HISTORY OF MATHEMATICS IN MATHEMATICS TEACHER EDUCATION PROGRAMS:
The development, implementation and evolution of a course

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ABSTRACT

Several countries including England, Scotland and Denmark have recently incorporated the history of mathematics in their school curriculum. This raises questions on the nature of teacher education and professional development programs in the history of mathematics courses and workshops. What are the examples of recent courses in the history for mathematics for teachers? In this paper, I present a case history of a mathematics course for teacher candidates at a Canadian University. I specifically studied the modifications which the course has undergone since its inception five years ago. The question at the center of this analysis is, in what ways has the course developed? And what are the reasons for the changes? I carried out a content analysis of the course's outlines, plans, lists of resources and teaching and assessment materials in order to study the changes and the possible reasons for the changes over the past five years. Interpretations of the results are framed by Furinghetti's framework, the role of history of mathematics in schools. I adapt the discipline of noticing as a method of inquiry. The discipline of noticing encourages teachers and educators to reflect on their practices. Results display major changes in course content, assignments and materials; while the course goals showed few changes. For example, new topics, such as, historical numbers and historical mathematics problems were introduced. Further, broader topics, such as, the evolution of specific concepts, vis-à-vis the narrower topic of history of number development were introduced. The nature of the course participants and the increased availability of multimedia texts appear to be the major reasons underlying the course changes.

Keywords: History of mathematics, teacher education, course implementation, history of mathematics-for-teachers tasks, practitioner research.

1 Introduction and Background

Several countries including England, Scotland and Denmark have recently incorporated the history of mathematics in their school curriculum in order to motivate and develop students' mathematical thinking (Marshall & Rich, 2000; Pope, 2010; Pritchard, 2010). This raises questions on the nature of history of mathematics courses and workshops offered in teacher education and professional development programs. What are the examples of recent courses in the history for mathematics for teachers? In this paper, I study modifications undertaken on a history of mathematics course for teacher candidates since its inception at a Canadian University. This paper relates to four themes of HPM 2012 meeting: Theoretical and/or conceptual frameworks for integrating history in mathematics education; History and epistemology implemented in mathematics education; Classroom experiments and teaching materials; and Topics in the history of mathematics education.

I developed a course the Historical-Conceptual Development of Mathematics, 18 hours, for elementary and senior teacher candidates in 2007 at a University in Canada. Since then, I have taught the course for three terms to an average of 20 teacher candidates who elect to take the course a term; increasingly a majority of the teacher candidates are from the secondary division. This elective course was designed to draw from the history and traditions of mathematics to inform the understanding and teaching of school mathematics. The course integrates the connections (and disconnections) between the historical and conceptual development of mathematics topics. It aims to: relate the history of mathematical concepts to sequencing topics in the school curriculum; assist teacher candidates to appreciate why concepts, such as integers, are better understood by learners in specific ways and not in other ways (say, integer multiplication in abstract ways and not in concrete ways); and illustrate the multiplicity of meanings and representations—including concrete, graphical and analytical—of several concepts. The classes are designed in the format of short lectures, problem solving activities of historical mathematics, acting out skits and plays, exploration of selected texts and documentaries, anecdotal displays on both conventional and non conventional mathematicians, researching the evolution of mathematics concepts in selected articles, and preparing creative teaching activities based on the history of mathematical concepts.

The format of the course in 2011 consisted of short lectures, problem solving of historical mathematics, acting out skits and plays, exploration of selected texts and documentaries, anecdotal displays on both convention and non conventional mathematicians, researching the evolution of mathematics concepts in selected articles, and preparing creative teaching activities based on the history of mathematics concepts.

In this paper, I study the course modifications that have occurred since the course's inception. I carry out a content analysis of the topical outlines, daily plans, readings lists and assignments in order to study how these areas have evolved over the past five years. I also share some classroom tasks. The question at the center of this analysis is, in what ways has the course evolved? And what might be the main reasons for the changes?

