SECONDARY SCHOOL STUDENTS' DIFFICULTIES WITH VECTOR CONCEPTS AND THE USE OF GEOMETRICAL & PHYSICAL SITUATIONS

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ABSTRACT

For many years, vectors used to be - and still remain - marginal to the Greek secondary mathematics education. For a long time, young students are left to form their own ideas about vector concepts only through physics courses and everyday life experience, up to grade 11 (16-17 year old), where they are introduced to vector algebra applied to geometry.

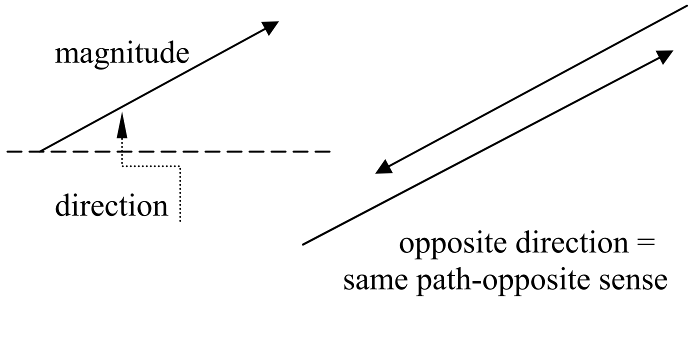
In our previous research with Greek students (9^{th}-12^{th} grades), we identified specific persisting difficulties, concerning certain epistemological aspects of vectorial concepts. Students' difficulties and the historical development of these concepts, led us to a teaching experiment with 8^{th} and 9^{th} graders (14-15 year old), where vector methods & concepts were considered as a new language to be learnt and explored. Inspired by the historical development of vectorial concepts, which was due mainly to problems and situations in physics and geometry, we used privileged situations, based on forces, velocities and displacements, to introduce the concept of vector, vector notation (symbolic representation), geometric representation, comparison between vectors and vector addition. Our experimental approach gave us the opportunity to face, handle and attempt to eliminate a variety of difficulties, related to the multifarious, composite nature of the vector language. The results indicate that our experimental teaching helped students to overcome some of their misconceptions and to proceed to a synthesis of partial concepts in a more coherent and abstract conceptual structure. However, due to space limitations, in this paper we present only some of the activities used in our teaching, focusing on students' misconceptions and difficulties and the role of physical and geometrical situations.

1 Introduction

Vectors in Greek secondary curriculum are marginal in school mathematics teaching. Students form ideas about vector concepts only through physics courses and everyday life experience up to grade 11 (16-17 year old), where they are introduced to vector algebra applied to geometry.

Teaching and understanding difficulties are related either to the epistemological nature of the vector concept because of its multi-dimensional and multi-level character, or/and to the didactical context, since the symbolism and terminology vary in different teaching contexts (geometry, algebra, physics).

For example, in Greek mathematics textbooks, a vector is determined by three components: *magnitude*, *path*, *sense*, whereas, in Greek physics textbooks, a vector quantity is determined by two components: *magnitude* and *direction* (including the concepts of path and sense).



Additionally, for a long time in Greek mathematics courses symbols like $\vec{a}$ or $\overrightarrow{EF}$ have been used, while in physics courses in lower secondary education the symbols used denote only the magnitude of vector quantities and not the vector quantities themselves[1]. This disagreement between mathematics and physics has a negative influence on students understanding of vectorial notation. In a vector's symbol they mainly recognize its "magnitude". For example, they use the notation $\vec{F} = \vec{F}_1 + \vec{F}_2$ for magnitudes of non-collinear forces.

Our previous research with Greek students (9^{th} - 12^{th} grades), has verified specific and persisting difficulties concerning certain epistemological aspects of vector concepts (Demetriadou 1994, 1999, 2002, Demetriadou & Gagatsis 1995, Demetriadou & Tzanakis 2003). Students' difficulties and strong preconceptions concerning vector quantities and operations have also been verified by other researchers, mainly in physics education (Turner 1979, Trowbridge & McDermott 1980, McCloskey 1983, Watts 1983, Aguirre & Rankin 1989, Eckstein & Shemesh 1989, Eisner 1991, Knight 1995).

