HISTORICAL MINI-THEORIES AS AWAY TO REFLECT ABOUT THE MEANING OF PROOF

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ABSTRACT

The paper examines in the first part the historical fact that the Greeks invented not only mathematical proof but also and simultaneously ‘theoretical physics’. This simultaneity was not accidental; rather, the two events were connected and influenced each other. The link between them was an idea in the Greek philosophy of science called ‘saving the phenomena’. The paper establishes a connection between this idea and the pre-Euclidean meaning of the term ‘axiom’. The astronomical problems by which the Greeks were led to the idea of ‘saving the phenomena’ are well suited to be explored in the teaching of mathematics at school with the intention to enter a discussion about the origins of the axiomatic method and the meaning of proof.

1 The origin of proof

In 1960 the Hungarian historian of mathematics Árpád Szabó proposed a thesis about the origin of the axiomatic method in ancient Greece (Szabó 1960). Szabó’s investigations are not only interesting from a historical point of view but they establish also a framework for a didactical conception of proof. In the first section of this paper we present those parts of Szabó’s study which seem especially interesting from a didactical point of view. In the second section we relate these ideas with reflections in Greek science about the relation between theory and empirical evidence. This leads to an overall picture of the genesis and meaning of proof whose didactical fruitfulness is shown by a teaching unit which was discussed in the workshop at ESU 6 and which is sketched in the third part of the present paper. For reasons of space we suppress most of the references and documentary evidence. The interested reader may consult (Szabó 1960), (Maté 2006) and (Jahnke 2009).

Euclid divided the foundations of the ‘Elements’ into three groups of statements: (1) Definitions (greek *Horoi*), (2) Postulates (*Aitemata*) and (3) Axioms or Common Notions (*Axiomata* or *Konai Ennoiai*). Earlier manuscripts contain the terms (1) *Hypotheseis,* (2) *Aitemata* und (3) *Axiomata*.

Szabó investigated the etymology of the terms and showed that *Hypothesis*, *Aitema* und *Axioma* were common terms of pre-Euclidean and pre-Platonic dialectics and still played a considerable role in Plato’s dialogues and Aristotle’s treatises. Dialectics was the art of exchanging arguments and counter-arguments in a dialogue debating a controversial proposition. The Greeks treated dialectics as part of rhetoric, Plato (427-348 BC) considered it as a method of philosophical debate and of generating new knowledge.

From the times of Plato and Aristotle up to the 19th century mathematicians and philosophers understood axioms as statements which are self-evident and absolutely true. The most important ancient advocates of this view were Aristoteles (384 – 322 BC), especially in his treatise *Analytica Posteriora,* and the Neo-Platonist commentator Proclus (412 – 585 AD) in his influential commentary to book I of Euclid’s ‘Elements’.

The view of axioms as self-evident truths undergoes a remarkable modification when Szabó’s studies are taken into account. At first, let us consider the term *Hypothesis*.

Literally, *hypothesis* is something which is *underlying* and consequently can function as a foundation of something else. Hence, it is an unproven assumption in a dialectic discourse whose validity is assumed in order to derive further statements. In Plato's works the reader finds a multitude of situations where this term is used. Plato knows *Hypothesis* as a concept of mathematical terminology, but he used it even more frequently as a designation of an assumption in philosophical discourses. E. g. Socrates "asked" his partners to "allow" him to begin with this or this hypothesis. The dialogue can only continue if the partners explicitly agree to this request. Therefore, Plato designated hypotheses frequently as *Homologemata* (the "things which were conceded").

As a rule a person will introduce in a discourse such hypotheses which he considers as especially strong and which he assumes to be accepted by his partners. At various places of Plato's dialogues Socrates reflected this use of hypotheses and asked his partners to take a great deal of care over the starting point of a debate.

