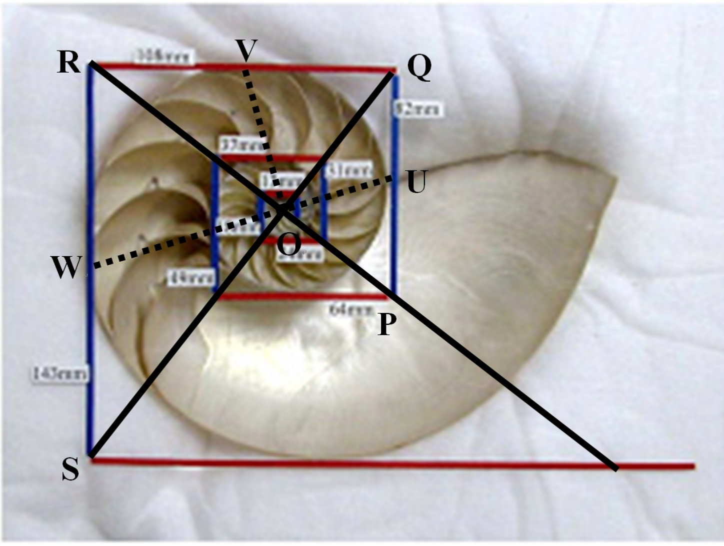

Fig. 3 Nautilus shell and its center

In the experiment, the procedures (2-3) and (2-4) can be attained on the hypothesis that the sipiral is a logarithmic. It is because all the points of contact could be arranged on the same segments only when the angle between the radius and each tangent are always equal to a constant value (an equiangular spiral). And only in this condition, the center of the spiral curve could be fixed (see the Fig. 4).

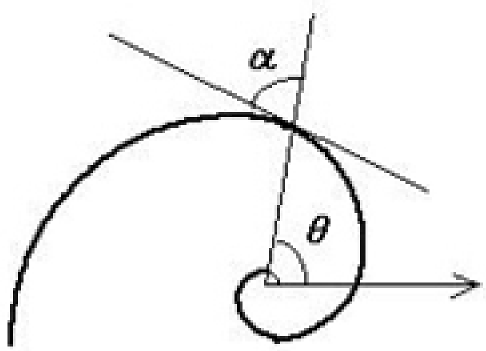

Fig. 4 Equiangular Spiral

In the Table 2, the length of each radius is related to the growth of the shell in every right angle. Thinking of the geometrical discussion mentioned above, the length of each tangent coincides with that of each radius. Then, we can investigate the growth rate of Nautilus shell by using the length of the tangents (Table 1).

Rewriting the Table 1 as follows:

Table 3 The length of tangents (rewritten)

No.	1	2	3	4	5	6	7	8	9	10
Length	13	18	24	31	37	49	64	82	108	143

(mm)

where each number of order signifies the value of the angle increasing in every right angle (here, we adopt the number of order for simplifying; i.e. the number k signifies the value $k * (\pi)/2$, for $k = 1, 2, \ldots, 10$).

When we approximate the relation between the number and the length to an exponential function, we can get the following graph (by means of Excel software).

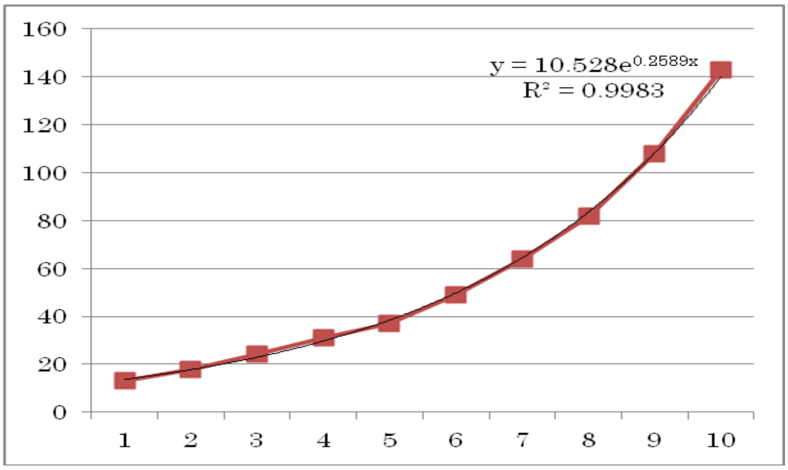

Fig. 5 Relation between the number and the length

This graph shows that the approximation is quite suitable, and then, the fact that Nautilus shell has a curve close to a logarithmic spiral can be clarified.

(4) From a viewpoint of the golden ratio

Considering the discussion of the fact that the spiral of *Nautilus* shell is logarithmic, we can continue to investigate if Nautilus shall has the golden ratio.

