HARRIOT'S ALGEBRAIC SYMBOLS AND THE ROOTS OF EQUATIONS

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ABSTRACT

Thomas Harriot(1560 1621) introduced a simplified notation for algebra and his fundamental research on the theory of equations was far ahead of that time. He invented certain symbols which are used today. Harriot treated all answers to solve equations equally whether positive or negative, real or imaginary. He did outstanding work on the solution of equations, recognizing negative roots and complex roots in a way that makes his solutions look like a present day solution. Since he published no mathematical work in his lifetime, his achievements was not recognized in mathematical history and mathematics education. In this paper, by comparing his works with Vié ta and Descartes who were mathematicians in the same age, I will show his achievements in mathematics.

Keywords: Harriot, algebraic symbol, the roots of equation

MSC 2000 : 01A40, 01A45, 0803

ZDM Classification : A30

1 Introduction

Although there have been constant disputes among scholars about the mathematician who first invented the notation of the algebraical symbol between Thomas Harriot and Viéta, François, Thomas Harriot, a British scientist and mathematician in the late sixteenth and the early seventeenth centuries, took algebraical symbols in the equation and some parts of them have been used up to this day. It is very progressive that Thomas Harriot received the idea of complex roots and negative roots in the equations and endeavored to generalize the form of equation.

Notwithstanding the achievements, his works have been dealt with carelessly. That is because he does not have any mathematical work but a posthumous book.

First of all, this paper will look into the mathematical history when negative numbers were not received as solutions of equation. A brief description of Harriot's life in connection with mathematical history, his scientistic and mathematical works will be dealt in the second part. Harriot's algebraical symbols and solutions of equation will be compared with Viéta, François and Descartes, René for bring out the differences. This will be a meaningful work and help to shed light on one of the foremost mathematicians.

2 Solutions of Equations and History of Algebraic Symbol

2.1 History of Negative Numbers

It took long time that mathematicians receive negative numbers thoroughly. Mathematicians found it difficult to adopt negative numbers. There was possibly difficulties in detecting a visual and geometric meaning and operating[8, 9]. It had reached the peak of disputes on an approval of negative numbers in sixteenth and seventeenth centuries. Even in eighteenth century, there were a lot of scholars who did not receive negative numbers for reasons of irrationality. The history of negative numbers, inversely, has showed farseeing intelligence of Harriot. Arcavi[1] made a study of history of negative numbers, which particularly had difficulties to be received, in his paper on the methods of instruction of it.

Diophantus, Greek mathematician in the third century, did not receive negative numbers as solutions of a linear equation because he regarded it as an absurd one. He only adopted positive numbers and nothing else is possible to extract. Brahmagupta in the early seventh century looked into multiplication between signs but he did not receive negative numbers as solutions of a quadratic equation. Since then, multiplication between signs became known to the whole India. Though Al-Khowarizmi in the ninth century made notes of negative and positive roots of a quadratic equation, it is doubtful that he understood it completely without any comments on this.

Fibonacci in the early thirteenth century rejected negative roots, but took a step forward when he interpreted negative numbers in a problem concerning money as a loss instead of a gain[1, 14]. In the fifteenth century, though Pacioli used a minus sign in the equation such as $(7-4)(4-2) = 3 \times 2 = 6$, he did not understand the meaning of negative numbers. On the other hand, French mathematician Chuquet may have been the first mathematician to recognize negative numbers as exponents[9]. Stifel, a well known German mathematician in the mid sixteenth century, recognized negative numbers as absurd ones saying 'negative numbers are smaller than nothing'. Cardano notes that the product of multiplication between two negative numbers has a positive sign in his work in 1545 but he was doubtful of negative numbers as a fictitious one. Bombelli also understood it insufficiently by appending a term 'm and n are positive numbers' to $m - n$ in 1572[1, 9]. Viéta explained some laws of algebra in <In Artem Analyticem Isagoge> (1591) but he left out a specific explanation of negative numbers. He set limits on coefficient in the equation as positive numbers[1].

Hudde in Germany in 1659 did not make a distinction between positive and negative numbers. Harriot consented to his idea but literature disputes whether Harriot received negative numbers as solutions of an equation and understood the meaning of it[1, 2, 8, and 9]. Descartes partly received negative numbers. He called negative roots as a fake one because he thought that negative numbers are smaller than nothing. He inferred that positive numbers would be genuine roots of the equation.

