USING HISTORY OF MATHEMATICS IN HIGH SCHOOL CLASSROOM:
SOME EXPERIMENTS IN TAIWAN

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1 Introduction

When did Taiwan start to develop the HPM? The date is hard to be definite. However, 1996 must be a decisive year. In this year, Prof. Horng Wann-Sheng accepted the task to host the conference HPM 2000 Taipei and initiated *HPM Taipei Tongxun* in order to promote HPM in Taiwan. There are two vital impacts on mathematics education of Taiwan. First, *HPM Taipei Tongxun*, now renamed as *HPM Tongxun*, has become the most important platform for high school mathematics teachers to acquire as well as to share knowledge, information, teaching skills, teaching experience, and materials of the HPM.[1]

Secondly, the HPM 2000 Taipei attracted many mathematics educators of every level in Taiwan, so that they became more familiar with the HPM and started to be involved in the HPM. For instance, there are one doctoral and thirty-two master's dissertations on the HPM in high school in the period between 2001 and 2011. In contrast, there are only two master these before 2001. Furthermore, authors of these dissertations are all in-service or pre-service teachers of mathematics, and they did their research and experiments in actual classrooms. Besides, from 2000 onwards, there are many articles about HPM, especially concerning historical materials and using history of mathematics in classroom. Now HPM comes to be a meaningful and legitimate subject in Taiwan.

In what follows, I will introduce some experiments and examples of using history of mathematics in high school, and share my own experience as well. Before that, I have to clarify that by "experiment" I mean in this article does not necessarily refer to a formal or an academic one, it may be an example or experience of a teacher using history of mathematics in his or her class.

2 Experiments and examples of using history of mathematics

2.1 Research projects conducted by Prof. Wann-Sheng Horng

In Taiwan, Prof. Wann-Sheng Horng is the first scholar to investigate how history of mathematics integrated with education of mathematics in all possible aspects. He led his team completing several

[1] In Taiwan, high school includes junior high school and senior school. The former is for students aged from 13 to 15, and the latter from 16 to 18.

research projects, like "Ancient Mathematical Texts used in the Classroom", "Teacher's Professional Development in Terms of the HPM" and so on. Since most members of Prof. Horng's team are teachers of high school mathematics, these research projects combined practical experience and aimed at realistic applications.

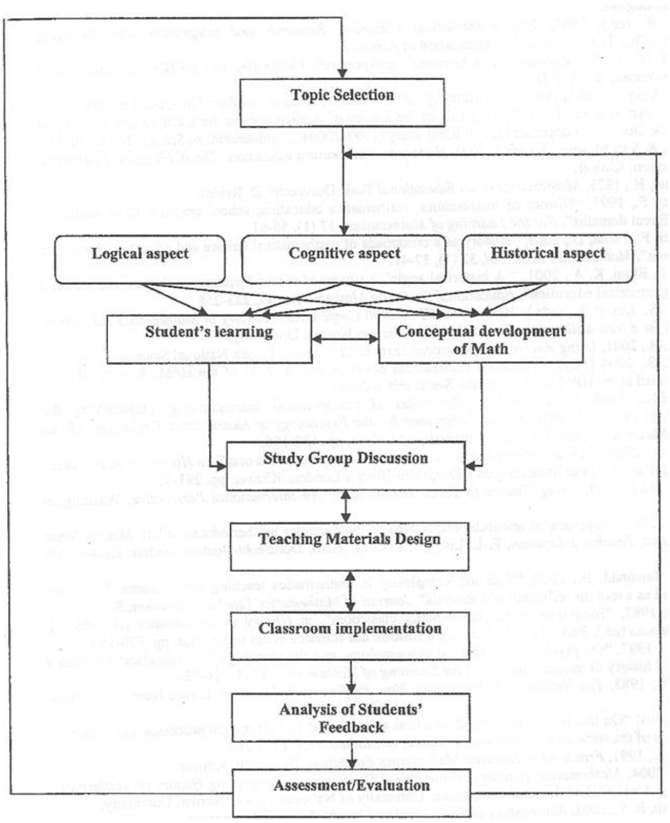
On the project "Ancient Mathematical Texts used in the Classroom", they have developed as many as twenty-nine teaching projects and worksheets/work-cards in terms of the HPM. "However, the participants were not aware that in this connection a subtle reconciliation of historical reflection with cognitive approach was necessary." (Horng, 2004) Jing-Ru Chiu and I took part in this project, and we designed three teaching projects and brought them into use in her classes. What we learned in the end of the project is that it is quite difficult to make a transition from history of mathematics to what students have learned or are going to learn. There are many interesting topics in history of mathematics, but what teachers think interesting is not necessarily suitable to students. For example, we developed a teaching project of Egyptian fractional numbers for 7th grade students. In the first class, students were all highly attracted by how ancient Egyptians wrote integers and used them to do addition, subtraction, multiplication, and division. Nevertheless, in the second class, students got confused with the reason why they needed to learn so complicated and useless Egyptian fractions. With these teaching experiences, we came to realize that Egyptian fractional numbers were fascinating to us, but they did not make much sense to some students. We appreciated the significance and the unique style of Egyptian fractions and their representations and arithmetical operations, but students did not. Moreover, did 7th graders need to know what Egyptian fractions were all about?

Through this research project we have learned that history of mathematics do bring to teachers as well as to students some benefit such as motivating learning of mathematics, appreciating cultural aspects of mathematics, seeing mathematics as human-being activities, and so on. However, history of mathematics is a double-edged sword, and it will do harm to math classes if teachers use it without second thought. How to use history of mathematics in teaching is a significant theme to which many papers have contributed. I will not go into the details. Instead, I am going to tell a story that has a significant impact on me for a long while.

When Jing-Ru Chiu and I participated in the research project, she was a junior high school teacher with practical experience of teaching mathematics. By contrast, I was a graduate student with lots passion in HPM but with little practical experience. When we cooperated to design teaching projects, we usually stood at the two ends of a balance. I wanted to put more and more history of mathematics into the teaching project while she concerned primarily not only about performance of teaching but about outcome of students' learning as well. Frankly speaking, at that moment I thought she was not capable enough to use history of mathematics in teaching. Nevertheless, in the end of this two-year research project, I had learned a lot from her about how to provide students proper and accessible materials related with history of mathematics.

