A HISTORICAL TEACHING MODULE ON "THE UNREASONABLE EFFECTIVENESS OF MATHEMATICS"

The case of Boolean algebra and Shannon circuits*

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ABSTRACT

As a reaction to E. P. Wigner's paper on *The unreasonable effectiveness of mathematics in the natural sciences* from 1960, R. W. Hamming wrote a paper in 1980 called *The unreasonable of effectiveness of mathematics*, where he expanded Wigner's discussion by looking into the use of mathematics in computer science, being able to draw on his own 40 years of experiences in the area.

In the setting of the Danish upper secondary mathematics program this paper reports on the design and implementation of a so-called HAPh-module (History, Application, and Philosophy), where students were to read the original text by Hamming as well as two original texts by G. Boole and C. E. Shannon, respectively, illustrating the main point of Hamming's paper. More precisely the students worked with Boole's *An investigation of the laws of thought on which are founded the mathematical theories of logic and probabilities* (1854) and Shannon's *A symbolic analysis of relay and switching circuits* (1938). The implementation of the teaching module took place in a third year upper secondary mathematics class (students age 18–19) in the fall of 2011. Besides discussing the design of the module, a selection of data gathered during the implementation will be provided to illustrate outcomes (positive as well as negative) of the module.

1 Introduction

The motivation for carrying out the study described and discussed in this paper is threefold. First, in Denmark a reform, initiated in 2005, of the upper secondary school led to a more serious inclusion of elements of history in the mathematics program. Through "modules in the history of mathematics" students now must be able to "demonstrate knowledge about the development of mathematics and its interplay with the historical, scientific, and cultural development" (UVM, 2008, appendix 35, articles 2.3 and 2.1, my translation from Danish). Further, actual applications of mathematics play an important role in the new program, e.g. it says that students also must be able to "demonstrate knowledge about application of mathematics within selected areas, including knowledge about application in the treatment of a more complex problem" (ibid., article 2.1).

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Second, in 2002 a Danish report was published on *Competencies and Mathematical Learning* (recently it was translated into English: Niss & Højgaard, 2011), which besides listing eight mathematical competencies that students of mathematics are to develop and come to possess as part of their training, also lists three types of 2^{nd} order competencies or types of *overview and judgment* (OJ):

- OJ1: the actual application of mathematics in other subject and practice areas;
- OJ2: the historical evolvement of mathematics, both internally and from a social point of view; and
- OJ3: the nature of mathematics as a subject.

Where mathematical (1^{st} order) competencies are a kind of "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge", the three types of overview and judgment are "'active insights' into the nature and role of mathematics in the world" which "enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (Niss & Højgaard, 2011, pp. 49, 73).

Finally, the purpose of including history above, both in the Danish upper secondary mathematics program and as OJ2 in the report on competencies, has to do with a use of 'history as a goal', rather than one for instruction, i.e. 'history as a tool' (Jankvist, 2009a). Something similar is the case for including aspects of the actual application of mathematics, both in the upper secondary program and as OJ1. And also for OJ3, which we may associate with a use of philosophy in mathematics education, the purpose is rather one of 'goal' than of 'tool' (for further discussion, see Jankvist, forthcoming). Often when original sources play a role in mathematics education, it is mainly in the sense of a tool (e.g. Glaubitz, 2011; Barnett et al., 2011; Kjeldsen & Blomhøj, In press). But as we also know, e.g. from Jahnke (2000) and Fried (2001), a study of history through original sources introduces many aspects which are not only related to the actual learning of some mathematical concepts, theories, or methods. The study of original sources bears with it considerations of many aspects, e.g. in relation to the three types of overview and judgment mentioned above, which are not only a matter of understanding the mathematics treated; this being the third part of the motivation.

Thus, the overall motivation for the presented study is to do with if and how we may design upper secondary level teaching modules that through a use of original sources take into account all three types of overview and judgment simultaneously.[1] My way of trying to address this question in the present paper shall be mainly through students' own reactions and responses to such a teaching module. Since the three types of overview and judgment may be said to deal with the *H*istory (OJ2), the *A*pplications (OJ3), and the *Ph*ilosophy (OJ3) of mathematics, respectively, I shall refer to such teaching modules as *HAPh-modules*.