But it is also possible to propose a hypothesis with the intention of its critical examination. In a philosophical discourse the participants derive consequences from this hypothesis. The consequences might be *desired* or *plausible* and then lead to strengthen or even accepting the hypothesis. Or they might be *not desired* or *implausible* and thus lead to rejecting it. In a debate on ethical questions a participant might propose a certain maxim of behavior. Then the participants jointly investigate consequences of the maxim. In case, it will imply not desired behavior patterns they will reject the maxim, in the opposite case they will judge it of being worth further consideration or even accept it. An extreme case of a not desired consequence is a logical contradiction. In that case mathematicians speak of an indirect proof.

All in all, Szabó found in regard to the term 'hypothesis' three different variants of meaning:

(1) Hypothesis = strong, unproven proposition. To convince his partners of a certain statement person *A* proposes a hypothesis which *A* assumes to be accepted by his partners and from which the statement in question can be derived;
(2) Hypothesis = proposition whose validity is assumed for a time in order to critically investigate it. The participants derive consequences and check whether these are desired or not desired. An extreme case of an undesired consequence is a logical contradiction.
(3) Hypothesis = definition. This variant of meaning is especially relevant to mathematics. E.g. somebody proposes definitions of say 'odd number' and 'even number' and in the course of work a whole bunch of theorems can be deduced which are considered relevant. This leads to accepting the definition as fruitful.

Szabó's investigations showed that the three terms *Hypothesis*, *Aitema* and *Axioma* had a similar meaning in pre-Platonic and pre-Arisotelean dialectics (with the exception of the meaning 'definition'). They designated those initial propositions in a dialectical debate whose acceptance is *required* by one of the participants. If the participants agreed on the proposition it was frequently called *Hypothesis*. If, however, acceptance was left undecided the proposition was called *Aitema* or *Axioma* (Szabó 1960, 399). This meaning of *Aitema* was still known to Aristotle.

From the study of the genesis of the term *Axioma* in Greek dialectics we see that, originally, it did not designate a statement which is absolutely true and cannot be doubted but, for a long time, had a connotation in the sense of the modern concept of hypothesis.

Probably, at the times of Euclid philosophers and mathematicians were still aware of this hypothetical connotation and thus used the term *Konai Ennoiai* (= *the ideas common to all human beings*) instead of *Axioma*.

In this way, the concept of an axiom made a career in Greek philosophy and mathematics whose starting point lay in the art of philosophical discourse; later it played a role in both philosophy and mathematics. More importantly, it underwent a concomitant change in its epistemological status. In the early context of dialectics, the term axiom designated a proposition that in the beginning of a debate could be accepted or not. However, axiom's later meaning in mathematics was clearly that of a statement which itself cannot be proved but is absolutely certain and therefore can serve as a fundament of a deductively organized theory. Since Plato and Aristotle this was the dominant view among mathematicians and philosophers to whom mathematics served as the ideal of a certain science for more than two thousand years.

2 Saving the phenomena

We can draw two consequences from Szabó's etymological studies:

> (1) The *practice of a rational discourse* served as a model for the organisation of a mathematical theory according to the axiomatic-deductive method. The terms *hypothesis*, *aitema* and *axiom* are crystallized rules of behavior in a dialectical discourse. They say that the participants have to accept and obey certain rules of behaviour which entail their obligation to exhibit their assumptions.
>
> (2) The second consequence refers to the universality of dialectics. Any problem can become the subject of a dialectical discourse, regardless whether it is a problem of ethics, of physics or mathematics. The axiomatic-deductive organisation of a group of propositions is not confined to arithmetic and geometry, but can, in principle, be applied to any field of human knowledge. The Greeks realized this insight at the time of Euclid, and it led to the birth of theoretical physics.

Within a short period of time Euclid and other mathematicians/scientists applied the axiomatic organisation of a theory to a number of areas in natural science. Euclid wrote a deductively organised optics and a music theory, whereas Archimedes provided axiomatic-deductive accounts of statics and hydrostatics. Also in astronomy the concept of hypothesis was systematically used.

