THE PLACE OF GEOMETRICAL CONSTRUCTION PROBLEMS IN FRENCH 19th CENTURY MATHEMATICS TEXTBOOKS

Guillaume MOUSSARD
IUFM du Mans - IREM des Pays de la Loire, Le Mans, France
Guillaume.Moussard@univ-nantes.fr

ABSTRACT

The place dedicated to construction problems in 19th century French textbooks shows a striking disparity. In fact, it turns out that the destination, and the notion itself, of the geometrical construction problem is radically different from one textbook to another. This article surveys a certain number of characteristic examples of such textbooks. For each one of them, the analysis ends up unveiling some of the author's fundamental conceptions about geometry and geometry teaching.

1 Introduction

At the dawn of 19th century, in 1794, Legendre writes his *Éléments de Géométrie* as a comeback to Euclid's rigour and exactness. Though, this strong reference to the Ancients is contradictory on the following point : the place given to construction problems is radically different from what occurs in Euclid's text. They are no longer part of the rational knowledge of the figures that constitutes geometrical truth, and this will remain true in most 19th century textbooks. Then, what place do geometrical constructions hold in those textbooks? For what reasons and with what intentions do authors integrate construction problems? These questions are analysed through seven different, and at the same time characteristic, textbooks of the 19th century. First of them, the *Elements* from Legendre, as we just mentioned, focus the geometrical truth on the discourse on the figures, and put aside, as far as possible, the construction problems. On the contrary, Lacroix brings them back into the discourse, for innovative pedagogical reasons. A glance to the primary degree reveals, in the 1830's and for the sake of industry, the introduction of geometry classes consisting in reproducing geometrical figures. This leads to the edition of specific textbooks in which construction problems are torn between the pursuit of material precision and the initiation to geometrical reasoning. On the other end of the scholarship, the entrance exam to the Ecole Polytechnique includes questions about construction problems, as can be seen in a published collection of given subjects. Actually, geometrical construction problems have taken an independent place in many textbooks since the curriculum is officially scattered in a list of items to be studied. In the work of Briot, they are dedicated a specific chapter. Anyhow, the crucial question of the methods available to solve construction problems is approached later in the century, and two textbooks focusing on this question will be presented.
As can be seen, the textbooks chosen correspond to different contexts and levels, nevertheless they all deal with school geometry teaching. And our analysis will, for each one of them, tell much about the underlying conception of geometry and of geometry teaching.
All the translations of historical texts given here are mine. In case of any doubt, please refer to the original texts.

2 The reference work : Euclid *Elements*

Euclid's *Elements* keep being a very strong reference in elementary geometry along the 19th century. In particular, it is evident that all the authors we will read know this book perfectly well. It is based on a logico deductive structure, whereby each new proposition is deduced from prior knowledge. So that all the propositions are deduced from a couple of basic assumptions, namely axioms and postulates. The propositions are of two types : the *theorems* are propositions that establish properties of given figures, they constitute the discourse on the figures, while the *problems* ask something to be done, that is, a construction fulfilling given conditions. Consequently, in Euclid's work, the constructions are a type of proposition, as are theorems. All the constructions are deduced from the three basic assumptions :

POSTULATES.

Let it be granted:

1. **That a straight line may be drawn from any one point to any other point.**
2. **That a terminated straight line may be produced to any length in a straight line.**
3. **That a circle may be described from any centre, at any distance from that centre.**

All along Euclid *Elements*, theorems and problems are interlinked, in the sense that a theorem about a figure shall only be expressed if the figure was first constructed, and conversely, a theorem shall be proven with the aid of constructions justified before. Geometry is then a rational body of propositions that tell on the one hand how to construct geometrical figures (problems) and on the other hand the properties of these figures (theorems).
Now, in this context, let's see how 19th century textbooks deal with construction problems.