[1] See also Jankvist (2012).

2 HAPh-modules—design and implementation

What then is a HAPh-module? Following the concept of a *guided reading* of primary original sources, as developed by Pengelley, Lodder, Barnett, and others associated with the NMSU group,[2] where the reading of the original text(s) is 'interrupted' by explanatory comments, tasks, etc. (see in particular Barnett et. al, 2011), the idea is to have one original source representing the historical, the applicational, and the philosophical dimension, respectively. For the HAPh-module to be discussed here, the three original texts which the students studied – in Danish translation – were:

- o GEORGE BOOLE, 1854: *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. (Boole, 1854)
- o CLAUDE E. SHANNON, 1938: *A Symbolic Analysis of Relay and Switching Circuits*. (Shannon, 1938b)
- o RICHARD W. HAMMING, 1980: *The Unreasonable Effectiveness of Mathematics*. (Hamming, 1980)

I shall explain in detail the contents of these three texts, their interrelations, and touch upon their exemplarity in relation to the three types of overview and judgment in the next section, but for now I shall focus on implementation and design.

During the three years of Danish upper secondary school, a class of 27 students were given two HAPh-modules, the first one on Euler's solution to the Königsberg bridge problem, Dijkstra's algorithm for finding shortest path, and Hilbert's 1900-lecture on mathematical problems (see Jankvist, 2011b; 2011c; forthcoming), and the one discussed in this paper. The first module was implemented in first year of upper secondary school (student age 16-17 years), and the second in their third and final year. After each implementation, the students were given a questionnaire containing both questions on the content of the modules and on their opinion about it. Half of the class was also interviewed about their questionnaire answers as well as their hand-in written material. During each implementation, I followed a focus group of 5 students in their work with the original texts and the associated tasks and assignments.[3] The duration of each teaching module was approximately ten 90-minutes lessons.

For each lesson, the students were to prepare in advance by reading a selection of the teaching material including the original texts. When meeting in class they then split into seven prefixed groups of approximately 3-4 students – the same groups during the entire implementation. Here they worked on the tasks given in relation to the texts as part of the guided reading approach. These tasks could be of various kinds, asking, for example: how we may be certain that Boole assumes associative and multiplicative properties for his classes; what Hamming means, when he says that science describes only 'how' and not 'why'; or in relation to the text by Shannon to have the students search the Internet to find out what a 'relay' is, a 'switch', a 'circuit', and to find examples of pictures of such. In general, the teaching material (Jankvist, 2011d) was designed for 'self-study', meaning that the students worked mainly by themselves in their groups, but with the possibility of asking their teacher

[2]NMSU is New Mexico State University. The group's teaching materials based on original sources may be found at: http://www.math.nmsu.edu/hist_projects/ and http://www.cs.nmsu.edu/historical-projects/ (Retrieved on February 15, 2012). In particular the projects by Janet Heine Barnett (2011a; 2011b) have served as a source of inspiration for the HAPh-module discussed in this paper.

[3]The focus group(s) were video filmed during the two implementations.

for help if needed. This also meant that the teacher would not give lectures at the blackboard. But by circulating among the groups during lessons, the teacher would still have an idea of how the students progressed with the material.

After having read and studied the three original texts and done the related tasks, the students were to do a collection of *essay assignments*. In a previous study I have found that this is a good way of bringing students to work with aspects of history as a goal (Jankvist, 2011a). For this reason, the same approach was taken to bring in the two dimensions of applications and philosophy. The particular setting creates a scene, where students near the end of the implementation of the teaching module are to discuss in their groups selected meta-perspective issues – or *meta-issues* – regarding the case. These meta-issues are chosen beforehand and included in the description of the essay assignments. More precisely, as part of the essay assignments, the more historical text by Boole and the application oriented text by Shannon were to be related to the philosophical discussion in Hamming's text, a task which to some degree also demands understanding of the inner mathematical issues – or *in-issues* – dealt with in the original texts. Furthermore, such discussions may force out some of the interplay between the three dimensions of history, application, and philosophy, although still exemplified by the concrete case. Once having outlined the content and context of the three original sources in the following section, I shall display an example of an essay assignment as well as an example of a student group 'essay', i.e. their answer.[4]