2.2 Dr. Su Yi-Wen's doctoral dissertation

Dr. Su Yi-Wen's doctoral dissertation, *Mathematics Teachers' Professional Development: Integrating History of Mathematics into Teaching*, is one of the major outcomes of Prof. Horng's research project, "Teacher's Professional Development in Terms of the HPM". It is about her school-based research during a two-year period. There were four participants including Su herself in the research, and they were all mathematics teachers in the same senior high school. They developed a HPM model for designing teaching materials (see the diagram below, cited from Su, Yi-Wen, 2006) and finally completed teaching worksheets for eight topics: complex numbers, Heron's formula, circles, the mathematical expectation, metrics, transformations of translation and rotation, the concept of limit, and applications of limit. The overall process of completing teaching worksheets can divide into four parts.



First, after selecting a topic of senior high school mathematics, each participant needed to read relevant papers or books on history of mathematics and mathematics education. He or she not only analyzed the logical, cognitive, and historical aspects of the selected topic, but also took students' learning and conceptual development of mathematics into consideration. Then he or she discussed those with other participants and started to design teaching materials. The second part was implementing designed teaching worksheets/materials into class. Next, the designer shared his/her reflections on performance and responses from students with other participants, and looked for advices from others. Finally, the designer wrote a final report on his/her worksheets, and proposed suggestions for future users. Hence, teachers who are interested in these worksheets can easily get into practice through these reports.

After making a comprehensive survey of these reports, I discover several interesting things from designers' reflections and suggestions:

(1) What students are going to learn is mathematics, not history of mathematics

In each report, I see many positive responses from students. However, not every student approved of history of mathematics. Some thought learning from history of mathematics was inefficient, and some thought history of mathematics made mathematics even more difficult to learn. Therefore, the designers remind readers that the key to integrating history of mathematics into teaching is selecting proper historical materials and adapting them into accessible materials for students.

(2) History of mathematics inspires students not only in mathematics, but also in personality.

Prof. Wann-Sheng Horng has pointed out that telling historical stories can inspire students in personality. After finishing the research project, "Meta-Development of Teachers' Beliefs and Knowledge on History of Mathematics", Feng-Jui Hsieh concluded that one of the significances of integrating history of mathematics into instructions is to develop positive values in life. (Su, Yi-Wen, 2004) In the report on metrics, the author/designer told a story that he successfully helped a depressed student getting through his hard time by means of the story of Cayley and Sylvester. After listening to the story, the student actively fitted into his class and organized a study group. In a period, all members of the group progressed in studies.

(3) Responses from students helped the designers' professional development.

In all of the reports, authors/designers expressed their pleasure and satisfaction with most students approving of learning mathematics through the worksheets. Those feedbacks encouraged them to design worksheets for the other topics. This virtuous circle enhanced their professional expertise in terms of the HPM in an efficient way. Yi-Wen Su puts it that "by the end of the two-year project, it is obviously that the participants, in particular, T_1, have enhanced their professional expertise in terms of the HPM in following ways, namely, (i) they can begin to write popular mathematics articles; (ii) they are more reflective into their teaching than ever; (iii) they are able to integrate their mathematics knowledge into a broad picture; and (iv) he starts to care about the students' thinking. As a conclusion, the outcome of the project indicates that HPM approach can help the participants' professional development in an efficient way and can be another way for the in-service training." (Su, 2006)

2.3 Jun-Hong Su's Award Winning Teaching Projects

Jun-Hong Su is an experienced teacher of high school mathematics, and he wrote many articles about the HPM in varied publications for high school teachers in Taiwan. In addition, he designed several teaching projects in terms of the HPM, and won the first prize of the contest of teaching projects of high school science in 3 consecutive years, 2006, 2007, and 2008. The SpringSoft Education Foundation held this contest from 2005 to 2008. It asked participants to use the PowerPoint software to design and to present their projects. Su combined his experience of teaching mathematics and knowledge of mathematics history to win the prizes. The topics of these three teaching projects are the cosine formula, irrational numbers and conic sections.

The teaching project of the cosine formula has a distinctive feature that it deeply connects to the Pythagorean Theorem. It not only shows the connection between the cosine formula and the Pythagorean Theorem, but also illustrates that we can modify Euclid's proof of the Pythagorean Theorem in *Elements* to prove the cosine formula. See Figure 1. It is analogous to the diagram with which Euclid gives his proof except that the triangle ABC is not right-angled. We have rectangle AJ equal to rectangle AK, and rectangle BL equal to rectangle BK. Therefore, the sum of square AD and BH exceeds square AF by the sum of rectangle CJ and CL. Actually, rectangle CJ and CL are both equal to $\overline{AB} \cdot \overline{BC} \cdot \angle ABC$, and then we have the cosine formula:

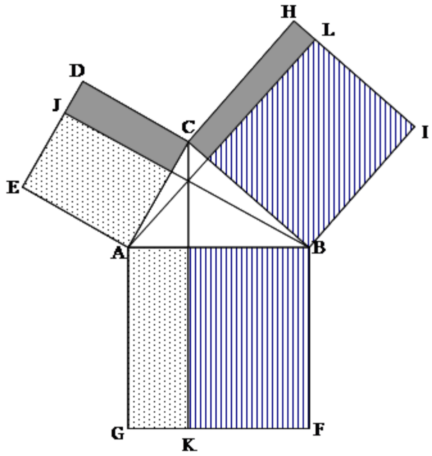


Figure 1

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2 \cdot \overline{AB} \cdot \overline{BC} \cdot \angle ACB$$

Su expects that through this proof, students can regard the cosine formula as an inherent logical entailment of the Pythagorean Theorem.

Su's second teaching project aims to make students truly "perceive" irrational numbers. Students are told that irrational numbers are numbers that are not rational numbers. However, this kind of definition or explanation gives students almost nothing as to what is irrational number. In order to improve the situation, Su introduces the concepts of "commensurable" and "incommensurable" of Euclid's *Elements*. First, he shows the connection between rational numbers and commensurable magnitudes, and uses the Euclid algorithm to find the greatest common measure of two commensurable magnitudes. Second, he demonstrates the diagonal and the side of a square are incommensurable to explain the square root of 2 is irrational. (See Figure 2) Finally, he concludes that irrational numbers are those numbers that cannot be written to be fractional, ratios of two integers.