3 Legendre

Legendre writes elements of geometry in 1794, after a long experience of teaching. His intention is to write elements that would be as perfect as possible in rigour and clearness. In this, he refers himself to the method and rigour of the Ancients, particularly Euclid. Though, there are no problems in his elements until the end of book II, which is a complete break with Euclid's text. According to Legendre, geometry is a discourse on the figures constituted only by theorems, and problems of construction are whether corollaries of these theorems, or means to reach their demonstrations.

We give here, to help the general understanding of the paper, a list of problems given in Legendre's *Elements*, which are also problems present in Euclid's *Elements*.

Problems related to the two first books :

- To divide the given line AB in two equal parts
- Through a point A, given on the line BC, draw a perpendicular to this line
- From a point A, given outside the line BD, draw a perpendicular to this line
- At the point A of the line AB, make an angle equal to the given angle K
- To divide a given angle or arc in two equal parts
- Through a given point A, to draw a parallel to the given line BC
- Being given two sides B and C of a triangle and the angle A they include, to draw the triangle
- Through a given point to draw a tangent to a given circle
- To inscribe a circle in a given triangle ABC

Problems appended to book III :

- To divide a given strait line in as many equal parts as required, or in parts proportional to given lines
- To find a fourth proportional to three given lines A, B, C
- To find a proportional mean between two given lines A and B
- To make a square equivalent to a given parallelogram or triangle

4 Lacroix

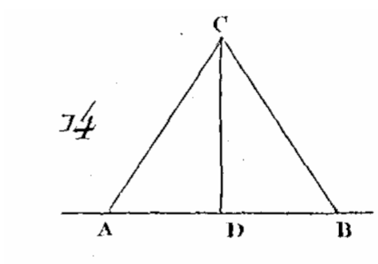
Lacroix has, at the turn of the 19th century, a strong experience of teaching and is very much involved in post Revolution mathematics education. He writes in 1805 an essay to expose his views on education, in which he takes up an explicit position on the place to give to constructions in mathematics textbooks. As for Legendre, problem is here synonymous of geometrical construction. Lacroix writes (1838, p. 298.) :

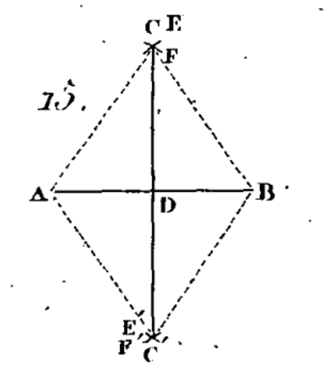
> The permanent custom of proposing problems to the pupils made me conscious of the inconvenient there would be in presenting a whole section of theorems, and report later, the problems that are their continuation. This arrangement, for the less singular, not to say more, that brings out the problem when the theorem on which it is based, and which it would have enlightened or confirmed, deprives the reader from the means to construct his figures with some care ; and although I know as well as anybody, that it is on the rigour of the reasoning, rather than on the exactness of figures

that is based the geometrical truth I nonetheless believe that the practice of the drawing is not less necessary in geometry, than that of the calculus in arithmetic, for the multiple usages of the former science depend on the construction of the figures, as those of the latter, on the practice of the rules [...] Guided by these motives and by experience, I placed the problems as soon as they resulted from the theorems, or were necessary for the construction of the figures ; and I always observed that this order was the most convenient for all minds.

As Legendre, Lacroix states that the discourse is the place where stands geometrical truth. Nevertheless, he attributes to constructions a main role as being the proper activity of those who employ geometry. Besides, his experience of teaching provides him with the intuition that problems enlighten and confirm the theorems. Consequently, he opposes explicitly to Legendre's exposition of theorems one after the other, and asserts that the problems must be placed in the textbooks along with the theorems on which they are based, or that they enlighten or confirm. This idea is not much developed by Lacroix, but it is very interesting from the pedagogical point of view.
Lacroix had actually written, in 1799, his own *Éléments de géométrie* (1808). An example will help us interpret what he means when saying that a problem "results from a theorem" and is "necessary for the construction of figures". There is, first, the theorem (Lacroix 1808, p. 15):

> Theorem. Lines AC and CB, fig. 14, that go from a given point C of line CD, perpendicular on AB, and that divert equally from the foot of this perpendicular, i.e. from point D, where it meets line AB, are equal [...]
> Corollary. [Line CD] has all its points equally distant from points A and B;





The theorem leads to the property that the perpendicular raised on the middle of a line AB has all its points equally distant from A and B. This property brings out the procedure that permits to draw a perpendicular on any given line.