It was the seventeenth century that mathematician received and applied freely. Practicalism was surged forth in the mid seventeenth century even in mathematics. It made an algebra design a consistent theory so that the application of negative numbers was unrestrained. However logical consideration about negative numbers was unsatisfactory. Since the notion and logic of negative numbers was unreliable, mathematicians evaded to comment or object to use of it. d'Alembert said 'if there was an equation having negative roots, there should be a misdirection' and 'the right answer was with a plus sign'[1, 8]. Maseres, a British mathematician, disregarded negative numbers as not understandable one in his book. Euler misjudged negative numbers as 'bigger one than infinity' in the letter to Wallis but corrected it later. This conclusion was deduced from the sequence — $1/4, 1/3, 1/2, 1/1, 1/0, 1/-1,$ $1/-2, 1/-3,$ — but later he changed position after the observation of the sequence — $1/9, 1/4, 1/1,$ $1/0, 1/1, 1/4, 1/9,$ — . When Pascal said 'subtracting 4 from 0 leaves 0', his friend Arnauld made an objection to it bringing $-1 : 1 = 1 : -1$, the ratio of smaller to bigger one was bigger to smaller one. De Morgan in the nineteenth century consented to d'Alembert saying 'there is no numbers smaller than 0'.

After adopting the logical formalism which excluded self-contradiction, mathematicians started to receive a negative number. Whitehead and Russel said that 'if receiving a symbol as an operator, there would be no restraint' in <Prinkipia>; as the solution of the equation $x + 1 = 3$ is $x = 3 - 1 = 2$, the solution of $x + 3 = 1$ is $x = 1 - 3 = -2$. That is, the solution of $x + a = b$ is $x = b - a$. They emphasized that an idea of $+$ and $-$ as an operator would remove absurd and irrational working. It was a consequence of mutual supplementation of Fibonacci's intuitive and unconscious point of view which regarded negative numbers as a meaningful one and the logical and consistent point of view which insisted an adoption of negative numbers or complex numbers as solutions of equation just like a Euclidean geometry. The former emphasized intuitive and comprehensive understanding and the latter stressed analytic and axiomatic formalism.

2.2 History of Algebraical Symbols

It was necessary to have mathematical signs for modern mathematics. Consequentially the development of mathematical notation brought about the development of mathematics. In the sixteenth century in Europe, mathematical notations were invented and applied. The sign of $+$ and $-$ were first appeared in <Mercantile> by Widmann in 1489. In this book, the 'excess' and 'shortage' were represented in equations instead of 'addition' and 'subtraction' or positive and negative numbers. Heocke was the first mathematician who used it as an algebraic symbol in 1514. Recorde, a British mathematician in 1557, first presented an equal sign '=' and he explained that equality means the parallel. At that time, the length of an equal sign was little longer. Or, two parallel vertical lines and '  ' also used as an equal sign. The multiplication sign '•' appeared Harriot's <Artis Analyticae Praxis> but Oughtred used '$\times$'[4] as the multiplication sign. Harriot, the initiator of an inequality sign, devised more convenient signs than Oughtred's one.  and  was appeared in Harriot's manuscript but we could find '$>$' and '$<$' in his book <Artis Analyticae Praxis>[11]. At that time, the present sign of division '$\div$' was appeared in Rahn's algebra book and a radical sign '$\surd$' was used in Rudolff's paper[4, 9].

3 Thomas Harriot

Thomas Harriot as the virtual first Britain algebraist introduced algebraic signs such as the interpretation of equation and inequality signs. An equal sign (=) generally known as the sign which a mathematician Robert Record had invented had become famous because Harriot aggressively used the sign.

Unfortunately, Harriot did not leave any of his books in mathematics in his entire life. After ten years since his death, his book <Artis Analyticae Praxis> was published by his colleagues in 1631. Furthermore, this fact had not been known to the world until Descartes's book referred to Harriot's thesis in 1637.