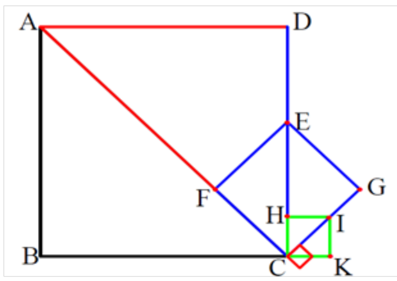


Figure 2

The third teaching project is about conic sections. The first part of it is Dandelin's theorem, and the most interesting thing is that Su uses the software, Cabri 3D, to display Dandelin spheres dynamically. The second part focuses on the meaning of the *latus rectum*, which Apollonius (ca. 262 BC 190 BC) called the upright side. In Taiwan, we have definitions of *latus rectum* for conic sections and formulas for calculating their lengths in senior high school mathematics textbook. However, that is all, nothing more. Therefore, students only memorize them without knowing how actually they meant. Hui-Yu Su, the editor of *HPM* Tongxun, wrote an excellent article to expound the original connotation of the latus rectum related to naming conic sections as *parabole*, *elleipsis*, and *hyperbole* in Apollonius' *Conics*.

Moreover, in the light of their original meaning, parabola, ellipse, and hyperbola may all have the same form of analytic expression, $y^2 = px \pm \frac{p}{d}x^2$. (See Hui-Yu Su, 2005) This article inspired Jun-Hong Su to integrate it into the second part of this teaching project (see Figure 3 below).

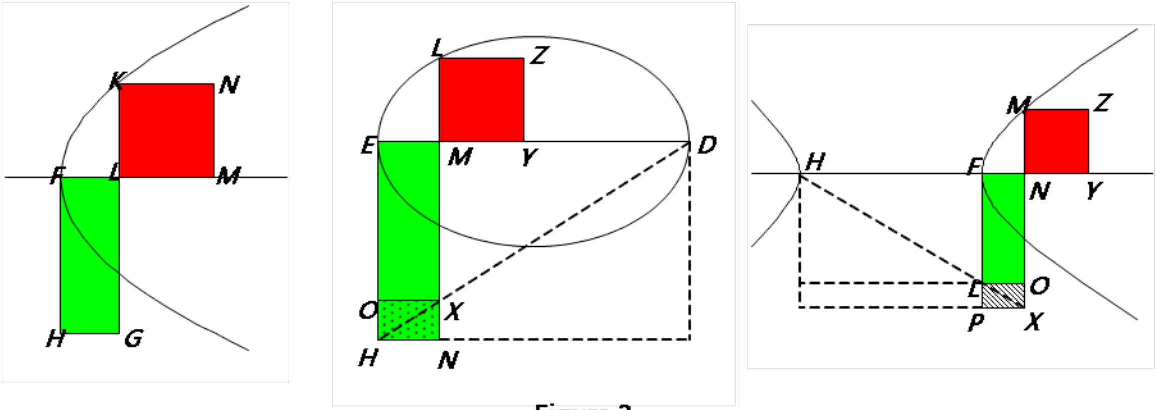


Figure 3

2.4 My Own Teaching Projects

Since 2007, I have designed several teaching projects, and two of them, tables of logarithm, and Cramer's rule, are illustrated in the following paragraphs. I used them in my classes of National Tainan First Senior High School.[2]

2.4.1 Tables of Logarithm

In Taiwan, every student has to learn logarithm in his or her first year of senior high school, and to memorize the number 0.301 as an approximation of log2. One day a question crossed my mind that if a student asked me about how to find out the approximation of log2, what my answer would be. The only method I remembered then was to approximate it by using Taylor's Formula, but Taylor's Formula is beyond what my students has learned. In addition, using Taylor's Formula to explain the "genetic" development of logarithms tables is anachronistic. Therefore, I looked into the history of logarithm and found something worth showing to students. Consequently, I designed a series of worksheets of logarithm tables and used them in my classes. Responses from my students were quite positive, so I wrote two articles about this topic, my worksheets and my students' feedbacks. Some teachers told me these articles not only inspired them but also intrigued them to take my worksheets into use. Those flatter me a lot indeed.

When I developed my worksheets of logarithm tables, I took some ideas and materials from two articles concerning using history of mathematics to teach logarithm in *HPM Tongxun*. They are "Shu Xue Shi Rong Ru Giao Xue: Yi Dui Shu Wei Li" (integrating history of mathematics into teaching: take logarithm for example) and "Dui Shu Sui Bi" (some things about logarithm) written by Jun-Hong Su and Zhi-Yang Horng respectively, who are both senior high school mathematics teachers. In his article, Su offers four worksheets of logarithm. His first worksheet is devoted to Nicholas Chuquet (1455-1488) while the others Napier's logarithm. My first worksheet essentially bases on his first

[2]Founded in 1922, National Tainan First Senior School is one of the most academically competitive senior high schools in Taiwan. It has 57 classes with around 2,300 students, and only around 10 students are girls. Roughly speaking, nearly each student's percentile ranke of academic performance in the Basic Competence Test for Junior High School Students is above 94%.

one. In Horng's article, he applies logarithm to find the relation between orbital periods and radii of planets, and in this way, the third Kepler's law of planetary becomes easier to follow. I adapted this application for my last worksheet.

1	0		
2	1		
4	2	1	
16	4	2	Tetras prima
256	8	3	
1024	10	4	
10,48576	20	7	
109,9511627776	40	13	Tetras secunda
12089,25819,61463	80	25	
12676,50600,22823	100	31	
16069,38044,25899	200	61	
25822,49878,08685	400	121	Tetras tertias
66680,14432,87940	800	241	
10715,08607,18618	1000	302	
11481,30695,27407	2000	603	
13182,04093,43051	4000	1205	Tetras quarta
17376,62031,93695	8000	2409	
19950,63116,87912	1000	3011	→ 10000
	Indices	Numerus notarum	

Except for the first and the last worksheet, the remaining included two parts. First, Henry Briggs (1561-1630) suggested John Napier (1560-1617) to change the logarithm into what we use today, and utilized his brilliant method to calculate approximations for the logarithm tables. The cited right tabulation is from Ian Bruce's translation of Briggs' *Arithmetica Logarithmica*, and there is a mark I made to show an error in the tabulation. This tabulation illustrates how Briggs approximated log2. The second column represents the degree of 2^{100}. Take 100 for example, the number 12676,50600,22823 in the first column is the first fifteen digits of , and the number 31 in the third column indicates there are 31 digits of 2^{100} . Then Briggs multiplied 12676,50600,22823 by itself to get the first fifteen digits and the total number of digits of 2^{200} . Briggs did not stop doing multiplication until he got the number 30,1029,9956,
6399, which is the total number of digits of $2^{10^{14}}$. Then Briggs obtained an approximation of log2 with high accuracy.