> Problem. Draw on line AB, fig 15, a perpendicular that divides it in two equal parts.
> Solution. From points A and B, taken successively as centres, and with a compass opening greater than the half of AB, will be described two arcs, CE and CF, that cross on a point C. The same thing will be done below AB; and joining points C and C', the line CC' will be the requested perpendicular.

The problem results from the theorem, and is necessary for the construction of any figure involving the draw of a perpendicular.

For Lacroix, geometrical truth is embedded in the discourse, but geometry's destination is the construction of the figures. Now the place of the problems is dictated by the pedagogical intuition that problem solving enlightens and confirms the theory. The following example shows that, in Lacroix's book, constructions are propositions that assume a pedagogical role. For this problem, Lacroix gives three solutions. This can be explained as each solution enlightens a different theorem. On the left side below stand the theorems, and and the right side the different solutions of the problem, each corresponding to the use of a different theorem (Lacroix, 1808, p. 38):

64. *Theorem.* When two triangles ABC and EDF have their angles equal one to another, their homologous edges are proportional, and they are consequently similar.	68. Problem. To draw on a given line EF, a triangle similar to triangle ABC.
65. *Corollary.* It follows from former proposition, that two triangles are similar, when they have only two equal angles one to another, for the third angle of the first one is necessarily equal to the third angle of the other one ;	*Solution.* It can be drawn a triangle similar to an other one, starting from the various characteristics by which the similarity of these figures may be established. If it is then requested to form on line EF, a triangle that be similar to triangle ABC, it could be achieved, 1°. drawing through points E and F, lines that make with EF angles E and F, respectively equal to angles B and C (65);
66. *Theorem.* Two triangles are similar when they have an angle equal one to another, situated between proportional edges.	2°. Drawing on point E, on line EF, an angle equal to angle B, and bringing on edge DE of this angle a distance DE fourth proportional to the three lines BC, EF, AB ; this way, the two triangles will still be similar, having, one to another, an equal angle placed between proportional edges (66);
67. *Theorem.* Two triangles that have proportional edges, one to another, are similar.	3° At last, searching for a fourth proportional to the three lines BC, EF, AB, an other to the three lines BC, EF AC, and drawing on the two resulting lines and on EF, a triangle DEF ; the triangles DEF and ABC will be similar, having their edges proportional (67).

The three solutions given by Lacroix to problem 68 refer explicitly to the three theorems 65, 66 and 67. This shows that the problem is placed here not only because it results from the theorem, but also to give sense to each of the three theorems by showing how they may be useful in the application of geometrical knowledge.

