THE CONCEPT OF TANGENT LINE

Historical and didactical aspects in Portugal (18th century)

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ABSTRACT

The concept of a tangent line to a curve is, from the beginning, present in the history of mathematics. The study of the historical evolution of this concept is a source of knowledge and enlightenment that can lead to a better understanding of the concept.

By 1772 Marquês de Pombal reorganized the University of Coimbra, establishing in Portugal the first Faculty of Mathematics in the world. The Statutes that ruled that Faculty stated, in particular, that the concept of a tangent line to a curve was to be taught in the first and second years of the Mathematics degree. The books for these years included the Portuguese translation of Euclid's *Elements* and translations, made by José Monteiro da Rocha, of the work of Etienne Bezout. By then, José Anastácio da Cunha wrote a manuscript "*Principios de Geometria tirados dos de Euclides*" where he strongly criticizes the Euclidean definition of a tangent line, presenting an alternative definition in his treat "*Principios Mathematicos*".

In this article we approached the teaching of the concept of a tangent line in the Faculty of Mathematics at University of Coimbra and we focus on the alternative solution presented by the Portuguese mathematician José Anastácio da Cunha.

Keywords: Concept of tangent line; 18th century; University of Coimbra.

1 The concept of a tangent line in Portugal in the 18th century

1.1 Context

The Faculty of Mathematics at University of Coimbra is the first Faculty of Mathematics in the world, being created in 1772, in the reign of D. José and under the influence of Marquês de Pombal, the Minister of the Kingdom, when this University was reformed. With this reform D. José was aiming to disseminate the knowledge in Liberal Arts and Sciences, creating five Faculties (Theology, Law, Medicine, Mathematics and Philosophy). The Portuguese mathematician José Monteiro da Rocha (1734 – 1819) was the responsible for creating the Statutes for the Faculty of Mathematics, in which it was stated that Mathematics is vital to strengthen both the spirit and the wisdom of mankind; as a consequence the study of Mathematics was made compulsory to mathematicians as well as to every student from the other faculties[1] (see [10], [14], [15]).

The degree in Mathematics was, according to the Statutes, composed by four courses: Geometry, Algebra, Phoronomy and Astronomy, each of them in every of the four years[2].

The textbooks used by that time were portuguese translations of the treats published in other languages, and the faculty teachers were strongly encouraged by the Statutes to write their own treats in Portuguese, so that they were used in classes.

[1] Do Curso Mathematico, 1772, p. 141 – 142.

[2] The distribution of the courses, subjects, teachers and textbooks through the years is presented in table 1 of the Appendix.

The concept of a tangent line, in particular, should be taught in the 1st year, in Geometry, as presented in Euclid's *Elements*[3], as well as in the 2nd year of the degree, in Algebra classes, using Bezout's *Elementos de Analisi Mathematica*, a translation into Portuguese made by Monteiro da Rocha from Bezout's *Cours de mathématiques à l'usage des gardes du pavillon et de la marine* that was published in Paris between 1764 and 1769[4]. José Anastácio da Cunha (1744 – 1787), the Geometry professor[5], also studied the concept of tangent line to a curve in a manuscript entitled *Principios de Geometria tirados dos de Euclides*[6] where he criticized the Euclidean definition. In his master work *Principios Mathematicos para instrucção dos alumnos do collegio de São Lucas, da Real casa Pia do Castello de São Jorge* (see [7]), Anastácio da Cunha offered an alternative definition of the concept.

We, therefore, faced a dichotomy: one the one hand the definitions for a tangent line presented in the textbooks chosen to be used in the newly created Faculty of Mathematics and, on the other hand, the alternative definitions as stated by the Geometry professor José Anastácio da Cunha.

1.2 The concept of a tangent line in the Geometry course

The definition of a tangent line, according to *Elementos de Euclides dos seis primeiros livros* (…) is, as expected, the usual "touch without cut", namely:

> *Elementos III, Def. 2:* A straight line is said it touches a circle or that it is a tangent to a circle when it is at the same level as the circle and it meets the circle without cutting it[7].