In what follows let me explain the procedure in today's notation:
$2^{10^{14}} = N \times 10^{30,1029,9956,6399-1}$,
$1 < N < 10 \Rightarrow 10^{14} \cdot \log 2 = (30, 1029, 9956, 6399 - 1) + \log N \Rightarrow \log 2 \doteq 0.30102999566398$

In my classes, I took $2^{10} = 1024$ to show my students how to find an approximation of log2:
$2^{10} = 1024 = 1.024 \times 10^3 \Rightarrow 10 \cdot \log 2 = \log 1.024 + 3 \Rightarrow \log 2 = \frac{1}{10} \cdot \log 1.024 + 0.3 \doteq 0.3$
Then I challenged them to get the approximation as accurate as possible by using electronic calculators. Actually, most calculators cannot show more than 13 digits which is the number of digits of 2^{40}. Moreover, the approximation of $\log 2$ coming from 2^{40} is 0.3 which is as same as the approximation coming from 2^{10}. This outcome somewhat depressed my students. Therefore, when they knew what Briggs had done without an electronic calculator, they all felt amazing and admired Briggs for his clever method and persistence as well.

Secondly, establishing logarithms tables was a slow, laborious job in the time without electronic calculators yet it brought much convenience for posterity though. I wanted my students to experience the process, so in the third worksheet of my teaching project, I asked them to calculate the approxi-

The first five digits of 7^n	n	The number of digits of 7^n	The first five digits of 7^n	n	The number of digits of 7^n
49	2	2		200	
2401	4	4		400	
	8	7		800	
	10	9		1000	
	20			2000	
	40			4000	
	80			8000	
	100			10000	

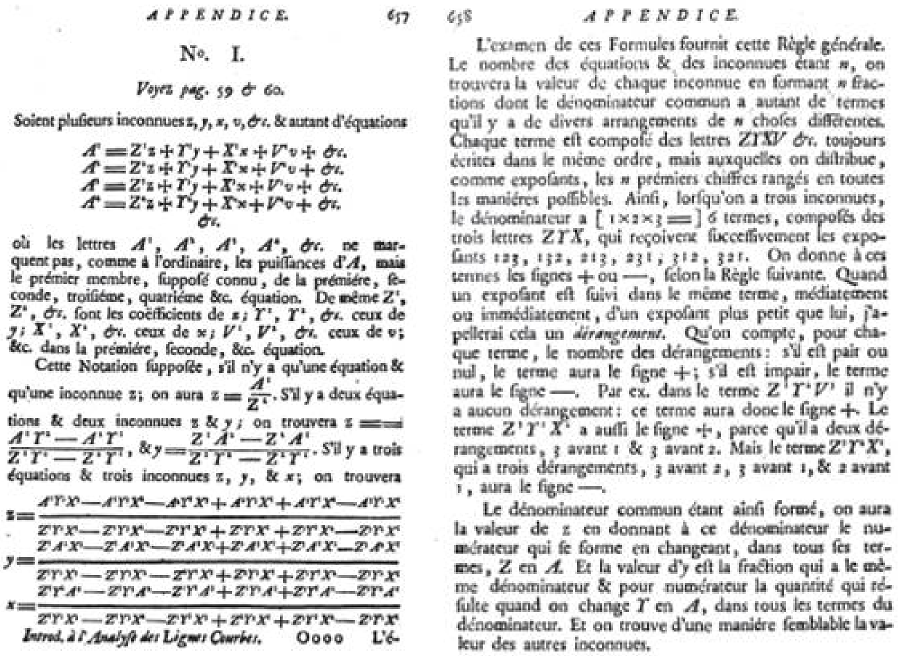
mation of log 7 by following Briggs' way, but used electronic calculators and wrote down the first five digits of 7^n (see the tabulation). Even using electronic calculators, they still spent some time to finish the task. Through this activity, they profoundly realized that to complete tables of logarithm was a huge task and people in that time definitely were in need of these tables for otherwise they would not need to do it. The following are my students' feedbacks:

* The method is amazing and unexpected!
* Briggs spent so many years on calculation and surprisingly, he could acquire so precise approximations without an electronic calculator. This shows that he was extremely persistent.
* It was very fortunate that Briggs altered Napier's logarithm with base $1 - 10^{-7}$. Briggs benefited later students. Thank you Briggs!!
* I finally realized that completing logarithm tables was a huge task, and surveys of astronomy and navigation would become easier with these tables. We should learn the predecessors' method well.
* I thought every concept of mathematics was easy to construct. Now I know that each concept we thought it as a matter of course came from hard work of mathematicians who even had devoted his whole life to it.
* It is more attractive to us to present mathematics in this way, and let us know development, application, and interesting things of mathematics.

2.4.2 Cramer's Rule

Hui-Yu Su, the editor of *HPM Tongxun*, selected 90 articles from volume 1 to volume 10 of *HPM Tongxun*, and sorted them by topics of senior high school mathematics. Actually, there are more than 120 articles relating to these topics in *HPM Tongxun* so far. Although the amount is huge, there are still some topics lacking research articles. Therefore, Su asked for them. As a deputy editor of *HPM Tongxun*, I chose the topic of Cramer's rule and designed a teaching project that I drew upon in two of my classes.

Nowadays, Cramer's rule is presented in the form of determinant. This, however, is completely different from the original version in Cramer's *Introduction à l'analyse des lignes courbes algébrique* (1750). In what follows, I take for example linear equations with three unknowns to explain Cramer's original



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No. I.

Voyez pag. 59 & 60.