5 The *dessin linéaire*

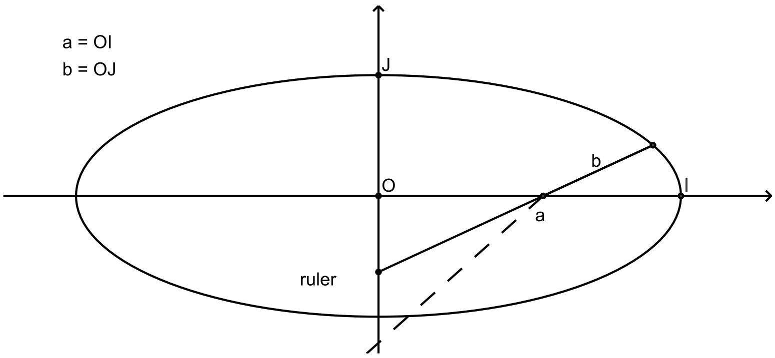
The *dessin linéaire* consists in drawing geometrical figures, first without instruments, and then with the instruments. This latter part concerns directly our subject as it is nothing else than geometrical constructions. The *dessin linéaire* is taught from the years 1820 in primary schools in France for the needs of industry and craftsmen. It appears in the official curriculum from the 1830's up to 1850. During this period, many textbooks are printed on that subject. It is interesting to note that many of them mix two different objectives, that are :
- to reach good precision in the drawing, from the concrete point of view, which is the primitive objective of this teaching. The geometry involved here consists in a set of rules saying how to use the instruments to obtain certain figures.
- to prepare to the geometrical reasoning. This objective is explicitly defended on the argument that to draw and then to construct the figures is a fruitful preliminary to the elements. This point is actually contested when the *dessin linéaire* is officially introduced in the primary school programs, and some basic elements of geometry are eventually placed ahead in the curriculum.
It should be noticed that these two conceptions are reverse one from another : the first one goes from the rational discourse to the constructions, and the other one from the constructions to the rational discourse. This questions us as mathematics teachers : should we go from practice to theory or from theory to its applications? What is of particular (historical) interest here is that many authors, like Lacroix in the preceding chapter, have an empirical approach : they *observe* that going from practice to theory seems fruitful.
Here is, as an illustration, an extract of the introduction to the *Cours méthodique de dessin linéaire* from the schoolteacher Lamotte (1832):

> 1. The linear drawing is, in a large sense, the art of representing the different bodies with the help of mere lines. That kind of drawing is based on the principles of geometry, and is mainly aimed at representing industrial arts productions, machines, etc., etc. [...] The experience showed that pupils whose eye and hand were exercised with linear drawing were improving fast in the drawing of the figure, and obtained a marked advantage on their mates in mathematical constructions. All industrial professions need the linear drawing, the workers to do their job, the workshop chiefs to prepare it. Which is the man, even in the highest social position, who does not need in a thousand occasions to pass on clearly his thought to an architect, a builder, a cabinet-maker, or any other worker, with a figure, with a quick draw? Long explanations, unintelligible for the worker, are replaced so much advantageously by a linear drawing ! But in this case the linear stroke does not need to be executed with mathematical precision, it might be an indication, an approximation. The accuracy, that is not essential here, must be supplied with rapidity. If on the contrary a master prepares his workers' task, measures must be perfectly exact. It is only with the instruments of geometry that we may reach a convenient degree of precision. We will consequently distinguish two kinds of linear drawing, the

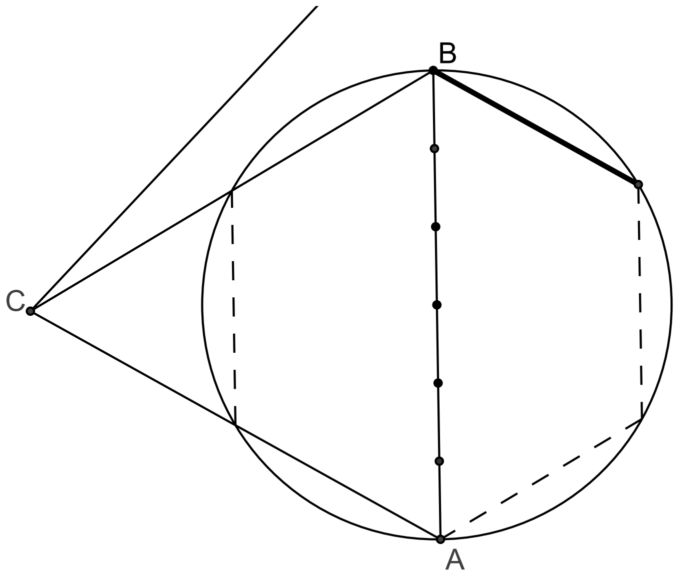
on-sight drawing or without instruments, and the geometrical drawing or with instruments. These two kinds of drawing are equally utile, depending on the circumstances. They shall be studied with same care.