From the argument in (3), we can put the following expression as the formula of the spiral of Nautilus shell,

$$r = Ae^{Kx} \text{ (where } r\text{=radius, } x\text{=angle; } A, K \text{ : constants)}$$

Here, suppose that the rate of the radius is dr $(= r'/r)$ depending on the difference of the angle $dx = x' - x$, and then,

$$dr = e^{Kdx}$$

therefore

$$\log dr = Kdx.$$

If the growing rate of the spiral of Nautilus shell is equal to the golden ratio, from the discussion in 3.1., the ratio between two tangents in every 180 degrees might be equal to the golden ratio; in consequence,

$$dr = 1.618\ldots,\ dx = 3.1415\ldots,\ \text{then } K = 0.1532$$

In the case of the expression in the graph shown above, the value of exponent is equal to 0.2589. Since this value is calculated with the numbers as variable, by converting this variable into the value of the angle, we get, in this case

$$K = 0.2589 * 2/3.1415\ldots = 0.1648\ldots$$

In consequence, we find that the rate of the growth is slightly larger than the golden ratio.

4 Observation

4.1 Historical Comment

The experiment conducted here in itself is quite biological, because it clarifies the feature of Nautilus shell: we can find that Nautilus shell possesses a logarithmic spiral in its structure. However, this fact is simultaneously mathematical, because it should be one of the conspicuous examples that the form (or the figure) of living things can be analyzed and explained by means of mathematical theory.

Historically, the form of living things has not always been explained by means of mathematics. Of course, we can found some specific cases; for example, it is known that Pappus (B.C. 3C.) tried to explain the hexagonal section of a honeycomb geometrically. But, especially as to a logarithmic spiral, it seems to have been discussed as a mathematical problem. Maybe Leonardo da Vinch (1452-1519) tried to discuss a kind of spiral, but it might be concerned with the flow of water and the women's hair. At that time, Albrecht Dürer (1471-1528) is known to discuss a logarithmic spiral implicitly. After Descartes' mathematical discussion on this spiral, in the 17th and 18th centuries, there appeared some trials concerning the form of shell.

However, it is just in the second half of the 19th century or even early in the 20th century that the form of living things had become a subject of mathematical research. We could say that some kind of mathematical morphology had just established at that time. Concerning the curves of living things, Theodore Cook and D'Arcy Thompson are evaluated as pioneer of this field.

Nowadays, it is well said that Nautilus shell has the golden ratio in its form. It is true that Cook and Thompson suggested this matter in their works. But their methods are not so clear. From a viewpoint of research, the form and the growth of living things are so complicated that we cannot discuss easily. On the contrary, just about Nautilus shell, the spiral appeared in it is so harmonic that we can understand its feature even mathematically. It is considered as a suitable material for children. In this paper, we tried to show how we can handle its spiral. Naturally, with the present advanced technology, we can make use of various kinds of devices (a photocopy, a scanner, a digital camera, etc.) to get the image of the spiral. But here, we adopted a way to trace the spiral by hand, because we think that a hands-on activity should be important for children.

4.2 Experiment in Mathematics Classroom

From the experiment, we can conclude that the growth curve of the Nautilus shell is almost exactly a logarithmic spiral. Moreover, we can get the same results by repeating the procedure with another type of Nautilus shell called *Nautilus macrompharus* native to New Caledonia, as well as other type of the shells (for example, *Argonauta argo* called a Paper Nautilus).

This is an example of the experiment in the mathematics classroom. Well, what is the benefit of the experiment in mathematics classroom? In conclusion, we show four points as follows:

(1) To understand the feature (the form) of living things mathematically:
It is a biological experiment, in itself. Each student handles each sample and then get each result. But when students accumulate each of their results, they can find that nautilus shell (as well as other kinds of shells) would have a proper feature that its spiral is close to a logarithmic curve. This is merely a kind of a scientific experiment, to be sure, but students can understand the validity and the advantage

of mathematics which can integrate what they acquired and after all even their thinking.

(2) To understand concrete features of mathematical subjects:
Normally, in the classroom, students try to understand mathematical concepts and theories. Since mathematics is often abstract, students are apt to feel that mathematics is far from their activities. On the contrary, through an experiment, students handle concrete objects, which include mathematical features. Well, what should students learn in mathematics classroom? It is not only how to understand mathematics but also how to think their activities and how to live by means of mathematical thinking. When we face some mathematical objects, we can understand various aspects hiding these objects through the experiment. It is because the experiment promotes various kinds of perspective.

(3) To understand mathematics through various concrete features of mathematics
How can we understand objects that we are facing? When student try to understand a abstract concept, usually some concrete examples might be quite helpful. Because, it is easier to understand at first some concrete matters and then to induce general matter. Therefore, experiments could give to students the first starting points and the direction of thinking.

(4) To encourage students to interest themselves to mathematics
From a viewpoint of mathematical education, teachers should discuss the methods to encourage children to interest themselves in mathematics. At present, we can find some mathematical museums, where many kinds of visual devices and hands-on exhibits are set up. Perhaps, many students are interested in such museums, and even some adult are still eager to visit. The experiment in mathematics classroom could play the same role as the devices and the exhibits in mathematical museums.

Therefore, it is necessary and important for teachers to consider that mathematics is related to many things around us, our culture, our everyday life, the nature, etc. The trial of some experiments could become one of key elements for mathematics education in the future.