Soient plusieurs inconnues z, y, x, v, &c. & autant d'équations

$$A^1 = Z^1z + Y^1y + X^1x + V^1v + \&c.$$
$$A^2 = Z^2z + Y^2y + X^2x + V^2v + \&c.$$
$$A^3 = Z^3z + Y^3y + X^3x + V^3v + \&c.$$
$$A^4 = Z^4z + Y^4y + X^4x + V^4v + \&c.$$
&c.

où les lettres A^1, A^2, A^3, A^4, &c. ne marquent pas, comme à l'ordinaire, les puissances d'A, mais le prémier membre, supposé connu, de la prémiére, seconde, troisiéme, quatriéme &c. équation. De même Z^1, Z^2, &c. sont les coëficients de z; Y^1, Y^2, &c. ceux de y; X^1, X^2, &c. ceux de x; V^1, V^2, &c. ceux de v; &c. dans la prémiére, seconde, &c. équation.

Cette Notation supposée, s'il n'y a qu'une équation & qu'une inconnue z; on aura $z = \frac{A^1}{Z^1}$. S'il y a deux équations & deux inconnues z & y; on trouvera $z = \frac{A^1Y^2 - A^2Y^1}{Z^1Y^2 - Z^2Y^1}$, & $y = \frac{Z^1A^2 - Z^2A^1}{Z^1Y^2 - Z^2Y^1}$. S'il y a trois équations & trois inconnues z, y, & x; on trouvera

$$z = \frac{A^1Y^2X^3 - A^1Y^3X^2 - A^2Y^1X^3 + A^2Y^3X^1 + A^3Y^1X^2 - A^3Y^2X^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$

$$y = \frac{Z^1A^2X^3 - Z^1A^3X^2 - Z^2A^1X^3 + Z^2A^3X^1 + Z^3A^1X^2 - Z^3A^2X^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$

$$x = \frac{Z^1Y^2A^3 - Z^1Y^3A^2 - Z^2Y^1A^3 + Z^2Y^3A^1 + Z^3Y^1A^2 - Z^3Y^2A^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$

Introd. à l'Analyse des Lignes Courbes. Oooo L'é-

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L'examen de ces Formules fournit cette Règle générale. Le nombre des équations & des inconnues étant *n*, on trouvera la valeur de chaque inconnue en formant *n* fractions dont le dénominateur commun a autant de termes qu'il y a de divers arrangements de *n* choses différentes. Chaque terme est composé des lettres *ZYXV &c.* toujours écrites dans le même ordre, mais auxquelles on distribue, comme exposants, les *n* prémiers chiffres rangés en toutes les maniéres possibles. Ainsi, lorsqu'on a trois inconnues, le dénominateur a [1×2×3 =] 6 termes, composés des trois lettres *ZYX*, qui reçoivent successivement les exposants 123, 132, 213, 231, 312, 321. On donne à ces termes les signes + ou −, selon la Règle suivante. Quand un exposant est suivi dans le même terme, médiatement ou immédiatement, d'un exposant plus petit que lui, j'appellerai cela un *dérangement*. Qu'on compte, pour chaque terme, le nombre des dérangements: s'il est pair ou nul, le terme aura le signe +; s'il est impair, le terme aura le signe −. Par ex. dans le terme $Z^1Y^2V^3$ il n'y a aucun dérangement: ce terme aura donc le signe +. Le terme $Z^3Y^1X^2$ a aussi le signe +, parce qu'il a deux dérangements, 3 avant 1 & 3 avant 2. Mais le terme $Z^3Y^2X^1$, qui a trois dérangements, 3 avant 2, 3 avant 1, & 2 avant 1, aura le signe −.

Le dénominateur commun étant ainsi formé, on aura la valeur de z en donnant à ce dénominateur le numérateur qui se forme en changeant, dans tous ses termes, Z en *A*. Et la valeur d'y est la fraction qui a le même dénominateur & pour numérateur la quantité qui résulte quand on change *Y* en *A*, dans tous les termes du dénominateur. Et on trouve d'une maniére semblable la valeur des autres inconnues.

Figure 1: *

Figure 4 Cramer's Rule of *Introduction à l'analyse des lignes courbes algébrique*

rule.

$$\begin{cases} A_1 & = & Z_1 \cdot Y_1 \cdot y + X_1 \cdot x \\ A_2 & = & Z_2 \cdot Y_2 \cdot y + X_2 \cdot x \\ A_3 & = & Z_3 \cdot Y_3 \cdot y + X_3 \cdot x \end{cases}$$

$A_1 \sim A_3$ are constants, $Z_1 \sim Z_3$, $Y_1 \sim Y_3$, $X_1 \sim X_3$ are coefficients of unknowns z, y, x respectively

$$z = \frac{A_1Y_2X_3 - A_1Y_3X_2 - A_2Y_1X_3 + A_2Y_3X_1 + A_3Y_1X_2 - A_3Y_2X_1}{Z_1Y_2X_3 - Z_1Y_3X_2 - Z_2Y_1X_3 + Z_2Y_3X_1 + Z_3Y_1X_2 - Z_3Y_2X_1}$$

Cramer gave a series of specific regulations to put down the values of unknowns:

(i) The values of unknowns have a common denominator. Each term of the denominator is in the form of $Z_aY_bX_c$, and a, b, and c are arrangements of number 1, 2, and 3: 123, 132, 213, 231, 312, 321. Therefore, the denominator has 6(=3!) terms.

(ii) We attach the sign "+" to the term with even number of "derangement", or we attach the sign "−". The "derangement" means infringing the condition $a < b < c$. For examples, the term $Z_3Y_2X_1$ has the sign "−" because there are two derangements: 3 before 1 and 3 before 2. The term has the sign " " because there are three derangements: 3 before 1, 3 before 2 and 2 before 1. After these, we have the denominator.