To illustrate the paradox mentioned before between the two objectives aimed at by such textbooks, namely the precision of the drawing and the preparation to the geometrical reasoning, let's detail two items of the textbook.

- Lamotte proposes the following method to check an ellipse drawn by a pupil : Mark, from one end, on the ruler, the lengths a and b (parameters of the ellipse) and make the corresponding points coincide with each of the axes; the end of the ruler must then coincide with the ellipse.



- Later comes a general construction to divide a circumference in as many divisions as requested.



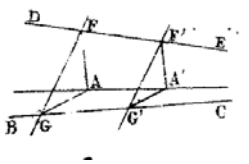
"From ends A and B of diameter AB, and with a compass opening equal to AB, describe two arcs that cross in C. ACB is an equilateral triangle. Divide the diameter in as many parts as you want divisions, and draw from point C a line that will pass through the second division of the diameter, and will reach the circumference, the chord of the intercepted arc is the side of the requested polygon"

Now if these two constructions give satisfaction from the practical point of view in many cases, one of the two is, in general, wrong from the deductive point of view. The reader probably knows which one?

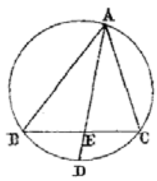
6 Construction problems as questions to be solved

Until the years 1870, the admission exam at the École Polytechnique is mainly an oral examination about mathematics. If analytic geometry, algebra and trigonometry are mostly represented, some subjects deal with constructions as we can see in the examples below. They are taken from a collection of questions given at the exam edited by Lonchampt, supervisor of the École Polytechnique (Lonchampt, 1865, p. 6):

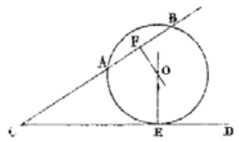
8. — Par un point A mener une droite qui passe par le point de concours de 2 droites BC, DE que l'on ne peut prolonger.



9. — Construire un triangle ABC, connaissant la base BC, l'angle opposé A et le rapport $\frac{m}{n}$ des deux autres côtés AB et AC.



10. — Décrire une circonférence qui passe par 2 points donnés A et B et qui soit tangente à une droite donnée CD.



Geometry is, in this context, the ability to work on any new question and find its solution or demonstration. That conception is challenging for the teacher : how to prepare somebody to work on any question ? Two answers can be found in 19th century textbooks. The first one consists in intensive training. Many collections of problems are edited in this intention. The second one consists in rationalizing the problem solving activity, i.e. to identify and describe general methods to solve whole classes of problems. We will come back on this point more specifically about construction problems in chapter 8.

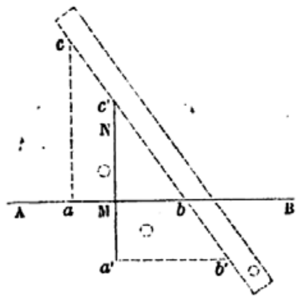
7 Construction problems as part of the curriculum

French Revolution settles a public instruction minister, in charge of defining the official program to be studied in schools. At first, the program simply refers to textbooks, in fact those of Lacroix (1808) and Legendre (1794) for elementary geometry. But from the 1830's on, the program is wholly detailed in a list of various items. From then on, textbooks are, conversely, written in accordance to the program. As a consequence, the knowledge is scattered in a succession of items, and is no more part of a coherent body. In particular, the axiomatico deductive structure vanishes. Concerning the construction

problems, they are departed from the rest and constitute somehow an independent part of elementary geometry. Besides, the paradigm of constructions is clearly enough that of practical applications.
Briot, teacher at the lycée Saint Louis in Paris, writes as an introduction to the chapter about geometrical constructions in his *Éléments de Géométrie* (Briot, 1865, p. 61.):

> Up to now [in the textbook], the figures that were used in the demonstration of theorems, were drawn with the hand only, with little precision ; to represent roughly the figure, to help the mind follow the reasoning. But, in practice, when it is asked to determine with precision, whether the position of a point, whether an unknown magnitude, the graphic constructions must be done on the paper with great exactitude.