In the latter domain, the Greeks discussed, in an exemplary manner, philosophical questions about the relation of theory and empirical evidence with influences even upon Kepler and Galileo. This discussion started at the time of Plato and concerned the paths of the planets. In general, the planets apparently travel across the sky of fixed stars in circular arcs. At certain times, however, they perform an irregular retrograde motion. This caused a severe problem; the Greeks held a deeply rooted conviction that the heavenly bodies perform circular movements with constant velocity. But this could not account for the irregular retrograde movement of the planets. Consequently the Greeks had to adjust the hypotheses about the movements of the planets in order to fit astronomical theory with empirical evidence. They did so by inventing new hypotheses, the so-called eccentric and epicyclic hypotheses. The process of adaptation was reflected under the heading „sozein ta phainomena“ (“saving the phenomena”) (Mittelstrass 1962). Saving the phenomena designated the process of devising hypotheses in a way that their deductive consequences agree with the observed data. Hence, saving the phenomena agrees exactly with the second variant of meaning Szabó has found for the use of hypotheses in dialectical discourses. A hypothesis is stated and the participants investigate whether its consequences are desired i.e. agree with the data, or not. In the former case it is accepted, in the latter rejected or modified.

Szabó doesn't hint at the fact that the investigation of a hypothesis in a dialectic discourse has a logical structure similar to the process of accommodating a hypothesis to empirical phenomena (see however Szabó 1974). Nevertheless, this is the case. In ancient times the latter process was coined 'saving the phenomena' and was, as far as we know, confined to astronomy. Since Galileo and Huyghens it was considered as the general method of research in the mathematised physical sciences. By the end of the 19th century scientists called the same procedure the 'hypothetico-deductive method'.

4 Reflecting about the meaning of proof by studying a historical mini-theory

In the teaching of mathematics at the secondary level it is not easy to answer the question of what a proof is. In fact, proof is not a stand-alone concept and cannot be explained without recourse to the notion of theory. But it is widely agreed that it does not make sense to treat axiomatic theories in schools. Nevertheless, it is an urgent requirement to provide also to students of the secondary level an adequate idea of what mathematical proof is and which meaning it has for man‘s understanding of the world.

Which knowledge about proof should students acquire? To say it in a few words, they should understand that a mathematical proof does not establish ‚facts‘ but ‚if-then statements‘. A mathematician does not proof a fact *B* but an *implication* „If *A*, then *B*“. E. G. we don't prove that all triangles have an angle sum of 180°. Rather, this is a consequence of certain hypotheses, namely the axioms of Euclidean geometry. If we suppose different axioms we arrive at different conclusions. The *certainty* of mathematics does not reside in the facts, but in the *inferences*. Whether a mathematician believes in the facts of his theory, e.g. that there are infinitely many prime numbers or that triangles have an angle sum of 180°, depends on his *evaluation* of the acceptability of the axioms of his theory. In professional mathematics this process of evaluation or assessment of a theory remains mostly implicit. *In teaching however students should be made aware of it*. For example, in arithmetic mathematicians deal with objects which are generated by a uniform process of counting. Hence, there is a high amount of control. But in geometry mathematicians touch the area of physics. If geometrical theorems are applied to physical space they are subject to empirical measurements. The fact that Euclidean geometry is extraordinarily reliable as an empirical theory of "medium-sized objects" is a matter of experience and not a fact of pure thought.

Basically, there are two ways of evaluating the axioms of a theory. Classically axioms are considered as self-evident and absolutely true. Consequently, the theory which is built upon them is true. The other way is to evaluate a theory on the basis of whether or not its consequences agree with what the theory is expected to explain. This corresponds to the above mentioned "hypothetico-decductive method". In contrast to traditional teaching of mathematics this paper supports the thesis that also the latter way of evaluating the axioms of a theory should be made a theme with students. This seems to be a promising way of explicitly reflecting about what axioms of a theory are and where they come from.