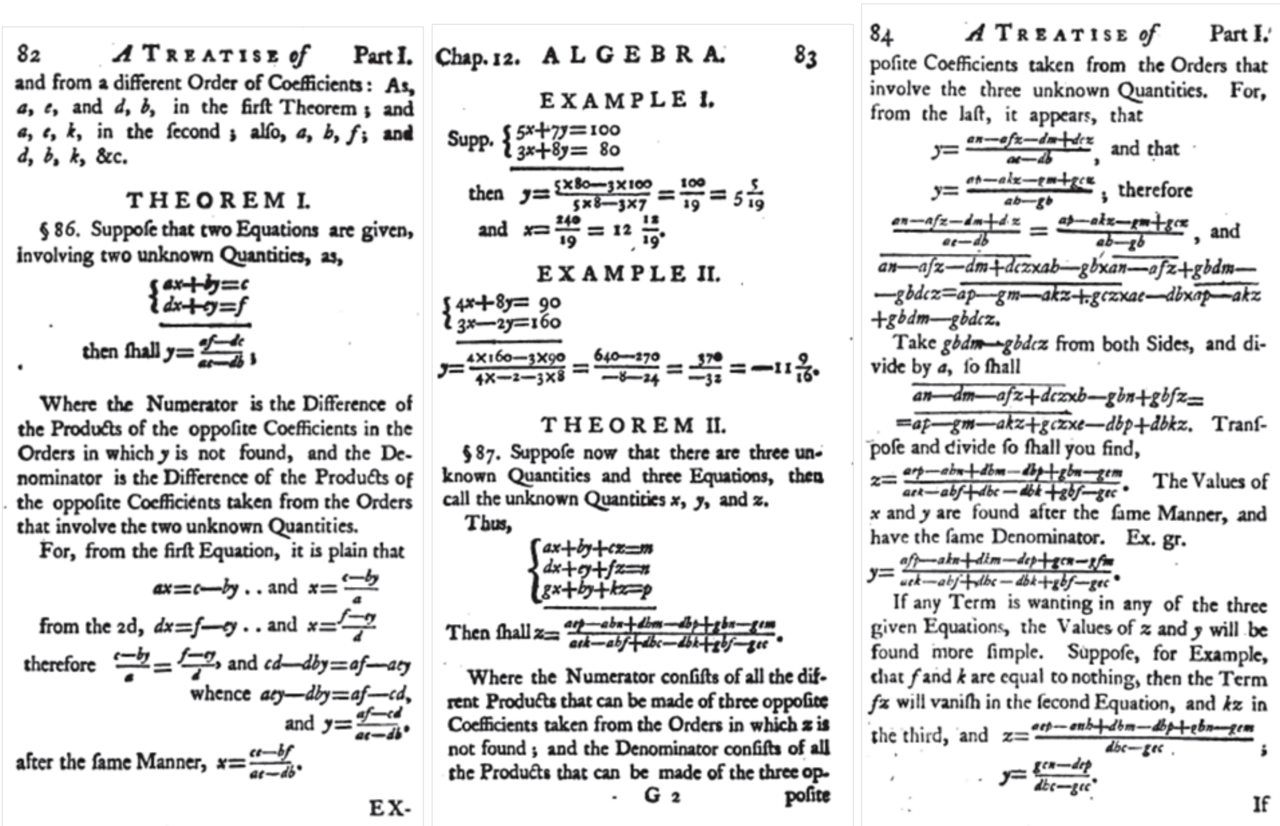
(iii) We change Z_1, Z_2, Z_3 into A_1, A_2, A_3 respectively, and then we find the numerator of the value of the unknown Z. We change Y_1, Y_2, Y_3 into A_1, A_2, A_3 respectively, and then we find the numerator of the value of the unknown Y. The value of the unknown X is obtained in a similar way.

In a word, Cramer used permutations to find the values of unknowns, rather than determinants. In my teaching project, I asked students trying to write down the Cramer's rule with four unknowns, and to imagine it with five unknowns. In addition to Cramer's original rule, I showed several pages of Colin Maclaurin's (1698 1746) *Treatise of Algebra* (1748) to my students, and asked them to explain what is the subject of these pages (see Figure 5). It was not difficult for them to identify the content of these pages as the so-called Cramer's rule. Moreover, after they knew Maclaurin already wrote the rule in 1729 before Cramer published his book in 1750, they answered the following three questions:

> Question 1: Which one do you like? Cramer's or Maclaurin's rule?
>
> Question 2: Should we call the name of the rule as Cramer's rule or Maclaurin's rule?
>
> Question 3: What are the advantages of representing the rule in the form of determinant?

There are 85 students in my two classes, and in the case of question 1, only 2 students liked Cramer' original rule, 30 students liked Maclaurin's, and 33 students disliked both of them. Most students disfavored Cramer' original rule because of its complexity of expression, and they preferred Maclaurin's rule because they comprehended what Maclaurin had done. When it came to question 2, needless to say, most students approved of the name, Maclaurin's rule. However, one student defended for Cramer because Cramer told us how to find the values of unknowns no matter how many unknowns, but Maclaurin only told us the rule with three unknowns. After listening to his explanation, many students became hesitant. I was very glad that some one could perceive the deep difference between them and triggered off other students' second thought.

Although I have some experience in terms of the HPM, this teaching project was not successful



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and from a different Order of Coefficients: As, *a*, *e*, and *d*, *b*, in the firſt Theorem; and *a*, *e*, *k*, in the ſecond; alſo, *a*, *b*, *f*; and *d*, *b*, *k*, &c.

THEOREM I.

§ 86. Suppoſe that two Equations are given, involving two unknown Quantities, as,

$$\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$$

then ſhall $y=\frac{af-dc}{ae-db}$;

Where the Numerator is the Difference of the Products of the oppoſite Coefficients in the Orders in which *y* is not found, and the Denominator is the Difference of the Products of the oppoſite Coefficients taken from the Orders that involve the two unknown Quantities.

For, from the firſt Equation, it is plain that

$ax=c-by$ .. and $x=\frac{c-by}{a}$

from the 2d, $dx=f-ey$ .. and $x=\frac{f-ey}{d}$

therefore $\frac{c-by}{a}=\frac{f-ey}{d}$, and $cd-dby=af-aey$

whence $aey-dby=af-cd$,

and $y=\frac{af-cd}{ae-db}$.

after the ſame Manner, $x=\frac{ce-bf}{ae-db}$.

EX-

Chap. 12. ALGEBRA. 83

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$

then $y=\frac{5\times80-3\times100}{5\times8-3\times7}=\frac{100}{19}=5\frac{5}{19}$

and $x=\frac{240}{19}=12\frac{12}{19}$.