3 The historical case(s)

As mentioned, the philosophical theme for this module was Hamming's (1980) comment to a paper by the physicist Eugene Wigner from 1960, in which he discusses the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960). Where Wigner's examples stem from the physical sciences, Hamming sets out to illustrate this unreasonable effectiveness of mathematics drawing on his own experiences from engineering – and aspects of what we today would consider to be computer science:

> During my thirty years of practicing mathematics in industry, I often worried about the predictions I made. From the mathematics that I did in my office I confidently (at least to others) predicted some future events – if you do so and so, you see such and such – and it usually turned out that I was right. How could the phenomena know what I had predicted (based on human-made mathematics) so that it could support my predictions? It is ridiculous to think that is the way things go. No, it is that mathematics provides, somehow, a reliable model for much of what happens in the universe. And since I am able to do only comparatively simple mathematics, how can it be that simple mathematics suffices to predict so much? (Hamming, 1980, p. 83)

As may be seen from the above quote, Hamming approaches his question from a rather constructivist point of view, which of course rules out some of the more Platonic explanations for the effectiveness of mathematics. This can also be seen from his statement that "Indeed it seems to me: The Postulates of Mathematics Were Not on the Stone Tablets that Moses Brought Down from Mt. Sinai" (Hamming, 1980, p. 86). Nevertheless, Hamming does point out that even though the standards of

[4]The two teaching materials may be found as texts 486 and 487 at http://milne.ruc.dk/ImfufaTekster/

rigor in mathematics may change over time, and with that definitions and proofs, the mathematical results often stay intact. After having discussed the effectiveness of mathematics and what mathematics is, Hamming (1980, pp. 88-89) goes on to provide some partial explanations for the unreasonable effectiveness of mathematics arranged under four headings, among these that "*We see what we look for*", meaning that we approach situations with an intellectual apparatus so that in many cases we can only find what we do – Hamming provides a parable by the physicist Arthur Eddington, saying "Some men went fishing in the sea with a net, and upon examining what they caught they concluded that there was a minimum size to the fish in the sea" – and that "*We select the kind of mathematics to use*", meaning that we select the mathematics to fit the situation, and that the same mathematics does not work in every place.

Boole's *The Laws of Thought*$\cdots$ from 1854 is an example of the latter, since he selects the elements from standard (arithmetic) algebra that applies to his logic system, the purpose of which he describes as follows:

> The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind. (Boole, 1854, p. 1)

In chapters II and III of his treatise, Boole considers the role of language in relation to the above and introduces a number of signs and laws to do so. More precisely, he introduces literal symbols x, y, etc. representing classes, and signs of operation $+$, $-$, $\times$ (times) and the sign of identity $=$ to be used on these classes. For example, if x stands for 'white things' and y for 'sheep', then the class xy stands for 'white sheep', similarly if z stands for 'horned things', then zyx stands for 'horned white things'. After associating the sign $+$ with the words 'and' and 'or', Boole deduces a number of laws which have their equivalent counterparts in standard arithmetic, e.g. the commutative law $x+y=y+x$, the distributive law $z(x+y)=zx+zy$, and the associative law (although this is not done as explicitly as for the others), and he deduces laws for the operation $\times$ (times) as well. The more interesting thing, however, is Boole's observation that in the context of his investigation we have that $xx=x$ (or $x^2=x$). If for example x stands for 'good', then saying 'good, good men' is the same as saying 'good men'. Boole then draws the consequence of comparing this to standard algebra:

> Now, of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2=0$, and that $1^2=1$; and the equation $x^2=x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values 0 and 1. Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. (Boole, 1854, pp. 26-27)

Boole's ideas went on to be adapted within mathematical logic and set theory, and the notion Boolean algebra was conceived. Some eighty years later, however, the ideas showed valuable in a very different setting than that of language and thought, namely design of electric circuits.