As a complement, the propositions involving our concept are the 16 to the 19, in Book III. One may notice however that proposition 16, a theorem, is in the Portuguese edition following the Robert Simson edition which is, in its last part, different of Heiberg's:

> *Elementos* III, 16 (*Theorem*): The straight line that from an extremity of the diameter of a circle raises perpendicularly above the same diameter, will fall out of the circle. And between this straight line and the circle it cannot exist any other straight line; this is the same as saying that the circle will pass between the perpendicular and the diameter, and the straight line that with the diameter makes an acute angle as big as it may be; or also, that same circumference will pass between that perpendicular and another straight line, that does the same perpendicular of any other angle, as small as it may be[8].

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[3] The Portuguese version of Euclid's *Elements* is a translation by João Ângelo Brunelli, dated from 1756, from the Robert Simson's edition of the geometrical books (I to VI and XI and XII).

[4] The first edition of this translation is dated from as soon as 1774 and further editions got a slightly different title, namely *Elementos de Análise*. All translations included in the notes and methods added by Monteiro da Rocha to the original text.

[5] We know that by 1776 José Anastácio da Cunha presented a geometry textbook to the Congregação da Faculdade de Matemática (a governing organ). This Congregação was supposed to have accepted or rejected every textbook presented to them by the professors, but unfortunately we know nothing of the solution given to Anastácio da Cunha's proposal.

[6] This specific manuscript is part of a larger set found recently in Portugal (see [6] and [8]).

[7] This translation, as well as all the others in this article, and the underlines is ours. In this particular case, in the original we can read: *Huma linha recta se diz, que toca hum círculo, ou que he tangente de um círculo, quando estando no mesmo plano do círculo encontra a circunferência sem a cortar* (Euclides, 1768, p. 81).

[8] *A Recta, que de huma extremidade do diâmetro de um círculo se levantar perpendicularmente sobre o mesmo diâmetro, cahirá toda fora do círculo; e entre esta recta, e a circunferência não se poderá tirar outra linha recta alguma; que he o mesmo que dizer, que a circunferência do círculo passará entre a perpendicular ao diâmetro, e a recta, que com o diâmetro fizer hum ângulo agudo por grande que seja; ou também que a mesma circunferência passará entre a dita perpendicular e outra recta, que fizer com a mesma perpendicular hum ângulo qualquer, por pequeno, que seja* (Euclides, 1768, p. 101).

We may conjecture that the author was, with this formulation, avoiding the contingence (cornicular) angle, present in the Heiberg's edition, which was a polemic subject for many years and therefore avoiding a potential problem to students (see [12],[13]).

1.3 The concept of a tangent line in Algebra course

Elementos de Analisi Mathematica, Bezout's work translated to Portuguese for the purpose of becoming a textbook in Coimbra, was divided in two parts "*Elementos de Álgebra*" and "*Elementos de Cálculo*".

In this work we cannot find a definition for tangent line, but the definition of circle is presented in Bezout's *Elementos de Geometria* (see [5]), namely:

> *Elementos de Geometria*, 46: It is called a tangent line that straight line that does not more than to lean to the circle[9].

In *Elementos de Analisi Mathematica* the concept of a tangent line is not defined but it is studied in its relationship with the conic sections. Since no definition is presented, we assumed that a generalization of the definition presented in *Elementos de Geometria*, which is similar to the definition presented in Euclides' *Elementos*, was considered.

The study of the concept of tangent line started with the presentation of a method for drawing and for determining the length of the subtangent. Here we may find that the tangent line was considered as the line that touched the curve in a single point (proposition 298):

> *Elementos de Álgebra*, 298: To draw a tangent line to any point M from the ellipse, produce the vector radius fM to G, such as $MG = MF$; and taking GF, produce by M the perpendicular MT to it by M, which is the tangent, this is, will meet the curve only at the point M.

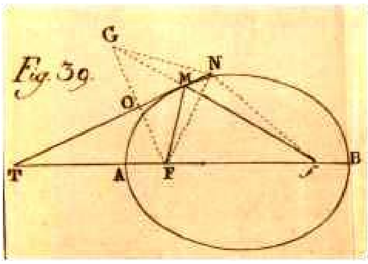

Fig. 1 – Construction of the tangent line from an ellipse.