2 Framework

A plethora of professional and scholarly publications exist on incorporating the history of mathematics into school and university mathematics. Questions; such as why, how and what of teaching history of mathematics are addressed in the literature. Liu (2003), Bidwell (1993), Ernest (1998), Freudenthal (1981), Furinghetti (1997) outline roles of learning history of mathematics in school teaching. The roles range from students' learning outcomes such as improvement in attitudes toward mathematics; to pedagogical outcomes, such as opening up a window into studying aspects of students' mathematical thinking; through to broader humanistic roles that the history of mathematics plays in showing the ways mathematics has been practiced over time, and even further to the argument that the history of mathematics is part of mathematical learning. Further, several forms of teaching the history of mathematics are explored in the literature including biographies of mathematicians, anecdotal and visual displays of mathematicians, their works and facts about their lives, and the exploration of original works and ancient textbooks (Bidwell; Fauvel, 1991; Kaye, 2010; Pritchard, 2010; Wilson & Chauvot, 2000). Several researchers recommend incorporating the history of mathematics in the teaching of the subject matter; say as mini-lectures on any topic of interest. Despite the plethora of literature on the importance of having history in a mathematics class, Marshall and Rich (2000) identified a lack of "empirical studies that discuss the use of history to teach mathematics" (p.704). Swetz (2003) notes an increased interest in offering history of mathematics in university mathematics departments. There is need for empirical studies on classes, courses and workshop where history of mathematics forms an integral part of learning.

Swetz (2003) observes that it is important to consider the goal of exploring the history of mathematics when deciding which resources to be used say in university courses on the history of mathematics. A course instructor might choose to focus beyond the Western origins of mathematics, whereas another instructors might choose to focus on doing historical mathematics problems from varies mathematics traditions. Furinghetti (1997) offers a conceptual framework for integrating history in mathematics education. Furinghetti describes four approaches to teach the history of mathematics: a) a history of mathematics for promoting the image of mathematics; b) a history of mathematics as a source of mathematical problems; c) a history of mathematics as a different approach to concepts; and d) a history of mathematics as a different approach to mathematical concepts. Furinghetti thus links the role of the history of mathematics to how it is taught or incorporated in school mathematics and in teacher education. Furinghetti further identifies two streams of intervention of the "history of mathematics into mathematics teaching …one stream is aimed at promoting mathematics, the other [is aimed] at reflecting on mathematics; the first is linked with the 'social' role of the discipline and its image, the second mainly concerns aspects interior to the discipline, such as development and the understanding of it" (p. 59). The study conducted by McBride and Rollins (1977), for instance, focuses on the changes in students' attitudes as a result of incorporating history into mathematics teaching. This study by McBride and Rollins belongs to the former category of the *social role*. D'Ambrosio (1997), on the other hand, promotes a cultural focus to the history of mathematics; he argues a cultural focus broadens students' understanding of mathematics. D'Ambrosio's focus aligns more with the latter category of the *mathematical role*, specifically the *ethno-mathematics* role. Swetz (1995) and Siu (1995) observe that exploring historical mathematical problems contributes to students' understanding of the processes, origins and development of mathematics; these researchers illustrate the latter category of the interior role to mathematics. Liu encourages incorporating the history of mathematics into teacher education because it expands teachers' mathematical knowledge. Exploring the history of mathematics in the context of the time, place, cultures and civilizations in which it was developed reveals much about the nature of how mathematics knowledge was develops and is learned (Bidwell, 1993).

3 Methods

Mason (2002) defines the *discipline of noticing*, what at other times he refers to as, *researching-from-the-inside*, as a method of inquiry for studying teaching and other practitioner practices. The *discipline of noticing* involves working towards improving one's ability to notice, mark, record, reflect, and analyze specific aspects of their professional practice. Mason encourages making a record of brief descriptions of a phenomenon while recording what one has noticed. These descriptions should be vivid and stripped off of any theory, judgment and personal views. He refers to such descriptions as *brief-but-vivid descriptions*, *accounts-of* an incident. These brief-but-vivid descriptions form the data to be analyzed. The data is then interpreted; the interpretations, explanations and application of theory follow after and remain separate from the descriptions. Data analysis, akin to several qualitative methods, involves searching for common themes among a collection of accounts. Mason outlines several aspects of noticing; which includes systematic reflection and the validating with others. Mason labels the reflection on the changes the *accounts*-for. To Mason and Spence (1999), systematic "reflection-on-action" promotes "reflection-in-action" (p.153). The discipline of noticing is useful at exploring more possibilities for acting when similar situations arise in the future. The discipline of noticing supports learning from experience and increases the possibilities to notice specific categories of a phenomenon.

In this paper, to study modifications which the course has have undergone since its inception, I study the documents for the course, the topical outlines, daily plans, reading lists and assignments specifications as the accounts of the course plans. I carry out the content analysis of both the 2007 and 2011 course plans to identify changes. Results of the analysis are summarized in tables. I share written plans of the classroom activities as brief-but-vivid descriptions of the course activities. The question at the center of this analysis is: in what ways have the course goals, content, teaching and assessment materials changed over the past five years? And what are the reasons for the changes?