3.1 Harriot's Lifetime

Harriot had been born in Oxford, England in 1560 and graduated from Oxford University when he was twenty years old. There was not known about his private life when he was young. He had two tutors in his entire life and those were Minister Walter Ralegh and Count Henry Percy. After he had graduated from university, he became Walter's private mathematics teacher and took part in the lesson of the reclamation of new world and exploration to the new world in the North America with Walter's companies. As he brilliantly had worked as an adviser of the American expedition, for instance he had done the design and production of the ship and recruitment; 'a report about the new world, Virginia' which had been issued in 1589 paid the highest tribute of admiration[4, 6, and 9]. In addition, the report was filled with the native's language, religion, the method of the trade; therefore it became a famous thesis of settlers to the new world. This book is the only book he made in person. After Walter had warped up in political chaos, Henry began to help Harriot since 1598. Walter unstinted in his praise of Harriot's work and he described Harriot as 'Count magician'. Thanks to Henry, Harriot was able to focus on a stable study in a science lab with mathematicians Walter Warner and Thomas Hughes.

As Count Henry underwent the hardships of prison life for political reason in 1605, Harriot was also suspected but released soon. After that time, he devoted himself to the study of natural science such as mathematics, astronomy, mechanics and optical science and left outstanding academic achievements which great scholars paid little attention to. Though he attained materials for figuring the sun's rotation period by observing a solar spot in 1613, this period was his last moment to have a passion for his academic work. He was under continuous adverse circumstance of Henry's unfortunate death, his colleague's death and his disease. After five years of struggle against his disease, he passed away in 1621[11].

Although he made splendid achievements in mathematics and science, he did not leave any publications in his life. Confusing socio-political atmosphere and his meticulous characteristics shown in the report about Virginia had not gained an opportunity to publish his work.

3.2 Harriot's Natural Science

Most of Harriot's manuscripts were about natural science. Looking into his achievements in natural science would help to understand Harriot.

Harriot was endowed with practical and profound scientific knowledge and he achieved results in the fields of astronomy, optical science and dynamics. He was interested in astronomy after the discovery of a comet at that time and he observed the movement of the comet, later called Halley's Comet, from 1607. He observed the comet using a telescope in 1609 and it was ahead of Galilei. He first discovered sunspots while he observed Jupiter and after that time he had 199 times of record of sunspots observation from 1610 to 1613[9, 11].

Earlier in 1597, he discovered sine rule about refraction of lens and it was twenty years ahead of Willebrord Snell who was known as the originator of the theory[11]. A multi-color spectrum of light inspired Harriot and he developed the theory of a rainbow. Kepler got the news and sent a letter to Harriot but they never exchanged their theories. Harriot might feel discomfort to deliver his idea to Kepler directly or he might plan to publish when he regained his health [14]. In dynamics, he studied free fall of a parabola in the no resistance condition ahead of Galilei. Harriot discovered a track of a shell which described a parabola and could be divided into horizontal and vertical components. He also theorized various problems of sea navigation and his calculation was so accurate that he won praise[6].

His strong will and untiring observation and study brought him achievements of natural science.

3.3 Harriot's Mathematics

Since Harriot published nothing in his lifetime, his achievements are underestimated. In 1631, after ten years from his death, his colleagues published < Artis Analyticae Praxis> which contained his achievements. He in fact wanted Nathaniel Torporley to publish his works but Torporley's intimacy with Viéta made him to hesitate. Manuscripts which are kept in the library in Britain lack consistency and were out of order. Further, there are differences between manuscripts and <Artis Analyticae Praxis>, though the book was based on the manuscripts[11]. It seemed that some mathematicians revised arbitrarily, in the process of editing, when they concluded his mathematical results were wrong. Such being the case, estimations of Harriot in mathematics history varied. Some literature mentioned his unacceptance of negative and complex roots. A recent research tends to give priority to manuscripts when they compare manuscripts and the book. In 1883, 260 years after his death, Sylvester showed his respect to Harriot as 'a father of modern mathematics who introduced algebra to analytics' in a letter to Cayley[5, 11].

Algebraic Symbols and Roots of Equations

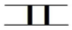
Even though most of literature mention Harriot as the originator of inequality signs, he used  and  instead of '$<$' and '$>$'. Moreover Robert Recorde's equal sign was used in his book and the equality sign spreaded among people.