EXAMPLE II.

$$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$$

$$y=\frac{4\times160-3\times90}{4\times-2-3\times8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\frac{9}{16}.$$

THEOREM II.

§ 87. Suppoſe now that there are three unknown Quantities and three Equations, then call the unknown Quantities *x*, *y*, and *z*.

Thus,

$$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$$

Then ſhall $z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$.

Where the Numerator conſiſts of all the different Products that can be made of three oppoſite Coefficients taken from the Orders in which *z* is not found; and the Denominator conſiſts of all the Products that can be made of the three oppoſite

G 2

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poſite Coefficients taken from the Orders that involve the three unknown Quantities. For, from the laſt, it appears, that

$y=\frac{an-afz-dm+dcz}{ae-db}$, and that

$y=\frac{ap-akz-gm+gcz}{ah-gb}$; therefore

$\frac{an-afz-dm+dcz}{ae-db}=\frac{ap-akz-gm+gcz}{ah-gb}$, and

$\overline{an-afz-dm+dcz}\times\overline{ah-gb}\times\overline{an-afz}+gbdm-$
$-gbdcz=\overline{ap-gm-akz+gcz}\times\overline{ae-db}\times\overline{ap-akz}$
$+gbdm-gbdcz$.

Take $gbdm-gbdcz$ from both Sides, and divide by *a*, ſo ſhall

$\overline{an-dm-afz+dcz}\times h-gbn+gbfz=$
$=\overline{ap-gm-akz+gcz}\times e-dbp+dbkz$. Tranſpoſe and divide ſo ſhall you find,

$z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}$. The Values of *x* and *y* are found after the ſame Manner, and have the ſame Denominator. Ex. gr.

$y=\frac{afp-akn+dkm-dcp+gcn-gfm}{aek-ahf+dhc-dbk+gbf-gec}$.

If any Term is wanting in any of the three given Equations, the Values of *z* and *y* will be found more ſimple. Suppoſe, for Example, that *f* and *k* are equal to nothing, then the Term *fz* will vaniſh in the ſecond Equation, and *kz* in the third, and $z=\frac{aep-anh+dhm-dbp+gbn-gem}{dhc-gec}$;

$y=\frac{gcn-dcp}{dhc-gec}$.

If

Figure 2: *

Figure 5

enough in practice. I discussed it with a teacher of reaching-practice status who went to the classroom and made a recording on digital video. We both primarily agreed that this teaching project has many merits. However, we could make it better in two aspects. First, I nearly adopted all of Cramer's symbols and notations in order to bring the flavor of history to students. However, they brought cognitive obstacles to students at the same time. It was not easy for students to learn new things with an unfamiliar system of symbol. In addition, there were so many symbols that Cramer's rule became an abstract monster. Therefore, I should use symbols that are familiar to students instead of Cramer's, and start the rule on two unknowns.

Second, it may be better to use this teaching project before students know Cramer's rule. I expected that students could appreciate mathematicians' efforts on developing this rule as students knew its modern form. However, it did not work. On the one hand, the original rule is much more complicated and troublesome than the modern one, so students became impatient and unwilling to follow it. On the other hand, the time I brought this teaching project into classroom was six day earlier than the final examination of the semester, and it was no wonder that a few students were not in the mood to know something unrelated to the exam.

Besides these two flaws mentioned above, there was an unanticipated and surprising gift from Cramer's original rule. In Cramer's approach, it is easy to count the number of derangements of any given term. However, if we ask the inverse question: how many terms have the same number of derangements, then we enter the realm of 20th century discrete mathematics. Actually, this question turns to be a problem about permutations and inversions (derangements). For example, permutation 31524 has four inversions, namely (3,1), (3,2), (5,2), and (5,4), and how many permutations formed by 1, 2, 3, 4, and 5 have four inversions? This line of research can trace back to Eugen Netto's Lehrbuch der Combinatorik in 1901. (Bóna, 2004) On the last day of the semester, I got together six students who are good at mathematics, and proposed two interesting methods to them to get start the study of this problem. (See Appendix) Although they all highly attracted by these two methods, they did not go further in their winter vacation. I wrote an article to show the way to solve this problem in detail, and two of my colleagues are very interested in it. We will work together to modify my teaching project to become a six-class lesson, including Cramer's original rule and modern discrete mathematics as well, for students in mathematics and science talented classes.

3 Concluding Remarks

From these experiments mentioned above, we can see that the HPM has rooted in these teachers' PCK (pedagogical content knowledge). When they integrated history of mathematics into instructions, they combined their expertise of the HPM and of the PCK to prepare practicable and suitable teaching materials for their students. Prof. Wann-Sheng Horng developed the hermeneutic tetrahedron from Niels Hans Janhke's hermeneutic twofold cycle to illuminate how the HPM enhances teachers' PCK. (Horng, 2004b & 2005) Being a mathematics teacher in high school, I propose my suggestions for teachers and scholars who are interested in using history of mathematics in classroom.

To design such kind of teaching materials, obviously, teachers need to know the history of the chosen topic in deep sense. It even requires an overview of history of mathematics. Take for instance Jun-Hong Su's award winning teaching projects, he cannot create them without recognizing the significance of incommensurable magnitudes, the Pythagorean Theorem in Euclid's *Element*, and the *latus rectum* in Apollonius' *Conics*. However, few teachers in high school are as good as Jun-Hong Su expertise in history of mathematics. So, how can we help them? To offer them a serious course of history of mathematics might be an option, but it is not attainable for most teachers. I asked some teachers and myself a question that *being a mathematics teacher in high school*, what kind of resources of the HPM is most helpful or useful for us.

I got two answers. One is popular articles about the history of topics in high school math textbooks. I have to stress the word "popular". There are many articles and books on history of mathematic in Taiwan, but most of them are written for mathematicians and historians, not for teachers in high school. Take logarithm for example, Napier is the main character in many articles and books, but Briggs is rarely mentioned or referred to. However, Napier's logarithm is unintelligible not only for students, but also for most teachers. Fortunately, there gradually come out articles and books basically written for high school teachers. *Math through the Ages: A Gentle History for Teachers and Others* was translated into Chinese by team members of *HPM Tongxun* in 2008. My wife Jing-Ru Chiu who is now a senior high school mathematics teacher told me she likes this book very much and thinks it useful. In addition, Hui-Yu Su, the editor of *HPM Tongxun*, continues to write a series of articles, named "HPM Gao Zhong Jiao Shi" (HPM in senior high school classroom), for senior high school mathematics teachers since 2011. Up to now, she has written seven series articles, and all of them are devoted to history of math topics in textbooks. These articles provide teachers abundant materials for designing teaching projects.