Mason (1994) observes that noticing is what teachers and educators do all the time. But to engage in a conversation with and about what teachers and educators notice is what requires a deliberate choice on the side of the teacher-or educator-researcher. Mason developed the discipline of noticing as a practice for working with noticing. The techniques from the discipline of noticing adapted here involves marking the changes over time by comparing and contrasting documents, offering a description of what the changes are, in Mason's (2002) terms, the *accounts-of* the difference, and then reflecting on the changes. The reflection on the changes is intended to offer justifications of what I—as the educator who is researching the development of her own designed and own taught course—thought were the impetus for the changes, the *accounts-for*. It is the accounts-of and the accounts-for that are at the center of the results and their interpretations. Presenting my reflections on the course at the conference is a way of validating with others what I notice about a history of mathematic course for teachers.

4 Analysis and Results

In tables 1 to 4, I offer analyses of the content in the course's goals, content, teaching and assessment activities, and daily class plans for the first year the course was taught, 2007; as well as the most recent year the course was taught, 2011. Text that is *italicized* in the tables marks the differences between the two years. Where no differences exist I have note, —same as in 2007—.

Table 1, where course goals for the two years are compared, shows one difference between course goals in 2007 and 2011: a new goal was introduced in 2011 which emphasized the use of "understanding and reflection" from the course to "inform professional judgment in practice". All other five goals remained the same as in 2011.

Table 1: *Comparing and Contrasting the Goals of the Course*

2011 Aims and Goals	**2007 Aims and Goals**
The course aims to 1. —Same goal as in 2007— 2. —Same goal as in 2007— 3. —Same goal as in 2007— 4. —Same goal as in 2007— 5. —Same goal as in 2007— 6. *Encourage understanding and reflection on student development, learning theory, pedagogy, curriculum and research on the history of mathematics, and to use such understanding and reflection to inform professional judgment in practice.*	The course aims to 1. Develop understanding of the evolution of mathematical ideas historically. 2. Explore the implications of this evolution on conceptual development and across grades. 3. Explore the implications of this evolution on sequencing of topics, selecting teaching models, and representations for teaching mathematics. 4. Introduce a range of strategies for teaching mathematics in ways connected to history, culture and new discoveries in mathematics. 5. Demonstrate how history of school mathematics is usefully integrated in school mathematics.

Table 2 shows a comparison of course content of the two years. Four differences are evident: (a) an introduction of an overarching topic, Topic 1 on *Introduction, overview and role of history of mathematics* in 2011; (b) a broadening of Topic 1 (2007 column) to include the *history* ⋯*of geometry* as well; (c) an amalgamation of topics 2, 3, 4 and 5 (2007 column) on specific concepts into two general topics, *Historical mathematics problems* and *Evolutions of specific concepts*; (d) the separation of the 2007 Topic 8 *Mathematicians in School Mathematics* into *Selected mathematicians' stories, lives and works* and *Women, prodigies and other interesting mathematicians*; as well as a shift from focusing specifically on, say, *Fractal Geometry* ⋯*Euclidean and non-Euclidean Geometries* to describing a general focus on *Selected new mathematics discoveries* . Further, a new topic was introduced in 2011, Topic *8, a brief history of school mathematics*.

Table 2: *Comparing and Contrasting the Course Content*

2011 Course Content	2007 Course Content
1. *Introduction, overview and role of history of mathematics* 2. History of Number and Geometry in teaching and learning 3. Western and non Western roots of the school mathematics 4. *Historical mathematics problems* 5. *Selected mathematicians: stories, lives and works in school mathematics* 6. *Evolutions of specific concepts e.g. History of zero, evolution of functions, and times of multiplication* 7. *Women, prodigies and other interesting mathematicians* 8. *Selected new mathematics discoveries and a brief history of school mathematics*	1. History of Number developments 2. Western and Non western roots of Mathematics 3. *Zero and Integers—Chinese rods and relation to use of colored chips* 4. *Multiplication—development across grades and variety of algorithms* 5. *Fractions—Egyptians fractions and other Fractions* 6. *Babylonian Algebra—polynomial, Geometry and the Polykit and Algebra tiles* 7. *Fractal Geometry, landscapes as Fractals, Euclidean and non-Euclidean Geometries* 8. *Mathematicians in School Mathematics — Gauss and number theory; Pythagoras theorem, its origin, proofs and the internet* 9. *Patterning: Pascal; Fibonacci and number sequences*