Shannon was a student at MIT when he got the idea for describing electric circuits by use of logic. With a set of postulates from Boolean algebra ($0 \cdot 0 = 0$; $1 + 1 = 1$; $1 + 0 = 0 + 1 = 1$; $0 \cdot 1 = 1 \cdot 0 = 0$; $0 + 0 = 0$; and $1 \cdot 1 = 1$) and their interpretations in terms of circuits (e.g. $0 \cdot 0 = 0$ meaning that a closed circuit in parallel with a closed circuit is a closed circuit; $1+1 = 1$ meaning that an open circuit in series with an open circuit is an open circuit), he was able to deduce a number of theorems which could be used to simplify electric circuits (see below). In an interview from 1987 in the magazine Omni, Shannon explained his use of Boolean algebra:

> It's not so much that a thing is 'open' or 'closed,' the 'yes' or 'no' that you mentioned. The real point is that two things in series are described by the word 'and' in logic, so you would say this 'and' this, while two things in parallel are described by the word 'or.' The word 'not' connects with the back contact of a relay rather than the front contact. There are contacts which close when you operate the relay, and there are other contacts which open, so the word 'not' is related to that aspect of relays. All of these things together form a more complex connection between Boolean algebra, if you like, or symbolic logic, and relay circuits.
>
> The people who had worked with relay circuits were, of course, aware of how to make these things. But they didn't have the mathematical apparatus of the Boolean algebra to work with them, and to do them efficiently. [···] They all knew the simple fact that if you had two contacts in series both had to be closed to make a connection through. Or if they are in parallel, if either one is closed the connection is made. They knew it in that sense, but they didn't write down equations with plus and times, where plus is like a parallel connection and times is like a series connection. (Shannon, 1987 in Sloane & Wyner; 1993, p. xxxvi)

For a given electric circuit $a - b$, Shannon defined the hindrance function X_{ab} to be 1 if $a - b$ is open and 0 if closed. For example, figure 2 (left) has the hindrance function $X_{ab} = W + W'(X + Y) + (X + Z) \cdot (S + W' + Z) \cdot (Z' + Y + S'V)$, where $+$ indicates series, $\cdot$ parallel and W' is the negation of W. Now, by means of manipulations according to his theorems of the expression for X_{ab}, Shannon is able to reduce this to $X_{ab} = W + X + Y + ZS'V$, the circuit of which is illustrated on figure 1 (right). (For exact reductions and theorems used, see Shannon, 1938b, p. 715 or Jankvist, 2011d, pp. 62-63.)

4 An example of an essay assignment and a student group essay

Having been introduced to the mathematical theorems and their proofs behind the reductions of the hindrance function above as well as the mathematics of Boole, the students were given the following essay assignment in order to relate the three original texts, and thus the three dimensions of history, application, and philosophy, to each other:

a. According to Hamming, what does it mean that a piece of mathematics is effective?

b. Do a comparison of the relative effectiveness of Boole's and Shannon's works (systems) distinguishing between effectiveness in terms of philosophy and effectiveness in terms of applications.

