THE PERFECT COMPASS: CONICS, MOVEMENT AND MATHEMATICS AROUND THE 10^{TH} CENTURY

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ABSTRACT

Geometry instruments certainly exist since men are interested in mathematics. These theoretical and practical tools are at the crossroads of the sensible world and mathematical abstractions. In the second half of the 10^{th} century, the Arabic scholar al-Sijzī wrote a treatise on a new instrument: the perfect compass. At that time, several other mathematicians have studied this tool presumably invented by al-Qūhī. Many works are now available in French and English translations. After an historical presentation of the perfect compass, this article deals with a few passages of al-Sijzī's treatise which show the importance of continuous tracing of curves and provide interesting elements on the role of instruments in the mathematical research process. All these texts can help to understand the importance of motion in geometry which can be easily simulated by geometry softwares and used in a geometry lesson or in teacher training sessions.

1 Introduction and historical context

It is well known that the Arabic medieval period has been marked by the creation of algebra. Between the 9^{th} and the 13^{th} centuries, many researches are engaged and the emergence of new theoretical questions is one of the main consequences of the elaboration and development of this new field.
For example, let us consider the equation:

$$x^3 + 2x^2 + x = 4$$

This equation is equivalent to:

$$x(x^2 + 2x + 1) = 4$$

That is to say:

$$x(x+1)^2 = 4$$

And thus (for $x \neq 0$):

$$\frac{4}{x} = (x+1)^2$$

The roots, if they exist, are the intersection points between the hyperbola $y = \frac{4}{x}$ and the parabola $y = (x+1)^2$ (figure 1). In this situation, the existence of the roots is based on the geometrical existence of the intersection points. Both curves are conics and

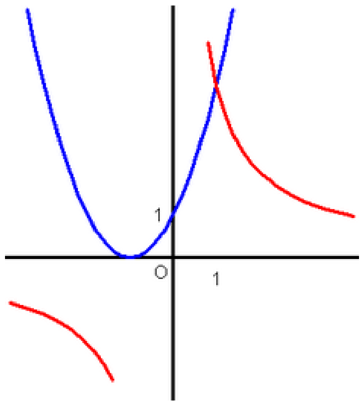

Figure 1: A simple example

these objects are complex enough to create a doubt on the reality of such intersections. Of course, the mathematician can not just say that the figure shows the trueness of the result and this simple example raises a crucial theoretical question. The point-by-point construction of conics has been well known since Antiquity (see the Apollonius' book entitled the *Conics*, for example), and that method is efficient enough for the analysis of the main properties of those curves. Algebraic equations can be solved by intersecting conics curves (ellipsis, parabola, and hyperbola) and the necessary taking into account of these intersections creates new difficulties. Indeed this possibility is based on the continuity of the different curves which is difficult to *prove*. The solution that has therefore been chosen is to associate the curve with a tool that enables a real construction. As the ruler and the compass allow straight lines or circles to be drawn and so justify their continuity, a new tool had to be invented to draw all the conics.

2 A new tool: origin and modelisation

As mentionned by R.Rashed[1], in the second half of the 10$^{\text{th}}$ century, a large research movement is engaged by the Arabic scholars on the continuous tracing of curves. In his treatise *On the perfect compass*[2], Abū Sahl al-Qūhī (about 922 - about 1000) presents the results of his own research on this question.

> Abū Sahl Wayjan ibn Rustam al-Qūhī said : This is a treatise on the instrument called the prefect compass, which contains two books. The first one deals with the demonstration that it is possible to draw measurable lines by this compass - that is, straight lines, the circumferences of circles, and the perimeters of conic sections, namely parabolas, hyperbolas, ellipses, and the opposite sections. The second book deals with the science of drawing one of the lines we have just mentionned, according to a known position. If this instrument existed before us among the Ancients and if it was cited and named, but if its names as well as the names of the things associated with it were different from the names we have given them, then we would have an excuse, since this instrument has not come down to us, any more

[1] al-Qūhī et al-Sijzī: sur le compas parfait et le tracé continu des sections coniques, *Arabic Sciences and Philosophy*, vol.13 (2003) pp.9-43