Table 3: *Comparing and Contrasting the Course Materials*

2011 Course Materials	**2011 Course Materials**
The course texts will include: A. Selected chapters from history of mathematics books, textbooks and magazines. Examples include: a. Joseph, Ghever Ghese G. (1991). *The crest of the Peacock: Non-European roots of mathematics*. London: Penguin Books. b. Burton, David M. (2002). *The history of mathematics: An introduction*. McGraw-Hill Math. B. Selected scholarly and professional/teacher journals articles. Examples include: a. Using history in mathematics Education. *For the Learning of Mathematics, 11*, 3–6. b. Liu, P. H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *The Mathematics Teacher, 96*, 416–421. C. *Selected web pages on the history of mathematics. Examples include:* a. *The History of mathematics—BBC documentaries* b. *History of Mathematics Wikipedia page*	The course texts will include: A. Selected chapters from books such as Chapter 1, Joseph, Ghever Ghese G. (1991). *The crest of the Peacock: Non-European roots of mathematics*. London: Penguin Books. B. Selected chapters from textbooks such as Howard, Eves (1990). An introduction to the history of mathematics, 6th Edition C. Selected articles from Professional/teacher journals such as Jarvis, D. (2007). Mathematics and the visual Arts: Exploring the golden ratio. *Mathematics Teaching in the Middle School, 12*, 467–471. D. Selected sections of Scholarly articles such as Section 1 and 6 of Furinghetti, F. & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: From phylogenies and ontogenesis theory to classroom practice. In: L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 631–654). New Jersey: Lawrence Erlbaum. Available online E. Selected websites such as The MacTutor History of Mathematics archive `http://www-groups.dcs.st-and.ac.uk/\~{}history/` Accessed January 4, 2007 F. Curriculum guidelines G. Curriculum Textbooks

The major change evinced in Table 3 is a move towards an increased use of nontraditional texts, especially to include visual-audio texts and texts published online. There was a resulting change of a reduction in the number of textual course materials listed.

Table 4: *Comparing and Contrasting the Course Assignments*

2011 Comparing and Contrasting the Course Assignments	**2007 Assignments and Other Course Requirements**
I. Attendance, classroom contribution and individual participation including small group work. *This will contribute 30% towards the final grade.* II. *One major assignment. You have a choice among A, B and C. This major assignment contributes 50% towards the final grade.* A. Mathematician assignment—Select a mathematician; research and write about one of his/her works and its relevance to school mathematics. Up to THREE pages double spaced. OR, B. Topical assignment—Select a mathematics topic or concept (e.g. integers, functions etc.); research and design a poster display that presents some of its historical, conceptual or curriculum (across grades) development. *Electronically designed posters are preferred but the instructor is open to other forms.* OR, C. *Small group performance project—in a group of 2 or 3 please plan a performance project that can be performed in class. The project may take the form of a lesson idea, a skit, a short video clip or any other agreed upon form. Samples will be shown in class.* III. *Presentations of the major assignment in seventh, eighth and ninth class. The presentation will contribute 20% towards the final grade.*	I. Classroom contribution and participation including homework, reading notes and assigned readings 25% II. Two written assignments, 2 pages double spaced each. • Topical assignment 25%—Select a mathematics concept (e.g. integers, fractions, angles etc.), research and write about some of its historical, conceptual or curriculum (across grades) development. • Mathematician assignment 25%—Select a mathematician and research and write about one of his works and it's the relevance of to school mathematics. III. III. A take home examination—25%

Three differences are seen in Table 4 which compares the assignment sections of the course: a) an introduction of a performance project; b) the provision of choice among the assignments on a mathematician, on a topic and a performance project; c) the inclusion of classroom presentation of the major assignment; d) the replacement of the take home assignment by classroom presentations based on a students' major assignment in the 2011 column.

The daily plans changed due to the changes in topical outline and due to the change in nature of participants. A majority course participants were from the IS division in 2011. This made it difficult to compare the class plans of the two years. I only carried out a holistic comparison of daily plans in 2007 as compared to those in 2011. In summary, the analyses of the class plans show more course readings are done in the context of in-class activities in 2011. Much of the teaching content and materials are still the same in 2011 as in 2007, but some content (say, evolution of specific concepts) is covered more in detail than other content (say, history of number) in 2011. Several more online, audio, and video texts are used during class in 2011, and more time in class is planned for students to present their work and assignments.