A possible idea for doing this is the development of *mini-theories* which are accessible to learners and sufficiently substantial to discuss meta-issues. The idea of such min-theories is not completely new. It bears resemblance with Freudenthal's (1973) concept of 'local ordering' or with using a finite geometry as a surveyable example of an axiomatic theory. Of course, finite geometries are not feasible in secondary teaching. In regard to Freudenthal's concept of local ordering the proposed idea of a mini-theory is different in two aspects. First, the teaching of a mini-theory would include phases of explicit reflection about the structure of axiomatic theories, the conditionality of mathematical theorems and the evaluation of its truth. Second, the proposal would also take into account 'small theories' from physics like Galileo's law of free fall and its consequences and other mathematised empirical sciences (Jahnke 2007).

In the following section a teaching unit for 9th graders on a historical mini-theory will be described which serves as an example of the conception sketched above. It concerns the so-called 'anomaly of the sun' which was discussed for the first time by the great astronomer and mathematician Hipparchos in the second century BC. Roughly speaking, the term referred to the observation that the half-year of summer is about one week longer than the half-year of winter. This contradicted the basic convictions of Greek astronomers that all heavenly bodies move with constant velocity in circles around the centre of the earth. Therefore Hipparchos invented a new hypothesis to "save the phenomena". Hipparchos' proposal can be found in Ptolemy's 'Almagest' (Toomer 1984). In the teaching unit we will not stick to the details of Ptolemy's exposition and especially will calculate with modern data.

To study a historical mini-theory seems to be especially rewarding in regard to the epistemological situation. The students can well imagine the situation that the ancient astronomers were forced to find out something about the structure and functioning of the universe without any direct access to it. This is reinforced by their knowledge of the fact that Hipparchos' theory does not agree with modern views.

5 A Teaching unit on hypotheses about the path of the sun

The following teaching unit is described by a series of worksheets for pupils of grade 9. The reader should take into account that the worksheets serve the twofold function of providing him with necessary informations and giving him an idea of possible work for pupils. Hence, they are not intended as the final sheets for the pupils.

5.1 Worksheet 1

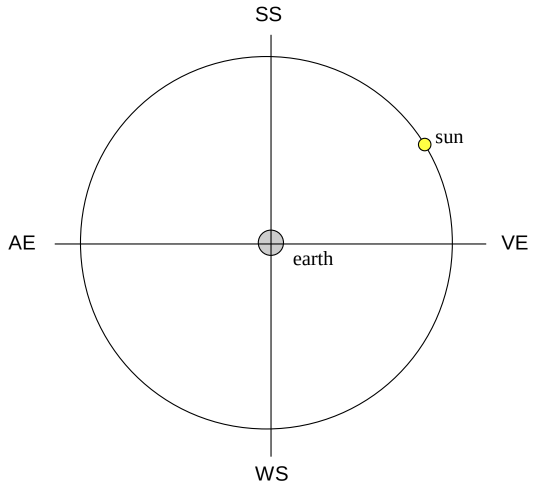
In ancient times Greek astronomers tried to understand the structure of the universe. Of course, they couldn't know how the sky was really built up but they could set up hypotheses and then examine whether the consequences of these assumptions agreed with their astronomical observations. Since the fifth century BC Greek astronomers supposed that the earth is a sphere and located in the center of the universe. Sun, moon and the stars were believed to move around the earth.

In regard to the yearly path of the sun they set up two precise hypotheses:

Hypothesis 1: In the course of a year the sun moves on a circle around the earth exactly one time. This circle is called **ecliptic**.

Hypothesis 2: The sun moves on the ecliptic with constant velocity.

There are four distinguished points on the ecliptic, vernal equinox (VE), summer solstice (SS), autumnal equinox (AE) and winter solstice (WS). When the sun is in one of the equinoxes day and night are equally long, summer solstice is the longest day and winter solstice the shortest. From their measurements the astronomers knew that these four points on the ecliptic are separated by angles of exactly 90°. They separate the four seasons spring, summer, autumn and winter from each other. Hence the path of the sun looked like this:



Tasks:

a) The length of a year is approximately 365,25 days. Derive from hypothesis 1 and 2, how many days spring, summer, autumn and winter comprise. Enter your results in table 1.

b) In table 2 you find the dates of vernal equinox (VE), summer solstice (SS), autumnal equinox (AE) and winter solstice (WS) for the present year 2010/11. Compute the number of days for the different seasons which result from these data. Enter your results in table 1.

c) Compare and comment about what you have found.