The text says clearly that geometrical truth belongs to the discourse, and that the constructions are only necessary to obtain precision in the practical applications.
Moreover, the construction procedures are freed from the restrictive frame of the Euclidean postulates (see p. 2) which request that any construction be drawn out of straight lines and circles. This allows new procedures and new instruments.
As example, here is the problem to rise a perpendicular to a given line AB on a given point Briot proposes after a classical method, another one that consists in sliding a set square along a ruler.

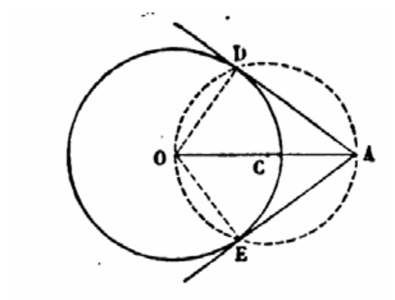


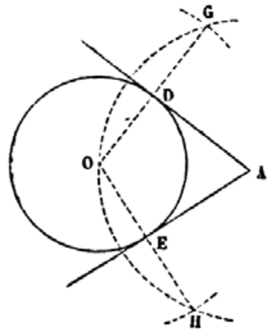
This is, in comparison to what Lacroix and Legendre wrote, a new instrument, as well as a new procedure as it involves movement.
Another example shows that what is searched for is the simplicity of the material realisation of the figure. Briot gives two different constructions to draw a tangent line to a given circle of centre O through a given point A.

The first one has two steps :
- draw the midpoint C of line OA
- draw the circle of centre C through O

The second one has four steps :
- draw the circle of centre A through O
- give the compass an opening twice the radius of the given circle
- draw the circle of centre O with the obtained opening
- connect O with the intersection points of the two drawn circles

Then Briot says the second construction is simpler, as it avoids the construction of the midpoint of line OA. The idea of simplicity involved here is the simplicity of the practical realisation of the figure, not the simplicity of the rational deduction of the solution to the problem.





8 Construction problems and methods

The question of the method has long been salient in solving construction problems of elementary geometry. In his essay of 1805, Lacroix writes (Lacroix, 1838, p. 299):

> Besides, the choice of geometry problems is more embarrassing than that of analysis problems, because the latter depend only on a small number of rather general methods, having together evident connexions, although the former require various constructions difficult to devise.

Indeed, the first half of 19th century sees a revival of synthetic geometry, in opposition with the analytic geometry of coordinates. In particular, new methods are developed. Consequently, some authors undertake to fulfil the long time noticed lack of general methods in synthetic geometry, notably in solving construction problems. The book of Julius Petersen (1880), the translation of which was successful in France, is of particular interest here. The preface says :

> Although modern mathematicians have not ceased to be interested in this branch of science [construction problems], the means to treat rationally this class of problems have developed relatively less rapidly [...] This situation has been the cause that many people have considered geometrical construction problems as kinds of enigmas whose solution could barely be attempted only by few minds gifted with very special faculties. It resulted that these questions have hardly penetrated schools where though they should naturally have been cultivated ; for there exist no problems that sharpen as well the observation and combination faculty and give the mind clearness and logic ; there are none that present as much attraction for the pupils. This book aims at teaching them how one should tackle a construction problem.

The practice of construction problems, writes Petersen, sharpens the mind, giving it clearness and logic. For him, construction problems have a main role to play in mathematics education. The book then exposes a range of general methods in solving construction problems, and tries to identify the class of problems solvable by each method. In this it is rather an innovative book. For example, Petersen (1880, p.71) develops a quick theory on rotation (we would say similarity) and immediately follows

> 4- By the mean of propositions we just developed, we are able to solve the following general problem : To place a triangle, similar to a given triangle, so that one of its vertices be on a given point and the two others be on two given curves.