To exemplify the teaching materials, I share these in the context of selected tasks used in class. These are presented in Figures 1 to 5. These task descriptions, showing the topic, materials, supplemental materials, school classroom relevance and specific tasks, are adapted from the daily class plans. Figure 1 is an example of a task in which I use online video clips from a 4 hour documentary, *The story of maths*. Students listen to an archived radio program on the story of two mathematicians, one from the European mathematics tradition and the other from the Indian mathematics tradition, G. H. Hardy and S. Ramanujan in the task shown in Figure 2. Figure 3 displays a task in which part of the video recorded lectures on *Historical numbers, The Story of Euler's e* is utilized. Figure 4 describes the task in which students, using a play script and additional materials, prepare to act a play based on a mathematician, Évariste Galois. Figure 5, is an example of a classroom activity which involves reviewing selected articles to explore the life and times of a concept; in this case the multiple meanings and representations of the concept, in this case functions. It is worth noting that course participants submit exemplary assignments. These are also presented in class. Here I only mention assignment projects. These include but are not limited to:

- A short 5 minute video on conversations between Robert Hooke, Gottfried Wilhelm Leibniz and Sir Isaac Newton. This video centers on the Feud between Leibniz and Newton
- A comic strip based on selected mathematicians,
- A manuscript on Zero and another on Paul Erdös,
- A script of a skit centered on Emilie Du Chatelet, Hooke, Leibniz and Newton, and
- A brochure on Fractals, a song performance based on selected mathematician and mathematics concepts, and a poster on prime numbers.

Topic: Western and non Western roots of math

Materials: History of Math documentary by Prof. Marcus D Sautoy, which is organized by mathematics tradition. Sautoy, P. M. (Director). (2009). The story of maths [Motion Picture]. Also available at `http://topdocumentaryfilms.com/the-story-of-maths/`

Task: In groups of 3 we are going to watch the first 14 parts covering 7 early mathematics traditions. This is going to take us up to 25 minutes per group

Group 1: Part 3, 4 (stop at 7 minutes)—Babylonia –17 minutes
Group 2: Part 4 (at 7 minutes), 5, 6 Greece –25 minutes
Group 3: Part 7, 8—China—18 minutes
Group 4: Part 8 (start at 8 minutes), 9, 10 –Indian –22 minutes
Group 5: Part 10 (starts at 8 minutes), 11 —Islam –12 minutes
Group 6: Part 11 (Start at 7 minutes), 12, please skip 13, we viewed it.
during the previous class, 14—West and Europe – 23 minutes

(You may view parts 15 onwards in your spare time.)

Please view the video part indicated for your group with an intention to list on the chart paper provided

- some stories, ___________________________
- pictures or video clips, ___________________________
- math problems, topic or examples, ___________________________
- mathematicians ___________________________
- or general themes ___________________________

Supplementary materials: You may supplement the information provided in the videos with published literature from the selected chapters and chronological charts and maps provided from books such as Joseph, Ghever Ghese G. (1991).

Classroom Relevance: These stories may be narrated, shown, connected to, mentioned or simply utilized when teaching mathematics in your future classroom.

Figure 1. Use of a documentary in a history of mathematics-for-teachers task.

Topic: Selected mathematicians: stories, lives and works in school mathematics: G. H. Hardy and S. Ramanujan, A mathematical romance

Materials: Archived Radio Program,

Sautoy, P. M. (Composer). (2011). A brief history of mathematics: Hardy and Ramanujan. [BBC Radio, Performer, & P. M. Sautoy, Conductor] London, UK. Accessed Feb 2011 at `http://www.bbc.co.uk/iplayer/episode/b00ss1j4/A\_Brief\_History\_of\_Mathematics\_Hardy\_and\_Ramanujan/` (14 minutes)

Supplementary materials: Selected sources include Google timelines and Google images; as well as selected pages from books such as Pickover, C. A. (2009). *The Math Book, from Pythagoras to the 57th dimension 250 milestones in the history of mathematics*. New York: Sterling.

Figure 2. Use of an archived radio program in a history of mathematics-for-teachers task.

Topic: Historical numbers: The Story of Euler's e

Materials: A Video Lecture, Burger, B. B. (2007). Science and mathematics, Part 2, history of numbers. (Taught by Prof. Edward B. Burger). Chantilly, VA: Teaching Company.

Task: We are going to view the lecture by Burger and review the pages provided in order to write a brief one paragraph story about the evolution of the number e

Figure 3. Use of a video recorded lecture in a history of mathematics-for-teachers task.

Topic: Selected mathematicians: stories, lives and works in school mathematics: Évariste Galois—A play

Materials: (1998).The life and times of Évariste Galois: A play in four scenes by the second-year class of Mariono Moreno School, Argentina. *Mathematics in Schools* (September), 12–13.