Warner, the editor of <Artis Analyticae Praxis>, wrote down comments under the error Harriot had made. Harriot used algebraic symbols except an exponent in an excellent way. Viete used vowels for unknowns and consonants for knowns and Harriot adopted it in his solution of an equation. Letters and abbreviations were also used in expansion of an equation. For instance, a^4 was represented as $aaaa$.

Following is quoted from his manuscripts[11]. For the expansion of the multiplication $(b-a)(c-a)(df+aa)$, we could find out that the symbol '$\lrcorner$' was used which was similar to the symbol in these days.

$$\left.\begin{array}{l} b-a \\ c-a \\ df+aa \end{array}\right| \sqsupset\!\sqsubset$$

$$\begin{array}{l} bcdf - bdfa + dfaa - baaa \\ \qquad - cdfa + bcaa - caaa + aaaa \\ \qquad\qquad\qquad\qquad\qquad\qquad \sqsupset\!\sqsubset\ 0000 \end{array}$$

A symbol  was an equal sign to represent the expansion of an equation and four 0s at the last line showed a homogeneous expression. This meant that he dealt a homogeneous expression emphasizing calculability.

The solutions of the equation $(b-a)(c-a)(df+aa)=0$ are $a=b$, $a=c$ and $aa=-df$. Harriot drastically represented $a=\sqrt{-df}$ from the solution $aa=-df$.

$$a \amalg b$$
$$a \amalg c$$
$$aa \amalg -df$$
$$a \amalg \sqrt{\quad} -df$$

This was a challenging idea to existing mathematicians who disregarded negative roots.

There are differences between his manuscripts and <Artis Analyticae Praxis> which his colleagues published in the way of dealing roots of an equation. As we could find out from his manuscript in advance, Harriot dealt with negative and imaginary roots as same as positive ones. He received negative roots without any comments and he added an explanation of complex roots as 'noetic by rationality'[13]. He presented four roots of biquadratic equation as follows.

$5 \amalg a, \quad -7 \amalg a$

$a \amalg +1+\sqrt{-32}, \quad a \amalg +1-\sqrt{-32}$

He received positive, negative roots and two complex roots without reluctancy. In his manuscripts, we could find many imaginary roots from equations.

The following biquadratic equation is quoted from his manuscript[14].

$$aaaa - 6aa + 136a = 1155$$
$$- - - - - - - -$$
$$aaaa - 2aa + 1 = 4aa - 136a + 1156$$

This showed the process of solution of a biquadratic equation $a^4 - 6a^2 + 136a = 1155$. In the left side of the equation, he changed to the form of perfect square of the second degree and, as a result, he had a form of perfect square of the first degree. Then, he extracted the two square roots with $\pm$. Surprisingly, he adopted complex roots at the last line of two equations without reluctances.

$$aa - 1 = 3a - 34$$
$$33 = 2a - aa$$
$$aa - 2a = -33$$
$$aa - 2a + 1 = 1 - 33$$
$$a - 1 = \sqrt{} - 32$$
$$a = 1 + \sqrt{} - 32$$
$$a = 1 - \sqrt{} - 32$$
$$- - - - - - - - -$$

However, editors of <Artis Analyticae Praxis> were reluctant to receive negative roots of the equations. They excluded square roots saying 'unexplainable and impossible'. We could infer that editors' lack of understanding of Harriot's mathematics brought about the wrong revision of manuscripts. This led to further literature to mention Harriot's neglect of negative and complex roots[9].

Further Mathematical Achievements

Among Harriot's manuscripts, the mathematical contents except algebraic signs and roots of equations has been recently studied. A conic section, its problems and an observation of celestial sphere and related problems drew from research on Archimedes. Though there have been controversies whether Harriot fully understood the concept of infinitesimal, many literature showed that he solved Alhazen problem by using infinitesimal concept ahead of Barrow[11]. We could also find out the Pythagorean number and some calculations similar to differential and integral in his manuscripts[7, 11].

4 Comparison with Viéta and Descartes

At that time in Europe, Viéta(1540 1603) was more recognized by public. He, who was acknowledged as an originator of algebra using letters, was an algebraist and also had interests in geometry. Contrary to former times to substitute numbers, he was the first mathematician using letters and generalizing a quadratic equation. The former algebra was called as 'Logistica numerosa' and the latter was as 'Logistica speciosa'. History of algebra, in fact, has divided before-and-after algebra on the basis of the advent of Viéta.