[2] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.726-796

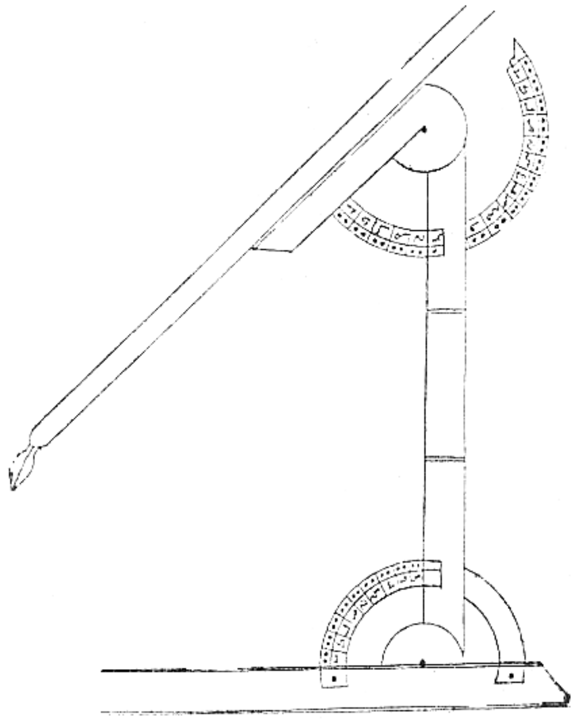

Figure 2: Sketch based on *Kitāb al-Qūhī fī al-birkām al-tāmm*, MS Istanbul, Raghib Pasha 569, fol.235^{v}.

> than has its mention; thus it is possible that this instrument, as well as the demonstration that is draws the lines we have just mentioned, may have existed without its use being the one we have made of it in the second book of this treatise.

In the second part of the short introduction (above quoted *in extenso*), al-Qūhī carefully explains that he has not found any texts on this instrument and that is why he wrote his treatise. The recent historical research seems to confirm that al-Qūhī's book is the first treatise on this tool[3]. Until now, no older descriptions have been found and there is no evidence of an implicite use of it before al-Qūhī. The new tool (figure 2[4]) is a kind of super-compass which can draw circles but also all the conic sections, even the degenerated ones like the straight lines.

> If, at a point of a plane, we raise a straight line that moves in one of the planes perpendicular to this plane, and if, throught another point of this straight line, there passes another straight line which has three motion - one around the straight line raised upon this plane, the second in the plane on which this straight line is situated, and the third on its extension simultaneously on both side - then if an instrument is described in this way, it is called a perfect compass.

The construction, in its principle, is very simple and other texts give lots of technical details that leads to think that some instruments have really been built. Unfortunately,

[3]For a complete analysis of the historical sources that have reached us, see R.Rashed (2003)
[4]Original picture in Rashed (2005) p.860

until now, no ancient perfect compass has been discovered. Nontheless, with this first description, one can construct such a tool. Some informations about modern replications of a perfect compass made for museum expositions, pedagogical or technical experiments, or just for pleasure can be found on the Internet[5]. In the educational context, a computer simulation is possible. A simple geometry software can give a good preview of what the tool can be. The example given in appendix produces the result below (figure 2). The first plane is (Oxy), the second (perpendicular to the first) is (Oxz). (OS) is the main line (the axis), the second one (SM) can move around the axis so that the point X leave a trace on the first plane. Let us call $\alpha = \widehat{SOx}$ and

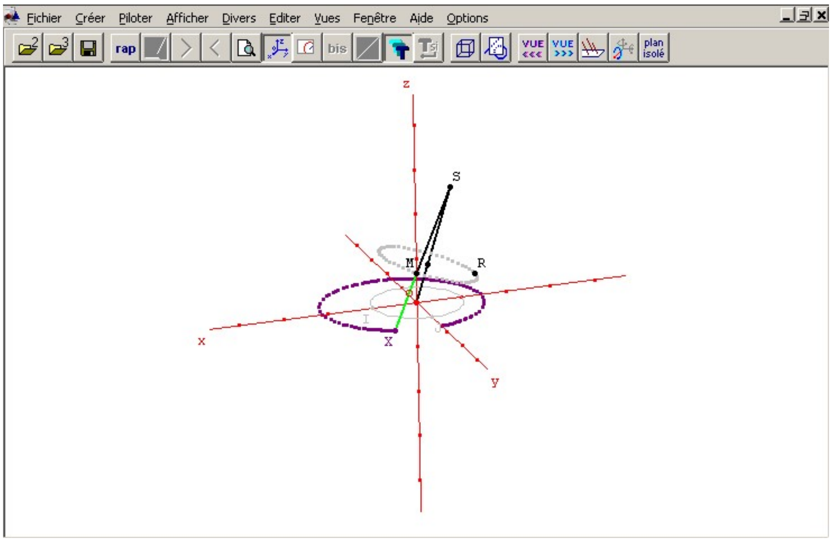

Figure 3: Perfect compass : a simulation with Geospace

$\beta = \widehat{OSM}$. Depending of the position of all the elements of the compass, the point X will trace:

- nothing, if $(OS) \perp (Oxy)$ and $(SM) \perp (OS)$
- a straight line, if $(SM) \perp (OS)$ with (OS) not perpendicular to (Oxy)
- a circle, if $(OS) \perp (Oxy)$ with (SM) not perpendicular to (OS)
- an ellipsis, if $\alpha + \beta < 180^o$
- a parabola, if $\alpha + \beta = 180^o$

[5] Some pictures of perfect compasses are available for instance in the virtual exposition *Theatrum machinarum* on Modena Museum website:
`http://www.museo.unimo.it/theatrum/macchine/017ogg.htm`
or on the pages about the exposition *Beyond the compasses* on the Garden of Archimedes Museum website:
`http://php.math.unifi.it/archimede/archimede_NEW_inglese/curve/guida/paginaindice.php?id=2`
A video of a pseudo-perfect compass *in action* is available on Professor Khosrow Sadeghi's personnal website:
`http://khosrowsadeghi.com/conic_compass.php#demo`

- an hyperbola, if $\alpha + \beta > 180^o$

Directly or not, this first description of the perfect compass is then reused by many other scholars. For instance, al-Bīrūnī (973 - 1048) in his book *Account of the perfect compass, and description of its movements*[6] explains:

> Abū Sahl has said: If, upon a point of a plane, we erect a straight line that moves on one of the planes perpendicular to this first plane, and if through another point on this straight line there passes another straight line, having three movements, of which one is around the straight line erected on this plane, the second is on the plane on which this straight line is situated, and the third is rectilinear in both directions; then if the instrument so described exists, we will call it a perfect compass.

And, in a same way al-Abharī (d. 1264) says in his *Treatise on the compass of conic sections*[7]:

> If, on a given straight line in a given plane, we erect a straight line, and if through the other end of the straight line we have erected there passes another straight line that comes to meet the given straight line in the plane, then we call these straight lines, in this configuration, the compass of the conic sections.

This last quotation gives the opportunity to read one of the other names of the perfect compass. Called by al-Abharī the *compass of the conic sections*, this instrument is sometimes simply named a *conic compass* or *cone compass*. All these treatises contain many mathematical propositions about the way to calculate the good angles corresponding to a given conic section. I do not detail this part but I strongly encourage the reader to have a look at these beautiful texts (see References).

3 Mathematical instruments in the research process

Ahmad ibn Muhammad ibn 'Adb al-Jalīl al-Sijzī (about 945 - about 1020) was born and lived in Iran. Son of mathematician, he worked between 969 and 998 and he wrote exclusively books on geometry. In all, he has written approximately fifty treatises and lots of letters to his contemporaries. Following his predecessors (Banū Mūsā brothers, Ibrāhīm ibn Sinān...) from whom he quoted in a precedent book on the description of the conic sections, al-Sijzī engages himself too in a treatise specifically on the *Construction of the perfect compass which is the compass of the cone*[8]. Like the other scholars, al-Sijzī wants to "construct a compass by means of which he shall draw the three sections mentionned by Apollonius in his book of *Conics*." He first notes that all the conics can be obtained from the right cone (depending on the position of the cutting plane), and afterwards he proposes three possible structures for the perfect compass. The beginning of the study gives technical recommandations. Here is a small quotation:

[6] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.816

[7] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.828

[8] R.Rashed, *Geometry and dioptrics in classical Islam*, al-Furqān, 2005, pp.798-806; the texts are also available in a French translation in *Œuvre mathématique d'al-Sijzi. Volume 1: Géométrie des coniques et théorie des nombres au Xe siècle*, Trad. R.Rashed, *Les Cahiers du MIDEO*, 3, Peeters, 2004.

> We must now show how to fashion a compass by means of which we may draw these sections. We fashion a shaft; such as AB. We place a tube at its vertex, such as AN, and to its extremity we attach another tube, such as AS. We can accomplish this with the help of a peg, or with anything else, so that the tube AN turns around shaft AB [...]