Task: Half of the class is going to prepare to act out a play about the life of Évariste Galoise Please review the script provided and prepare to act.

In groups of 4, the other half of the class is going to review the additional materials on Évariste Galoise provided online, at the BBC I player website, and selected pages provided.

Figure 4. Use of plays and drama in a history of mathematics-for-teachers task.

The Life and Times of Functions: f, $f(x)$

Materials: 4 readings that give the time line and mathematicians involved in the life and times of, $f(x)$ are:
(a) Earliest uses of function symbols `http://jeff560.tripod.com/functions.html` ; and (b)The history of the concept of functions and some education implications published in The Mathematics volume 3 Number 2. Also Available at PROQUEST database (c) Historical and pedagogical aspects of the definition of a Function by M A. Malik published in the *International Journal of mathematics education, science and technology, volume 11 number 4, pp. 489–492* also available at PROQUEST Database; (d) Functions (mathematics) at Wikipedia.
[In your groups (in 2011) or at home (in 2007)], please read one of the above articles assigned to you. You may review and complete the chart. We shall take up the chart together as a whole class.

Different meanings Today
What do functions mean for you?

1. ____________
2. ____________
3. ____________

Where is the function in the sequence 2, 4, 6, 8, ⋯⋯?

Functions were not explored by Ancient Math Traditions
But some of their mathematics areas might look like functions. Name three such areas

- ____________
- ____________
- *Cubics and cubic roots*

The **Greeks** studied distances and time but not speed, a relation between these two varying quantities.

Arabs and other earlier traditions did not study motion.

Euclid studied geometry of points, lines and planes by construction, devoid of any motion and formulae

Evolution of functions
The evolution of functions in mathematics is a recent concept dating to the end of the **17^{th} century**

Early notion of Function 17th and 18th century

Galileo (1564-1642)—(in his quantitative study of Nature with **Kepler (1571–1639)** measured quantities and sought to identify patterns and regularities. The study of falling bodies, motions of planets, motion along curves lead to the rigorous study of proportions, polynomial and trigonometric equations

Descartes (1596–1650) and Fermat (1601–1665) introduced analytic geometry—curves in a plane described by equations.

Descartes stated an equation in two variables & geometrically represented it by a curve. He used this to show dependence between variable quantities (x, y). He described curves by motion/locus and formulae.

Newton (1642–1727) —showed functions in infinite series. He devised some terminology: Fluent for independent variable, *relata quantitas* for dependent variables and *genita* for constants

USE-Explore-Define-Apply

D' Alembert (1717–1783)—vibrating string theory; Fourier (1768–1830)—heat flow in material bodies.

Dirichlet (1805–1859)—studied **Fourier series** defined function as a <u>unique correspondence</u> between variables representing numerical sets thus separated it from analytic representation by formulae.

This fact is contested by George Hardy who reviewed Dirichlet's work and did not find any mention of this by Dirichlet.

Dirichlet also worked with functions that could be <u>discontinuous</u> at some points. As well this definition was for a long time rejected for being too broad.

Cantor (1845–1918)—his work on set theory contributed to the definition by correspondences between sets, <u>numerical or non numerical</u>

Caratherdory in 1917 defined function as <u>a rule</u> of correspondence from set A to real numbers.

Functions have origins in intuitive geometry and intuitive **calculus**

To begin with functions were used to designate correspondence between geometric objects e.g. curve and its quantities, the slope, tangent, gradient, the area under a curve, limits, etc.

What other contexts or topics are functions used in

____

Viete (1540–1603) with his influence in the creation of **Algebraic symbolisms** increased expressive possibilities in mathematics.

Graphing calculators and computers might be very helpful when studying functions.

Discuss the assertion in light of the evolution of functions

____

Origins of function terms:
Set theory—Domain-range;
Computing—algorithm, input, output; function machine;
Cartesian geometry— formulae, plots, gradient and graphs
Calculus—_____________
Applied mathematics—tables of values; t-tables; differences;

PTO

Modern function Definitions

The **word function** is traced back to **Gottfried Leibniz (1647–1716) in 1673 (others say 1694)**. He used it in Calculus to describe quantities related to curves (graphs with no corners), specifically the gradient/ slope of a curve at a point.

Bernoulli (1718) defined function as a quantity composed from variable and constant.

More study of curves by the use of algebra and the use of analytical expressions became necessary thus fueling the need for function as a tool.