Then many examples of applications of this general method are given. They begin with [1] :

> 349. To place an equilateral triangle, so that its vertices be on three given parallels.
> One of the vertices might be put on any point of one of the lines ; let it be the centre of rotation ; $\tau = 60°$, $f = 1$.
> 350. To place an equilateral triangle so that its vertices be on three given concentric circles.

Another example is the textbook from frère Gabriel Marie, member of a congregation devoted to scholar education that edited a complete collection of mathematics textbooks in the second half of 19th century. The textbook mentioned here is a collection of a large number of questions to be solved. It begins with an important part exposing various methods, that are referred to along the questions treated in the body of the text. One of the methods presented is the similarity [2]:

> To solve a problem with the assistance of similarity, one constructs a figure similar to the given one, and one compares a single dimension to its given homologue. This is to be done especially when the problem requested depends on a single given line.

The method is illustrated with the following problem :

> 207. Problem. To construct a square, knowing the sum or the difference of its diagonal and its edge.

This problem may be solved by the mean of a synthetic, rather elaborated, construction, as it is done in another place in the textbook. But the solution proposed here follows from the observation that the problem depends on a single given line. It suffices to draw any square, to construct the sum, or difference, of its diagonal and its edge, and finally to enlarge or reduce the whole figure to make the obtained line coincide with the given.

These two textbooks highlight the importance of the method. They say that the ability to work on any new question in geometry rests on the knowledge of various methods together with the skill to use them adequately.

9 Conclusion

Throughout the various textbooks surveyed here, it appears that depending on the conception one has of geometry, and on what are the pedagogical objectives, the status and the role attributed to the geometrical construction problems can be totally different. From an epistemological point of view, Legendre considers these construction problems as corollaries, or lemmas of the discourse on the figures. Lacroix, on the contrary, claims they are the final outcome of geometry. Other authors consider them whether as precise drawings, or theoretical questions to be tackled.

[1] The problem 350 was given in the *concours général des collèges de Paris* in 1814 and the students renounced, while the problem given in replacement does not seem simpler to us.

[2] FGM, 1896, p. 100.

From a pedagogical point of view, construction problems are not only mere items of the curriculum, since various authors consider them as a pedagogical tool. Lacroix says they enlighten the theorems, many primary degree teachers remark that drawing geometrical figures improves the geometrical reasoning, FGM and Petersen employ them to illustrate general methods, and the latter even claims that the practice of these problems develops mathematical faculties more than any other type of problems.
Thus, even deprived from their deductive necessity in the discourse, geometrical construction problems keep occupying a determinant place in main textbooks. Certainly because, as writes R. Bkouche, they "guaranty the coherence between the theoretical discourse and the matter-of-fact situation"[3].

REFERENCES

- Briot, Ch., 1865, *Éléments de Géométrie*, Paris: Hachette, 5e édition.
- FGM, 1896, *Exercices de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, Paris: Poussièlgue, 3e édition.
- Lacroix, 1838, *Essai sur l'enseignement*, Paris: Bachelier, 4e édition.
- Lacroix, 1808, *Éléments de Géométrie*, Paris: Courcier, 7e édition.
- Lamotte, 1832, *Cours méthodique de dessin linéaire*, Paris: Hachette, 2e édition.
- Legendre, 1794, *Éléments de Géométrie*, Paris: Firmin Didot.
- Lonchampt, A., 1865, *Recueil de Problèmes*, Paris: Gauthier-Villars.
- Petersen, J., 1880, *Méthodes et théories pour la résolution des problèmes de constructions géométriques*, Paris: Gauthier-Villars, réédition Jacques Gabay, 1990.

[3] Bkouche, Rudolf, *La géométrie entre mathématiques et sciences physiques*, http://michel.delord.free.fr/rb/indexrb.html, p. 14.