Viéta represented a cubic equation as C, a quadratic equation as Q, and the unknown as N which were picked out from initial sounds of Latin. His use of letters and abbreviations was an epoch-making event in history. However, Harriot used more developed representation of a repetition of the same letters for expressing the degree[9]. Decartes(1596 1650), who was born thirty six years after Harriot, showed more progressive form of mathematics. Mathematics after the seventeenth century has developed by logic itself. Descartes, one of the prominent mathematicians at that time, showed his mathematical sense through his three paper <geometry> as appendixes of <Discours de la methode> dealing with philosophical problems. He was interested in mathematics because of its certainty definitude of an inference and he tried to apply rational considerations to studies of natural science. The invention of analytic geometry was the greatest achievement of Descartes. Analytic geometry, which related algebra and geometry, made our knowledge of space and spatial relations transfer to the language of numbers, and this allowed us to grasp the logic of geometric idea[3]. Though Descartes invented analytic geometry with Fermat, he was not ready to receive negative roots.

The representation of equations has used the abbreviation from Diophantus era and it has developed to easier way to deal with in the sixteenth and seventeenth centuries.

Viéta represented a cubic equation $x^3 - 8x^2 + 16x = 40$ as

$$1C - 8Q + 16N \text{ aequ. } 40.$$

and Harriot represented it as

$$aaa - 8aa + 16a = 40.$$

On the other hand, Descartes represented it as

$$x^{3^*} - 8x^{2^*} + 16x \propto 40$$

and it seemed like more modernistic one. Yet he represented the third and forth degree as x^{3^*}, x^{4^*} or x^3, x^4 the second degree was represented as xx for a long time[4, 9].

Viéta partly discovered the relations of roots and coefficients; he said 'solutions of a cubic equation

$$x^3 - (u + v + w)x^2 + (uv + vw + wv)x - uvw = 0$$

were u, v, w'. However, it was not complete that he only took positive roots in the actual extraction.

Harriot found out that 'if a, b, c were the solutions of a cubic equation, it could be represented as $(x - a)(x - b)(x - c) = 0$' and he showed the logic of the generalization of an equation of higher degree. The generalization of degrees of an equation was the progressive idea ahead of the times.

One equation could be represented in various ways. A sign could be changed when it transposes, and similar terms could be confused in order. However, the right side of an equation would be 0 when similar terms are put together and listed in descending order at the left side of an equation. This would be a standard type of an equation and a solution would be determined. It is no wonder in these days but in the sixteenth or seventeenth centuries people had various ways of solution. They had difficulties to receive negative numbers so that they had to transpose all negative numbers to the other side for a representation of positive numbers. Frend presented four types of quadratic equations in his book <The Principles of Algebra> (1796) as follows.

$$\begin{aligned}
&x^2 = b \\
&x^2 + ax = b \\
&x^2 - ax = b \\
&ax - x^2 = b
\end{aligned}$$

These four different types showed his difficulties to adopt negative numbers and a trial to remove it. Since Harriot dealt negative numbers the same as positive numbers, he had only one type of an equation. A standard type of an equation is the one and he only needed to factorize one. Thereafter, Descartes praised his achievement and called it as 'Harriot principle'.

5 Conclusion

Since Harriot had never published his achievements in lifetime, his works has gone unnoticed. He was one of the prominent mathematicians at that time in the aspect of discoveries in methodology and a mathematical sign and he led Britain mathematics to developed European one. His works were quoted by Stevin, Bombelli, Stiefel and Viéta at that time and later by Wallis and Descartes.

This paper revealed Harriot's mathematical achievements, especially on algebraic symbols and negative and complex roots in an equation, and restored his status by comparing it to Viéta's and Descartes'.

In the unacceptable atmosphere on negative roots in an equation, Harriot received even complex roots and said it was 'the perceptible roots only by a rational sense'. He also simplified the relations between roots and coefficients and generalized it. The representation of algebraic symbol was more modernistic than Viete's one which was famous at that time. A lot of literature mention that the first use of inequality signs of ‘>’ and ‘<’ was by Harriot but it has been controversial between Harriot and Viete.

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