Leibniz and Jean Bernoulli, in their correspondences, adopted function to further study curves by using algebra

In mid 18^{th} century, the word function would later be used by **Leohard Euler (1707–1793), a former student of Bernoulli** to describe an expression or formulae,

PTO

Hardy in (1908) defined a function in the modern terms as "**a relation between two variable x and y such that to every value of x and any rate correspond values of y.**"

Bourbaki in 1939 defined function as a rule of correspondence between two sets.

Later in computation theory a function came to be understood as a computation.

In computing science as an Algorithm—including recursive functions/ spread sheet functions or non-analytic functions such as the one that maps natural numbers onto 0, 1, 1, 2, 3, 5, 8, ⋯.

Partial functions defined as functions for which some x-values have no y-values defined.

How is a function defined in the curriculum?
Narrowly or broadly?

____

Clearly functions are now formal mathematical objects and physics tools. The function concept moved away from its origin in calculus.

PTO

Algebraic— equations, expressions, or generalizations; **Motion Geometry**— Transformation, mappings; object and image? *I recall learning functions in the context of domain and range, mappings in grade 8, what earliest meaning of functions do you recall learning?* ____________________	**Euler later** proposed an alternative definition to broaden the definition—but this was ignored in favor of the analytic expression definition for the whole of the 18^{th} century. Beginning the 19^{th} century onwards definitions would be elaborated especially by **scientist/mathematics who sought its use in other theories**. **As well** the impetus to formalize of mathematics using set theory during the 19^{th} century helped. It is claimed; **Dirichlet and Lobachevsky** broadened and formalized the definition of a function. Now functions are used in many areas including Dynamical systems—partial differential equations, topology, and probability functions in stochastic models.	*What other Mathematics topics and symbols is **Leibniz** famous for* ____________________ *How many years did it take between early inventions and more formalized acceptance of functions?* ____________________

Lessons for teaching (generated from classroom discussion)
Mathematics concepts evolve over time.
Meanings and representations of concepts vary—some are more elementary than others.
Studying several contexts for functions (e.g., sets, machines, speed and velocity, graphing/analytic geometry, series) might guide us in not mixing meanings of a concept when teaching it.
In mathematics, multiple representations might help get to several meanings of a concept.
Rigor and generalizability were valued during later traditions of mathematics.
Need/or relevance to use a concept influences recognition or acceptance of a concept.
Practical uses of mathematics are important as mathematical and scientific uses of mathematics.

Figure 5. An example of a task to illustrate the origin of multiplicity of meaning of functions.

5 Interpretations

In this section I account for, to the extent that it is possible, the possible catalysts for the differences shown among the course aims, content, teaching and assessment materials over the course of the 5 years. My reflections are first and foremost informed by the literature reviewed and theoretical framework identified. But they are also informed by my experiences teaching the class and by the survey and written feedback solicited from the students at the beginning of the term, half-way through the term and at the end of the term. At the beginning of the course, in 2007 and 2011 I requested students to complete a questionnaire about themselves and their expectations from the course. Half-way through the term I solicit for written feedback on the course, and at the end of the course the students complete university wide course and instructor evaluation. Teacher candidates' abilities and expectations as well as the availability of a variety of useful and captivating resources appear to have been the major impetus for the changes.

The difference between goals was primarily in order to match the program requirements which were introduced since 2011. The goal in 2011 focused on professional judgment in practice. The reduction in course readings and the coverage of more readings in class were another difference accounted for in the 2011 program: Elective courses were to reduce the amount of readings and assignments offered beginning 2010.

Several difference show changes to do with the course itself. The difference in course content reflects a move toward the amalgamation of topics on specific mathematics concepts such as evolution of functions toward general topics such as on the evolution of specific topics. On the other hand, the separation of a topic on mathematicians into mainstream mathematics and other mathematicians was evoked by the need to explore mathematicians that represented various groups of course participants including but not limited to gender, race and age. Also, the introduction of women, mathematicians from other cultures and prodigy mathematicians was intended to introduce the cultural and social justice aspects into the course. The introduction of the new topic historical mathematics problems such as those on prime numbers, equations, matrices, and the golden ratio, was evoked by the possibility of exploring mathematics in historical texts that are of direct relevance to school mathematics. Along these lines of relevance to school mathematics, a new topic on the history of school mathematics was introduced. This topic was intended to make connections between the history of mathematics and pedagogical and curriculum reforms supported by the Ministry of Education and School Boards in the province.