The instructions should enable the reader to really build such a compass. But for al-Sijzī, the aim of his work on the perfect compass is not only to draw conics. In *Sur la description des sections coniques*[9], the text[10] shows that this compass is also a theoretical tool and a tool for the discovery of new concepts.

> Mais puisque les propriétés de l'hyperbole et de la parabole sont proches des propriétés du cercle et que les propriétés de toutes les autres figures composées de manière régulière à partir de droites et qui ne subissent ni révolution ni rotation sont éloignées des propriétés du cercle, il est donc nécessaire que ces deux figures aient un rapport au cercle et une similitude à celui-ci, comme il en était pour l'ellipse. J'ai toujours réfléchi à l'existence de ce rapport entre elles et le cercle et à leur similitude et cherché à saisir ce rapport ; or la connaissance de ceci ne m'a été possible qu'une fois appris comment faire tourner le compas conique suivant les positions des plans. En effet, cette existence s'ordonne à partir de la rotation du compas conique sur la surface latérale ; la rotation régulière convient au cercle et cette rotation est commune au tracé du cercle sur une surface plane et au tracé de toutes les autres sections coniques ; étant donné que le cercle provient du tracé avec ce même compas si la position du plan est perpendiculaire à son axe, alors que pour les autres sections, leurs formes diffèrent suivant la position du plan par rapport à l'axe du compas. Quant à l'ellipse, sa conception est facile de plusieurs manières, soit à partir d'une section du cylindre soit à partir de la projection des rayons traversant une ouverture circulaire sur un plan de position oblique qui tient lieu aussi d'une section du cylindre ou d'une section du cône. Ce que nous voulions montrer.

Al-Sijzī explains that the link between the circle and the ellipsis is quite obvious. Indeed, the construction of the ellipsis by orthogonal affinity and the formula for the area are both well known. But what are the links between the circle and the parabola or the hyperbola? Now oriented towards the exploration and the solving of new problems, the practical tool becomes an instrument of discovery and as stated by al-Sijzī himself,"I always thought that there was a relationship between these two figures and the circle and their similarities and tried to get it but the knowledge of this has only become possible to me once I had learned how to turn the perfect compass following the positions of the plans". Confronted with the theoretical problem of the continuity of curves, the scientist suggests the use of a new instrument. The experimentation with this instrument creates new theoretical results that create new questions and so on and so forth. In his text, through the comings and goings between theory and practice al-Sijzī clarifies the role of mathematical instruments. They are objects as much as models and this dual status facilitates the theory-experiment passage.

[9]R.Rashed, *Les Cahiers du Mideo*, 3, Louvain-Paris, Éditions Peeters, 2004.
[10]Œuvre mathématique d'al-Sijzî, p.254

4 Conclusion

Halfway between philosophy and science, the acceptance of movement as a valid principle in geometry is one of the important topics[11] during the 10^{th} and 11^{th} centuries. Not only interesting from a mathematical point of view, the perfect compass is also useful in technological areas such as the construction of astrolabes and sundials where conics are essential. At the end of the Middle-Age, this instrument disappears and comes back at the Renaissance as a drawing tool (see figure 4[12] and Raynaud (2007)). The mathematics have changed and such an artefact between theory and practice is

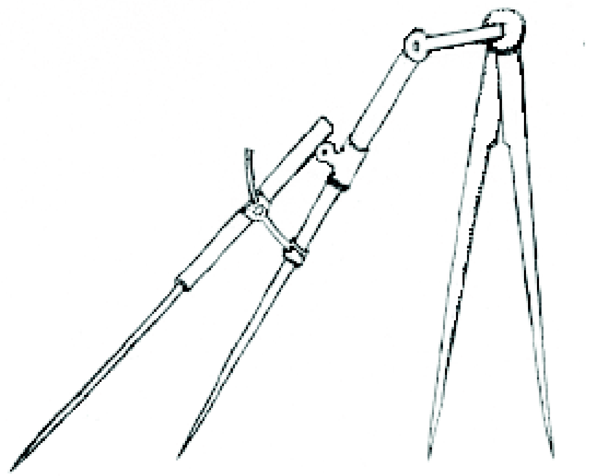

Figure 4: Renaissance : the perfect compass as a drawing tool

now useless. Mathematicians rarely expressed themselves on their relationships to the experiments. However when they did so, they gave us the opportunity to see the complexity of the links between theory and the use of technical instruments. The history of science assures us that: mathematical theories never emerge from nothingness. The scientist describes, builds and explores multiple examples before proposing an analysis or a system. The Arabic developements around the perfect compass is a model of such a process. In education in France and in many countries, the recent official instructions claim the importance of investigation in the learning process. The perfect compass can give a good entry point for an activity that helps the students to understand the way a new mathematical theory is elaborated. In secondary school, the conics are often studied only from a cartesian point of view. For students, and for teachers too[13], the