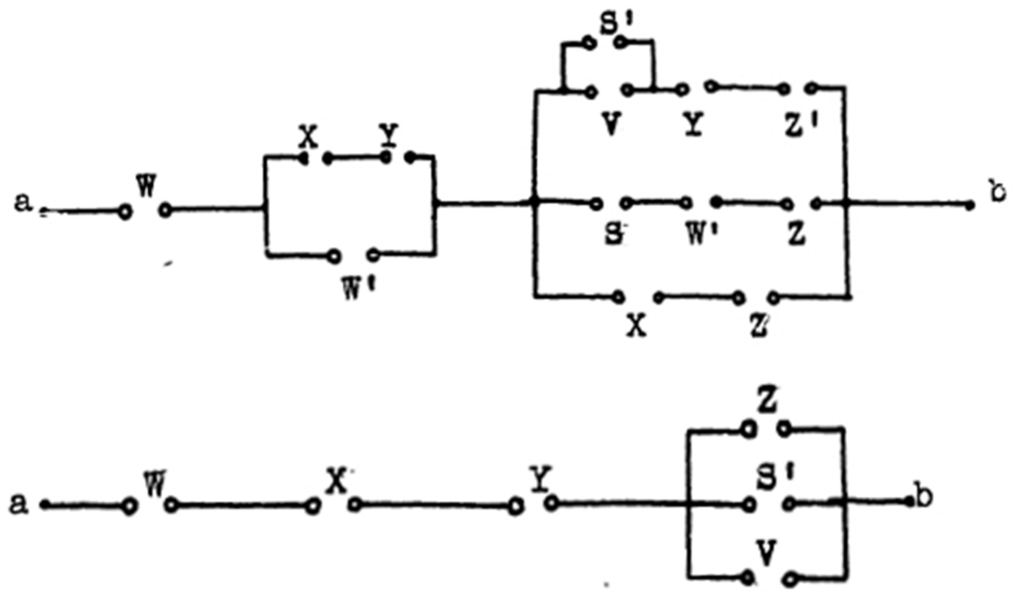

Figure 1 Left: The circuit to be simplified. Right: The simplified circuit after reductions on the hindrance function. (Shannon, 1938a).

c. Based on your answers to the above questions, discuss different types of 'the effectiveness of mathematics'. Recapitulate what Hamming means by the title of his paper *The Unreasonable Effectiveness of Mathematics*, and why it may be seen as 'unreasonable'.

d. Do you consider Boole's introduction and Shannon's application of the idea of an algebra only operating on the elements 0 and 1 along with the mathematical interpretation of 'and' and 'or' as an example of Hamming's 'the unreasonable effectiveness of mathematics'?

As an illustrative example of the students' work with this essay-assignment, I provide the following translated excerpt from one out of the seven groups:

> According to Hamming a piece of mathematics is effective when it can describe and predict natural phenomena. He finds mathematics puzzling in the sense that it can describe Nature using relatively simple formulas and expressions, practically without doing any experiments. [⋯]
>
> The philosophical effectiveness we see with Boole is connected to thoughts and the philosophy behind mathematics. It can say quite a bit about how we understand and confine mathematics with axioms [laws], for example when Boole makes it an axiom that x must be either 1 or 0. On the basis of philosophy he concludes something about mathematics' ways of thinking and methods. Shannon, however, on the basis of Boole's philosophical effectiveness, uses a method leading to his own effectiveness regarding application when transferring the [mathematical] theories to real life, where he at the same time tests them and thereby obtains an evaluation of the effectiveness. This is seen from the system [of electric circuits] on which his theorems are used.
>
> What Hamming believes is that mathematics is unreasonable because you are able to describe real events by simple mathematics and that we as human beings are finding it difficult to comprehend that apparently there are no limits to the range of mathematics regarding use in everyday life. By 'unreasonable' he means that it seems illogical that nature can be described by such simple

operations, since nature for us seems complex, incomprehensible, and unpredictable – i.e. that we 'just' cannot plot them on a graph and get a result.

> Yes, we do consider Boole's introduction and Shannon's application as being examples of the unreasonable effectiveness of mathematics, since Boole shows that mathematics can [help] explain composition of language and that you can translate language directly into mathematics. In addition to this, Shannon shows, by means of Boole's introduction, that [the idea of using] the elements 0 and 1 can be applied on a circuit and hereby find the most simple [circuit]. These two examples fall outside what is usually considered the main field of mathematics and should, following Hamming, be described by more complex systems – but since this is not the case, it is natural, according to Hamming's line of thought, to call this mathematics unreasonable. [⋯] (Group 2, excerpt from hand-in)[5]

I'll return to the above excerpt in the discussion section of the paper, but first let us turn to the students' own reactions to module.

5 Students' reactions

It is difficult to draw a completely conclusive picture of which of the three dimensions the students' preferred and found to be fulfilled best in the module. When asked about this in the post-interviews, some would say the historical and some the philosophical, even though the majority of the students did claim that for them personally it was important to see an actual application of the mathematics they had to learn, i.e. the application of Boolean algebra to electric circuit design. Of course, the students' answers to this question are to some degree dependent on which of the three texts they personally preferred. When asked about this, as part of the essay assignments, many would lean towards Shannon's text because it was closets in presentation to what they were used to, for example one group wrote:

> In Shannon's text there was more mathematics than text, which made it easier to picture, more palpable. There were examples of what he said, which helped us to understand his conclusions, his intermediate results, and purpose. Of the three texts this was the most accessible, because it was more visual than the other two. Our favourite! (Group 6)

Not surprisingly, when studying original texts, language becomes a major factor, one to which students immediately refer:

> We find Hamming's text more relevant for teaching in the way that he manages to explain the limits of mathematics without drawing a conclusion, thus leading us to think further for ourselves. For that reason we perceive Hamming's text as more open and accessible. Besides, it is easier to read in terms of language, which makes it possible for us to focus on the mathematics, while working with it. [⋯] (Group 2)

This group, Group 2, on the other hand found Boole's text to be close to inaccessible for the same reason, i.e. language, which is completely in opposition to the evaluation of Group 4:

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[5] All excerpts from student group's hand-ins or from student interviews have been translated from Danish.