The experiences teaching the course for two years meant that I increasingly, through my explorations as well as my interactions with students in the course and with history instructors, became aware of nontraditional texts and non-traditional approaches in the area. Also, some new texts were published after the first year of the course. For instance, the story of mathematics documentary was produced in 2009. Further, students' ongoing informal feedback during and after classes, and formal written feedback half-way during the course and at the end of the course indicated that teacher candidates, given their overall course load, found less traditional course materials to be more captivating. The inclusion of multimedia and online texts is at the same time accounted for by the increase in non-traditional texts readily available online as well as improvement in classroom technologies available at the university over the past three years.

The differences seen in the assignment materials of the course are mainly due to the three reasons: to reduce on the course load for students, to give choice for the students to explore topics of interest, and to leave room for the students to use formats of interest, especially those that were in line with the non-traditional texts explored in this class and in other mathematics methods classes.

6 Conclusions

My sense is that the changes implemented, since the course was introduced, have made the course more interactive for the students and has given them more choice to explore their topics of interest in formats that are more appealing to them. Several of the students who still find course readings and assignments that are in the format of manuscripts appealing still get the opportunity to use these formats. Most importantly, the results show that the course is evolving from the social role of using history for the purposes of promoting mathematics and improving on its image towards the mathematical role of encouraging teacher candidates to reflect more an understanding of mathematics, its origin and history more deeply (Furinghetti, 1997). The mathematical role of teaching the history of mathematics appears more appropriate for a course in which a majority of the members who elect to take the course are increasingly those that have interest, comfort and strengths in mathematics. Should the course have consisted of majority class members who profess discomfort and lack of strength in mathematics, the course's focus would have been interventions, using the history of mathematics, as described by Kaye (2010). Further, the inclusion of more diverse texts and the amalgamation of topics is supported by the possibility, as noted by Pritchard (2010), of the availability of thoughtful radio, television, and other multimedia texts on the history of mathematics. A major lesson for me from this analysis is that although the course goals remained the same, an instructor's interaction with the available resources and with the students was the key at forming modifications in the written plans of the course.

For further inquiry, it would be interesting to assess the course on the basis of the activities, resources, approaches, foci that have been noted in the literature on the use of history of mathematics in education. Further reflections on this, in addition to a content analysis of the course as planned, could also focus on the course as taught and experienced by the students.

REFERENCES

- Bidwell, J. K., 1993, “Humanize your classroom with the history of mathematics”, *Mathematics Teacher*, **86**, 461-–464
- Ernest, P., 1998, “The History of Mathematics in the classroom”, *Mathematics in School*, **27**, 25
- Fauvel, J., 1991, “Using history in mathematics education” *For the Learning of Mathematics*, **11**, 3-–6
- Furinghetti, F., 1997, “History of mathematics, mathematics education, school practice: Case studies linking different domains” *For the Learning of Mathematics*, **17**, 55-–61
- Furinghetti, F., 2000, “The history of mathematics as a coupling link between secondary and university teaching”, *International Journal of Mathematical Education in Science and Technology*, **31**, 43-–51
- Kaye, D., 2010, “Penetrating the inevitable certainty of error: Using the history of mathematics in teacher training for the post-16 sector” *Mathematics in School*, **3**, 9–11
- Liu, P. H., 2003, “Do teachers need to incorporate the history of mathematics in their teaching?” *Mathematics Teacher*, **96**, 416–421
- Marshall, G., Rich, B. S., 2000, “The role of history in a math class” The Mathematics Teacher, **93**, 704–712
- Mason, J., 1998, “Enabling teachers to be real teachers: Necessary level of awareness and structure of attention” *Journal of Maths Mathematics Teacher Education*, **1**, 243–246
- Mason J., 2002, *Researching your own practice: The discipline of noticing*, London: Routledge Falmer.
- Mason, J., 1994, “Researching from the inside in mathematics education: Locating an I-you relationship” in *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*, J. P. da Ponte, & J. F. Matos, (eds.), Lisbon PT: PME, pp. 176–193.
- Mason, J., Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, **38**, 135–161
- Pope, S., 2010, “The opportunities of the revised secondary national curriculum in England for using the history of mathematics” *Mathematics in School*, 6–7
- Pritchard, C., 2010, “Early ideas on the history of mathematics in the Scottish curriculum for excellence” *Mathematics in School*, 29–33
- Swetz, F., 2003, “Teaching history of mathematics, choosing a text “whither thou goest?” *Primus*, **13**, 165–173
- Wilson, P., Chauvot, J., 2000, “Who? How? And what? A strategy for using history of mathematics” *Mathematics Teacher*, **93**, 642–645