> ***Proof:*** Suppose MT is not the tangent line, this means that it meets the curve in another point N.
> By the property of the ellipse $FN + Nf = FM + Mf$, or (by Euclid I, 4 e 5) $NG + Nf = Gf$ which is absurd (by Euclid I, 20).
> Therefore, the point N does not belong to the ellipse.■

In addition to this direct method to determine the tangent line to the ellipse the author presents a second method to draw the tangent line, determining the length of the subtangent. Similar methods are presented for the parabola and the hyperbole.

The concept of a tangent line is approached again when, in the second part of *Elementos de Analysi Mathematica,* the differential calculus is presented. In this part, it is stated a general method to draw the tangent line to a curve. This new method, presented in proposition 30, stands from the concept of

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[9] *Chama-se tangente aquela que não faz mais do que encostar-se à circunferência* (Bezout, 1817, p. 25).

difference[10] (or “differential” or “fluxion”) and uses the similarity between the infinitesimal triangle $[Mrm]$ and the finite one $[TPM]$ to prove that the subtangent PT is given by $y\frac{dx}{dy}$ where *dx* and *dy* are the differences between the abscissa and ordinate, respectively.

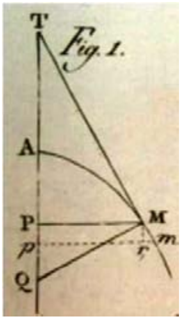

Fig. 2 – Construction of the tangent line using the infinitesimal triangle.

Several examples are presented, such as the conic sections, all genders of parabolas and a logarithmic curve.

The tangent line is not only determined by points on the curve but also by points outside the curve, being presented a general method to draw it.

Elementos de Cálculo, 32: To draw a tangent line to any curve from a point outside the curve[11].

Proof: Consider the curve AM and the point D outside the curve for which it is required to draw the tangent line.

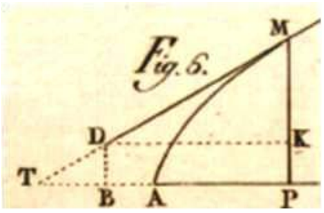

Fig. 3 – Construction of the tangent line to a curve from na outside point.

Considering the abscise $AB = g$, the ordinate $BD = h$ and if M , the contact point, has as coordinates $AP = x$ and $PM = y$ then, because the triangles $[TDB]$ and $[TPM]$ are similar, comes that:

$$\frac{TB}{BD} = \frac{TP}{PM} \Leftrightarrow (y-h)\frac{dx}{dy} = g + x \blacksquare$$

The following propositions studied the properties of the tangent line in curves depending on other curves, stating different methods for the determination of the tangent line for all types of curves, presenting as examples different types of Cycloids, Archimedes’ Spiral and Nicomedes’ Conchoid.

From proposition *42* to *46* the concept of a tangent line is used to state a method of *Maximis & Minimis*[12] where vertical tangent lines are accepted. Multiple points are also studied, being presented examples for the existence of more than one tangent line at a point. The study of tangent lines is

[10] Differential is the difference between two consecutive moments of a varying quantity (Bezout, 1774, vol. 2, p. 10).

[11] *Tirar uma tangente a qualquer curva de um ponto dado fora dela* (Bezout, 1774, vol. 2, p. 34).

[12] Bezout, 1774, vol. 2, p. 46 – 49.

finally concluded with the analysis of inflection points where one finds an example of the tangent line cutting the curve.

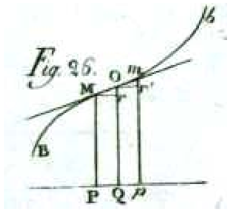

Fig. 4 - Tangent line to a curve at inflections points.

The concept of a tangent line is presented in a contradictory way in Algebra where, in the first part of the textbook, it is assumed as the "touch without cut" principle of a tangent to a curve , or that the tangent cuts the curve at a single point; and, in the second part, there are examples where the tangent line cuts, effectively, the curve at the tangency point and curves that it has more than one point in common with the tangent line, without any reference to a different definition for the concept of tangent line.