The other answer is teaching projects with guidelines in detail. Teachers who are not capable of designing teaching projects of history of mathematics are capable of integrating history of mathematics into instruction, as long as there are some things like reports written by participants of Dr. Su Yi-Wen's research program, or articles about my own teaching projects. Through these reports and articles, teachers acquire not only guidelines and advices about implementation, but knowledge of history of mathematics. Moreover, teachers can easily adapt these teaching projects for their students. In other words, these teaching projects are prototypes that can produce many teaching projects. As a teacher adapts more and more teaching projects, he or she would acquire more and more expertise in terms of the HPM. One day he or she may be able to create a new prototype for others.

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Appendix

The symbol $f_m(n)$ denotes the number of permutations formed by 1, 2, 3…m with n inversions.
For example, permutations formed by 1, 2, and 3 have three numbers of inversions:

Number of inversions	0	1	2	3
Permutation	123	213、132	231、312	321

We write $f_3(0)=1$, $f_3(1)=2$, $f_3(2)=2$, and $f_3(3)=1$. Let start the observation from $m=1$.

$m=1$：

1 ⇒ number of inversions：0

n	0
$f_1(n)$	1

$m=2$：

1 2 ⇒ number of inversions：0
2 1 ⇒ number of inversions：1

n	0	1
$f_2(n)$	1	1

$m=3$：

□ □ 3 ⟨ 1 2 3 ⇒ number of inversions：0
2 1 3 ⇒ number of inversions：1

⇒ The numbers are as the same as $m=2$.

□ 3 □ ⟨ 1 3 2 ⇒ number of inversions：1
2 3 1 ⇒ number of inversions：2

⇒ Each number increases 1.

3 □ □ ⟨ 3 1 2 ⇒ number of inversions：2
3 2 1 ⇒ number of inversions：3
⇒ Each number increases 2.

n	0	1	2	3
	1	1		
		1	1	
			1	1
$f_3(n)$	1	2	2	1

$m=4$：

□ □ □ 4 ⇒ The numbers are as the same as $m=3$

□ □ 4 □ ⇒ Each number increases 1

□ 4 □ □ ⇒ Each number increases 2.

4 □ □ □ ⇒ Each number increases 3.

n	0	1	2	3	4	5	6
	1	2	2	1			
		1	2	2	1		
			1	2	2	1	
				1	2	2	1
$f_4(n)$	1	3	5	6	5	3	1

⋮

⋮

$$\{f_5(n)\}_{n=0}^{10} = \{1,\ 4,\ 9,\ 15,\ 20,\ 22,\ 20,\ 15,\ 9,\ 4,\ 1\}$$

$$\{f_6(n)\}_{n=0}^{15} = \{1,\ 5,\ 14,\ 29,\ 49,\ 71,\ 90,\ 101,\ 101,\ 90,\ 71,\ 49,\ 29,\ 14,\ 5,\ 1\}$$

$$\{f_7(n)\}_{n=0}^{21} = \begin{Bmatrix} 1,\ 6,\ 20,\ 49,\ 98,\ 169,\ 259,\ 359,\ 455,\ 531,\ 573, \\ 573,\ 531,\ 455,\ 359,\ 259,\ 169,\ 98,\ 49,\ 20,\ 6,\ 1 \end{Bmatrix}$$

These sequences constitute the Triangle of Mahonian Numbers (or Mahonian Triangle). The name shows the respect for Percy Alexander MacMahon (1854~1929):

1

1 1

1 2 2 1

1 3 5 6 5 3 1

1 4 9 15 20 22 20 15 9 4 1

1 5 14 29 49 71 90 101 101 90 71 49 29 14 5 1

1 6 20 49 98 169 259 359 455 531 573 573 531 455 359 259 169 98 49 20 6 1

Now, we introduce discrete mathematics. Rewrite the tabulation of $f_4(n)$ as follows:

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6
x^0	1	2	2	1			
x^1		1	2	2	1		
x^2			1	2	2	1	
x^3				1	2	2	1
$f_4(n)$	1	3	5	6	5	3	1

Then we see the procedure of finding $f_4(n)$ from $f_3(n)$ is multiplication of two polynomials:

$$(1+2x+2x^2+x^3)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+1\cdot x^6$$

In the similar way, we have $1+2x+2x^2+x^3 = (1+x)(1+x+x^2)$. Therefore, we find the generating function of $f_m(n)$:

$$F_1(x) = 1$$

$$F_2(x) = 1\cdot(1+x) = 1+1\cdot x$$

$$F_3(x) = 1\cdot(1+x)\cdot(1+x+x^2) = 1+2x+2x^2+1\cdot x^3$$

$$F_4(x) = 1\cdot(1+x)\cdot(1+x+x^2)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+1\cdot x^6$$

$$F_5(x) = 1\cdot(1+x)\cdot(1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)$$

$$= 1+4x+9x^2+15x^3+20x^4+22x^5+20x^6+15x^7+9x^8+4x^9+1\cdot x^{10}$$

$$F_6(x) = 1\cdot(1+x)\cdot(1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4+x^5)$$

$$= 1+5x+14x^2+29x^3+49x^4+71x^5+90x^6+101x^7+101x^8+90x^9+71x^{10}+49x^{11}$$

$$+29x^{12}+14x^{13}+5x^{14}+1\cdot x^{15}$$

$$\vdots$$

$$\vdots$$

$$F_m(x) = 1\cdot(1+x)\cdot(1+x+x^2)\cdots\cdots(1+x+x^2+\cdots\cdots+x^{m-1}) = \prod_{k=0}^{m-1}(1+x+x^2+\cdots\cdots+x^k)$$

The coefficient of x^n of the expansion of $F_m(x)$ is $f_m(n)$. However, it still requires some work to write down the expression of $f_m(n)$.