Table 1

	Duration of the seasons calculated from hypotheses 1 and 2	**Duration of the seasons calculated from the data for 2010/11**
Spring	91,3 d	92,7 d
Summer	91,3 d	93,7 d
Autumn	91,3 d	89,9 d
Winter	91,3 d	89 d

Table 2

Vernal equinox 2010	March 20, 2010, 8 pm
Summer solstice 2010	June 21, 2010, 2 pm
Autumnal equinox 2010	September, 23, 2010, 6 am
Winter solstice 2010	December 22, 2010, 2 am
Vernal equinox 2011	March, 21 2011, 2 am

5.2 Worksheet 2

We have found that the four seasons are not equally long. The “half year” of summer (= spring + summer) is about a week longer than the “half year” of winter (= autumn + winter). This contradicts hypothesis 1 and/or 2. What can we do? What are your ideas?

The Greek astronomers had an ingenious solution for this problem. They kept the idea of a circular path and the constant velocity of the sun and nevertheless provided for the apparent variation of its velocities. The new hypotheses were:

Hypothesis 1': In the course of a year the sun moves on a circle around the earth exactly one time. The center of this circle is **not** in the center of the earth. This circle is called ecliptic.

Hypothesis 2: The sun moves on the ecliptic with constant velocity.

Hypothesis 1' has been proposed by the Greek astronomer Hipparchos (2^{nd} century BC). Therefore we call it Hipparchos' hypothesis.

Of course, it is still true that, observed from the earth the four seasonal points on the ecliptic are separated by right angles.

Tasks:

Below you find four different cases of relative positions of the earth (E), the vernal equinox (VE) and the center (C) of the path of the sun.

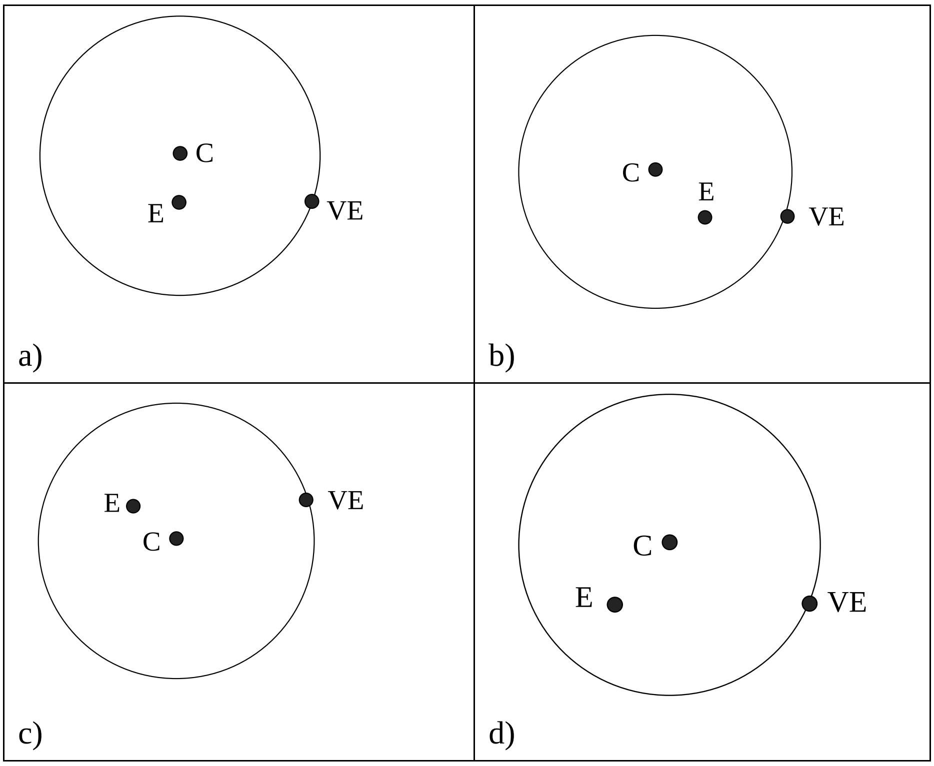
a) Please, add in each case the positions of the missing seasonal points.