1.4 The concept of a tangent line in Anastácio da Cunha's *Principios de Geometria tirados dos de Euclides*

In the manuscript *Principios de Geometria tirados dos de Euclides*, it was thought to have been composed by Anastácio da Cunha in 1778. There one may find profound reflections about geometry, its teachings, its peculiarities and many historical details concerning, in particular, the importance of definitions.

In this work the author criticizes the lack of generality in definitions, using as an example the definition of the tangent line, referring the traditional definition, "touching without cutting", according to which several curves would not have a tangent line.

> The lack of generality in definitions has also been the cause of remarkable embarrassments and errors, and still remain to prove several relevant theories. For example: all that followed Euclides have put the contact of two lines in meeting without cutting each other, and did not notice the several lines (and common lines), which due to that do not admit tangents. In fact, the well known algebraic and fluxionary theorems solves this problem; but to do so it is necessary first to reject the imperfect definition, that contradicted them[13].

Anastácio da Cunha comments, as well, the proposition 16 from Book III of Euclid's *Elements* (where, in Heiberg's edition, the angle of the semicircle is said to be greater than any acute angle, and the remaining angle less that any rectilinear angle), referring that the angle between the circle and the tangent line should not exist because it goes against the definition 8 of Book I[14] of the same treat.

In this manuscript it is not presented the solutions to the detected problems.

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[13] *Falta de generalidade nas definições também tem sido causa de notáveis embaraços e erros, e de estarem ainda por demonstrar varias relevantes theoricas. Por exemplo: todos, seguindo Euclides, põem o contacto de duas linhas em concorrerem sem se cortarem mutuamente, e não reparam na affinidade de linhas (e linhas vulgares), que por esse modo não admitem tangentes. Na praxe os sabidos theoremas algebraicos e fluxionarios emmendam este defeito; mas para se adoptarem he necessário primeiramente rejeitar a imperfeita definição, que os contradiz* (Cunha, 1778, p. 16).

[14] *Elements* I, Definition 8: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line (Heath,1956, p. 176).

1.5 The concept of a tangent line in Anastácio da Cunha's *Principios Mathematicos*

The exact title of Anastácio da Cunha's main work is ***Principios Mathematicos*** *para instrucção dos alumnos do collegio de São Lucas, da Real casa Pia do Castello de São Jorge*, which means that we are dealing with a textbook expressly written to be used for teaching mathematics to the pupils of Collegio de São Lucas in Lisbon, where there were, at the time, 2 different classes of teaching (an elementary one and a so-called scientific, aiming to prepar pupils to enter the university of Coimbra or abroad). This treat, with 300 pages covering a vast range of mathematics fields, came to be finally published, in 1790, two years after Anastácio da Cunha's death.

The first time the concept of a tangent line appears in *Principios Mathematicos* is in the second book, where the third definition is Anastácio da Cunha's alternative definition to our concept. Anastácio da Cunha presents a general definition for tangency that he applies in his work to tangent lines to curves and to tangent curves, namely:

> *Principios Mathematicos* 2, Definition III: If the sides of an angle meet at the vertex such as it is not possible to draw two straight lines between them it is said that the sides are tangent to each other[15].

This definition settled in the new definition of angle has already been presented in the first book.

> *Principios Mathematicos* 1, Definition VII: Angle is the figure that two competing lines form at a point[16].

The concept of a tangent line is used again in proposition *VI* of the second book[17] where, in the second corollary, Anastácio da Cunha states that the tangent line to a circle is perpendicular to the diameter through the contact point (the proof, left as an exercise according to the authors' methodology for the whole book, uses the fact already proved that if the angles *LIN* and *LMI* are equal then *IN* is a tangent line to the circle).

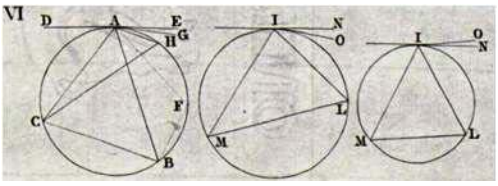

Fig. 5 - – Proof of the perpendicularity between the tangent line and the diameter through the contact point.

The method to draw a tangent line to a circle from a point outside the circle is given in the third proposition[18] of Book 7, being the method presented in Euclid's *Elements*.