[11]The movement as a theoretical geometry principle appears a first time in Ibn al-Haytham's tries of a new definition of the geometrical space (see Rashed 2002 or de Vittori 2009)

[12]Sketch. Original pictures can be found in Raynaud (2007), for instance : (a) Venise, B. Naz. Marciana, 5363 (olim Ital. cl. IV 41), fol. 18r, P. Sergescu, "Leonardo da Vinci et les mathématiques", Leonardo da Vinci et lexpérience scientifique (Paris, 1952): 73-88, C. Pedretti, Studi vinciani (Genve, 1957), Idem, Leonardo da Vinci architecte, op. cit., p. 336, Idem, "Leonardo discepolo della sperientia" ,F. Camerota, d., Nel Segno di Masaccio (Firenze, 2001), p.184-185. (b) Vienne, Albertina, Inv. 22448 (olim 164). O. Kurz, "Dürer, Leonardo and the invention of the ellipsograph" , Raccolta Vinciana, 18 (1960): 15-25, sur les problèmes dattribution, cf. infra IV, note 51. (c) Sienne, Biblioteca deglIntronati, ms. L. IV. 10, fol. 92r-98v, G. Arrighi, "Il compasso ovale invention di Michiel Agnelo" , Le Machine, 1 (1968): 103-106; P. L. Rose, "Renaissance Italian methods..." .

[13]Al-Sijzī's texts and the Geospace model have been successfully used during a teacher training workshop (Colloque IREM, Brest 2008).

work on the perfect compass restores the links between solids and curves, practice an theory, real world and mathematics models... Mathematics teaching is always renewing itself and the historical sources give many elements that enrich it and give it sense.

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Some websites:
http://www.museo.unimo.it/theatrum/macchine/con1_04.htm
http://php.math.unifi.it/archimede/archimede_NEW_inglese/curve/guida/paginaindice.php?id=2
http://khosrowsadeghi.com/conic_compass.php#demo

5 Appendix

Geospace[14] figure *(text file[15])*

Figure Géospace
Numéro de version: 1
Uxyz par rapport à la petite dimension de la fenêtre: 0.1
Rotations de Rxyz: verticale: -72 horizontale: 19 frontale: 1
Repère Rxyz affiché
Dessin de o: marque épaisse
I point de coordonnées (1,0,0) dans le repère Rxyz
    Dessin de I: gris
J point de coordonnées (0,1,0) dans le repère Rxyz

[14]This geometry software is available on http://www.aid-creem.org/telechargement.html
[15]Figure available on http://devittori.perso.math.cnrs.fr/sijzi/compas_parfait.g3w

Dessin de J: gris
C cercle de centre o passant par I dans le plan oxy
Dessin de C: gris
S point libre dans le plan ozx
Objet libre S, paramètres: 2.5, 0
Dessin de S: marque épaisse
Segment [So]
Dessin de [So]: trait épais
K point libre sur le segment [So]
Objet libre K, paramètre: 0.5
Dessin de K: marque épaisse, nom non dessiné
P1 plan passant par K et perpendiculaire à la droite (So)
D1 droite d'intersection des plans ozx et P1
Dessin de D1: non dessiné
R point libre sur la droite D1
Objet libre R, paramètre: 0.5
Dessin de R: marque épaisse
C1 cercle d'axe (So) passant par R
Dessin de C1: gris, trait épais, points non liés
M point libre sur le cercle C1
Objet libre M, paramètre: -2.4
Dessin de M: marque épaisse
X point d'intersection de la droite (SM) et du plan oxy
Dessin de X: rose foncé, marque épaisse
Segment [SM]
Dessin de [SM]: trait épais
Segment [MX]
Dessin de [MX]: vert, trait épais
Sélection pour trace: X
Fin de la figure