b) What can you say about the length of the seasons in each case?

c) Write down your observations concerning the position of the center of the eccentric circle in relation to the lengths of the seasons.

d) The Greek astronomers called the angle between the vernal equinox (VE), the earth (E) and the sun (S) the **"true sun"**. They called the angle between the vernal equinox (VE), the center (C) of the path of the sun and the sun (S) the **"mean sun"**. Please, add in one of the figures the sun (S) at an arbitrary position and draw true and mean sun.

e) Please, speculate why these angles were called "true sun" and "mean sun".



5.3 Worksheet 3

Now we are going to investigate the properties of the system which is created by Hipparchos' hypothesis. The system consists of the earth *E*, the eccentric center *C* and a circle around *C* on which the sun *S* moves with constant velocity. It is useful to connect with *E* two orthogonal axes which represent the observed positions of the distinguished points on the ecliptic. By moving the earth with this cross you can experiment with different relative positions of *C* and *E*.

For your investigation you can use a software tool like Geogebra (or Cabri, DynaGeo) or you take two transparencies. On the one you draw the center *C* with the circle on which the sun moves. On the other you draw the earth *E* with the orthogonal axes connecting the solstices and the equinoxes. Then you can move the transparencies and observe what happens in different relative positions of *C* and *E*.

You may investigate the following questions:

1. What can you say about the distances between earth and sun during a full circle of the sun?

2. What can you say about the varying velocities of the true sun?

3. What are possible lengths of seasons which are consistent with our theory. Or, to put it differently: is it possible to prescribe arbitrary lengths of seasons (of course, their sum has to be 360°) and find a center *C* which fits to these lengths?

You are asked to make as many observations as possible. If you are able to derive these observations from the hypotheses they will become theorems in our theory.

What do you think about the truth of these theorems? Write down your opinion.

5.4 Worksheet 4

Now we will investigate "how good" the hypothesis of Hipparchos is.

a) Please, guess in which quadrant the center C of the eccentric circle will lie if we take the data of table 1.

b) For the construction of the exact position of C we use a software like Geogebra.

First of all, we apply a trick physicists often use. The Greeks didn't know the (mean) distance of the sun from the earth. Therefore we set this distance equal to 1. Then we can represent the distance of C from E by a fraction like for example 1/3. This fraction would say that the distance of *C* from *E* is 1/3 of the distance of the sun from the earth. Already Hipparchos applied this trick and today's astronomers call the distance earth – sun the "astronomical unit".

Since working with real data involves numbers with a lot of digits and clumsy conversions of days in angles and vice versa we simplify the situation by using "easy numbers". That is we suppose that we have already done the conversion of the numbers of days of spring and summer in angles and that we have got 95° for spring and 105° for summer. In sum, the "half year" of summer lasts 200°. If you now consider that the equinoxes VE and AE lie at the endpoints of this arc of 200° you should be able to construct the position of the earth *E* relative to the position of the center *C*.

5.5 Worksheet 5

The same with real data.

Results:

With the center *C* in the origin of a Cartesian system we get the coordinates of the earth E

$$E = (0{,}007801; -0{,}032506).$$

When E is in the origin we get

$$C = (-0{,}007801; 0{,}032506).$$

The distance between E and C is

$$e = \underline{0{,}033429.}$$

This is approximately 1/30. Ptolemy got with the data of his time 1/24.

The angle between the apogee and the vernal equinox is with our modern data 103,475°. Ptolemy got with his data ca. 69°.

When we take 12 exact dates (one per month) and compare modern data for the positions of the sun with data calculated by means of Hipparch’s hypothesis we get a mean deviation of 3’ and a maximal deviation of 20’.

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