> Boole's text from 1854 is the one you remember the best. He makes it simple, and then be builds on top and on top, so that we continuously and gradually become wiser. He understands how to create images in our heads that are easy to remember and with which you can easily identify. (Group 4)

Thus, it is important to remember that the experience a student has with reading different original sources is individual, and that this experience may have an effect on their preference of one of the three dimensions over the others. Actually, this may not be so surprising, because original sources often are discussed in their role of 'interlocutors' (e.g. Jahnke, 2000; Kjeldsen & Blomhøj, In press). So in the same way that we as individuals may communicate better with some persons than others, we may simply 'communicate' better with one original source than another. Perhaps because we can relate better to its author, its objective, or the language it is written in. Nevertheless, some students were able to look past the language barriers. For example, although Group 3 also found Boole's text difficult, they stated:

> Relatively strange, but good because it provides us with a new way of thinking. (Group 3)

The idea of the reading of original sources providing the students with something else than just a knowledge of the mathematical in-issues of the text, is, however, something that several students bring up themselves in the interviews. One of the focus group students (Group 7), Sophia, said:

> Well, it's been dry getting through it, it has, but it's also been very⋯Well, it has provided insight, I think, on how mathematics has been used before and how it has come into being, quite precisely. That was really cool, I think. Even though it wasn't mega exciting and even though it was enormously difficult to interpret, it also gave something in a sense. You got a lot of information about how mathematics was applied or about how some clever fellow formulated it back then and so. That was exciting, I think. Okay, maybe not necessarily exciting, but I think it was really cool to see that, how it worked. (Sophia, post-interview, November 3^{rd}, 2011)

Another focus group student, Nikita, provides a much more elaborated account:

> Well, first of all, I think that language wise it was very, very different from what we normally read. That is, what we normally read is much milder, academically speaking. It is described in basic words, or how you say it, it is almost 'baby talk', so you really can follow. Whereas this, not only did you have to understand what it was about, you also had like the language of it, and it has been a different way of thinking compared to the mathematics we are usually taught, where we have this formula and it works like this, this, and this. Here you got all the background knowledge, and how he arrived at it, etc. For me, I personally think that I get much more interested, when I see it all, than if I'm only told that now we are studying vectors and we must learn how to dot these vectors and then we must be able to calculate a length, right. That's all very good, but what am I to use it for? Whereas, when you know about the background, the development up till today, that I think was exciting. Because when we began with the first text [Boole's text], it was kind of like, yeah, that's alright, he can figure out this thing here, and this equals that, I can follow that, and 'white sheep' and so⋯That was good for starters. Then more is built on top, and all of a sudden we see: Why, it's a [electric] circuit we are doing! You could begin to relate it to

your own reality; that is, something you knew already. So, the thing about starting from scratch, which I kind of felt I did, and suddenly seeing it form a whole, what it was used for today, and be able to relate it to something. Something you knew about. That, I think, was way cooler. (Nikita, post-interview, November 3^{rd}, 2011)

6 Discussion and conclusions

As evident from the above quotes, language is indeed a factor when using original sources in mathematics education. But despite the difficulty of reading old style language (even in translation), the students seem to find that the reading of the original sources in itself provides them with something, this 'something' ranging from: insights into a, for them, new way of thinking (Group 3); knowledge about the application of mathematics and original formulation of mathematical ideas (Sophia); historical background knowledge, the origin of mathematical ideas, and actual modern applications (Nikita). Results from previous studies also indicate that students may find a use of history more relevant if either there is an applicational side to it, or if the history is not too remote in time from themselves – because they feel that they can relate to this better than to something from "···before Christ was born···" (Jankvist, 2009b). And the student quotes above do seem to confirm this.