Anastácio da Cunha has criticized the 16th proposition of Euclid's *Elements*, namely the problems with the curvilinear angle. In his opinion, and since the only definition of angle presented in the Euclides' *Elementos* was definition 8 of Book I, the concept of curvilinear angle could not exist. To solve the problem with this proposition, Anastácio da Cunha defines curvilinear angle using the concept of tangency already presented.

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[15] *Se os lados de hum ângulo concorrerem no vértice, de sorte que deste se não possão tirar duas rectas entre elles, dir-se-ha , que cada hum dos lados he tangente ao outro, ou, que o toca no dito vértice* (Cunha, 1790-1987, p. 13).

[16] *Ângulo é a figura, que duas linhas formam concorrendo em hum ponto. Este ponto chama-se Vértice; e as linhas lados* (Cunha, 1790-1987, p. 1).

[17] Cunha, 1790-1987, p. 15 – 16.

[18] Cunha, 1790-1987, p. 86.

Principios Mathematicos 15, Definition VIII: Quantity of a curvilinear angle is the angle that straight lines do when touching the sides[19].

In book 17 Anastácio da Cunha returns to the study of the tangent line, presenting, in the first proposition[20], a method to draw a tangent line to a conic. In this case we can see the author using the *fluxionary calculus* together with the fact that the subtangent is given by $y\frac{dx}{dy}$. This method is also used to determine the tangent line to Huygens Cycloid[21] (*Principios Mathematicos 17, Proposition* IV) and to draw tangents to an algebraic curve at multiple points[22] (*Principios Mathematicos 17, Proposition* V).

In this treat, Anastácio da Cunha presents a general definition of tangency that has as objective avoiding the lack of generality of the Euclidean definition for tangent line.

2 Final Remarks

The concept of a tangent line was considered, for many centuries, as a “touch without cut” line. This definition, probably very much in use within our teaching/learning strategies of the present times, was also the one taught to students at the Faculty of Mathematics in the University of Coimbra by the end of the eighteenth century. The differential calculus, which was also studied at the University, fomented the appearance of tangent lines that may, in fact, cut the curves[23] going against the usual definition. This contradiction was not referred to in the textbooks used in the Faculty but the professor and Portuguese mathematician José Anastácio da Cunha confronted the “conflict” in his master treat, presenting an alternative definition to our concept, unifying, in addition, Geometry and Differential Calculus. The solution presented by Anastácio da Cunha is very much tuned with the standards of rigor that he got us used to, that is, we are faced with a definition that is, in our opinion, by all means remarkable: it is a skillful definition for the tangent line that being geometrical is also consistent with the use of the differential calculus.

Acknowledging the evolution of our concept of a tangent line in Portugal enables us to better understand the teaching of the concept as we perform it nowadays and to prepare for future research using new methods, new materials, new examples all “imported” from old/renewed mathematics.

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[19] *Quantidade de um ângulo curvilíneo he o ângulo que fazem as rectas que no vértice tocam os lados* (Cunha, 1790-1987, p. 229).

[20] Cunha, 1790-1987, p. 236.

[21] Cunha, 1790-1987, p. 239.

[22] Cunha, 1790-1987, p. 240.

[23] We are, in the case of dealing with tangents within the field of Differential Calculus reporting, as well, to a situation very much common to nowadays classes.

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APPENDIX

Table 1 – The Mathematics Degree in Faculty of Mathematics at University of Coimbra

Year	Course	Subject	Teacher	Textbook
1st	Geometry	Elements of arithmetic, geometry and trigonometry	José Anastácio da Cunha	*Elementos*, Euclides *Elementos de Arithmetica* and *Elementos de trigonometria plana*, Bezout
2nd	Algebra	Literal calculus, differential and integral calculus	Miguel Franzini	*Elementos de Analisi Mathematica*, Bezout
3rd	Phoronomy	Movement science applied to all branches	José Monteiro da Rocha	*Tratado de Mecânica*, Marie *Tratado de Hidrodinamica*, Bossut *Optica*, La Caille
4th	Astronomy	Movement of the stars, the practice of calculation and the observations	Miguel Ciera	*Astronomia*, Lalande