Now, regarding the module's philosophical dimension, to which the title of this paper refers, this was probably the one of the three dimensions which caused the students the most trouble. When asked about this particular dimension of the module, two students, Katharine from Group 4 and Sophia from the focus group, replied:

> Hamming was a little like being on the moon for me. Well, I understood it, but I had to read it twice before I could do the connection. [···] of course he [Hamming] taught me something, but for me it was on this high strange level, because I'm like; I just want the math, and then calculate and stuff, right. So, it was a bit high floating, I preferred the other two better [Boole and Shannon]. But it did provide an incredibly good connection between the parts, that there were the three dimensions. (Katharine, post-interview, November 3^{rd}, 2011)

> Well, I found it difficult. I found it *very* difficult. It was difficult to think philosophically like that. I don't think that I've been asked to think in that way before. [Usually] it is more like; that's the way it is, now try and work on it. I found it challenging. (Sophia, post-interview, November 3^{rd}, 2011)

When discussion falls on the use of original sources, it is often argued that although this may be one of the most ambitious, demanding, and time consuming ways of teaching mathematics/introducing history, it is also one of the most rewarding (e.g. Jahnke, 2000; Glaubitz, 2011). Something similar may be the case for the introduction of a philosophical dimension into the teaching of mathematics, because even though most students agreed to this being very challenging indeed, the essay assignments along with their questionnaire answers and post-interview utterances bear witness to this dimension having actually 'moved' something—not least in relation to the students' knowledge of meta-issues of mathematics, that is 'overview and judgment'. Focus group student Jean was even able to articulate some of his newly gained insights as well as pinpointing these to the presence of the philosophical dimension:

> I also think that the philosophy part was exciting, but rather abstract⋯It may sound a bit strange, but I do find it kind of cool to sit and think about mathematics [the subject], that it's not only a tool. Or yes, it is a tool, but you can kind of view it from different perspectives and, yes, view it more philosophically. (Jean, post-interview, November 3^{rd}, 2011)

When asked if this module (as well as the first one) had any impact on the way in which he perceived mathematics as a discipline, Jean replied:

> Yeah⋯definitely the view of mathematics has been altered because you've had the philosophical dimension as part of it [the modules], a fairly big part, I think. So, you gained a different insight into this than you had before. You kind of feel that you've reached a higher level⋯Yes, because you are able to see mathematics in a different way. And that has sort of surprised me; that you can view it in this way⋯that mathematics also has a philosophical side to it. That it is not only, as I said before, numbers. (Jean, post-interview, November 3^{rd}, 2011)

As promised, let us return to the excerpt from Group 2's essay assignment. Now, this essay may be considered a fairly deep answer for students at this particular level. In a few paragraphs, and particularly the last one, they are able to coin the essence of interplay between the three dimensions in the HAPh-module by providing their own sound argumentation for Boolean algebra and Shannon's use of it in electric circuit design being an example of Hamming's unreasonable effectiveness of mathematics. Admittedly, this essay answer is one of the better from the seven groups, but even so, it provides an *existence proof* of it being possible to have students reach the intended level of meta-issue abstraction, i.e. relating the three original texts to each other, and thus also the three dimensions of history, application, and philosophy.

That it is possible to introduce a historical dimension into mathematics teaching is well known from four decades of HPM research.[6] That it is also possible to introduce an applicational dimension into mathematics teaching is equally well known and documented. And that it is possible to introduce a historical dimension and an applicational dimension at the same time, has also been shown (e.g. Jankvist, 2009b; 2010; 2011a). But as of yet, only very few studies address the introduction of a philosophical dimension in mathematics teaching (see Jankvist, forthcoming, for a list), meaning that it is not a priori given how to do so. The above description of a concrete HAPh-module provides an example of this, one in connection with the dimensions of history and application where each of these three dimensions is introduced through a guided reading of an original source and the interplay of the dimensions (and sources) is dealt with in essay assignments. Judging from the student groups' essays and the student interviews, some of which were displayed above, a development of the three types of 'overview and judgment' does appear to be present. All in all, this points in direction of the laid out scheme of design indeed being 'marketable'.

REFERENCES

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[6]HPM is International Study Group on the relations between History and Pedagogy of Mathematics.

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