In our opinion, vectorial notions are too complicated to be introduced in secondary education in their abstract mathematical form. On the other hand, physics is a suitable field to introduce vector concepts and operations in a more intuitive and natural way. Moreover, historically, it was the interplay between physical and purely mathematical situations that led to the emergence of vectorial concepts, operations and methods. In fact, such concepts and methods and the corresponding notation were first established in physics; mathematical practice followed once the efficiency of vectorial methods became clear. This historically undoubted influence of physics is ignored in the Greek secondary education curriculum.

2 The teaching experiment

Based on students' difficulties concerning vector concepts and **implicitly** influenced by the historical development of the subject, we designed and implemented a teaching experiment with 8^{th} and 9^{th} graders (14-15 year old). Vector methods and concepts were treated as a **new language**, which should be learned and its virtues should be explored. Our teaching approach was designed along the following three axes:

1. A historical-genetic approach (Arcavi 1985, Fauvel 1991, Tzanakis 2000) inspired by key issues that were central to the historical development of vector calculus. They are related both to physics and geometry: (a) composition of motions, (b) composition of forces and (c) composition of displacements.
2. Didactical approaches connected to understanding and learning procedures of vector language (Vygotsky 1993, Donaldson 1995, Booth 1981). Vectorial concepts were faced as a language built upon/taking into account pre-existing conceptions and creation of intuition strategies.
3. Pedagogical approaches based on active participation and classroom communication during the learning procedure, for the dynamics of the group/classroom discourse (Piaget 1969, Cobb et al 1992, Radford 2011, Schwarz et al 2009).

The experiment included two phases: (a) a **pilot** teaching of 16 hours on 30 students, 14 year-old, before they had attended any systematic physics course and (b) a **main teaching** experiment of 14 hours with 58 students, 15 year-old, after they had followed part of the

[1]Only in the recently revised version of the curriculum of lower secondary education and the corresponding official textbooks (published in 2007), the vector notation with an overhead arrow is used only for displacements!

physics course. The results of the experimental teaching were compared with those of conventional teaching (following the official curriculum) for an equivalent control group consisting of 53 students.

Due to space limitations, we do not give a detailed account of the whole teaching experiment: In the next section we present only some of the activities used in our teaching experiment, with focus on students' misconceptions and difficulties and the role of physical and geometrical situations. In the last section, we summarize some of the main results that came out of the analysis of our experimental teaching, some of which are based on the comparison with a similar analysis for the corresponding teaching to the control group.

3 Some indicative activities

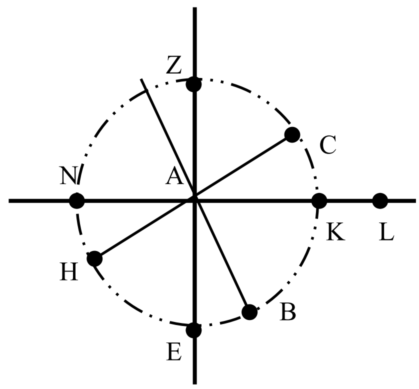
We used didactical activities based on geometrical and physical situations (involving displacements and velocities & forces, respectively), where vector concepts and operations are handled in the context of different conceptual frameworks (Brousseau, 1997). Vectorial methods were treated not as an abstract tool to express, handle and develop logically geometrical and physical concepts, but rather, as a means that clarifies (or even partly determines) their content and meaning, thus becoming crucial for the creation and development of new mathematics (Douady, 1991).

3.1 Introducing the concept of a vector

***1. The introductory activity: An insect's displacement on a plane surface**:*

Peter observes an ant A on his desk, trying to guess where it is going to move[2]. It was easily verified by the students that Peter couldn't do this, since there is an infinite number of possible directions. Given that in 1sec the ant covers 5cm, students were asked to compare possible displacements / trajectories with:

- ✓ Equal magnitude – opposite sense
- ✓ Equal magnitude – different path
- ✓ Same sense-different magnitude

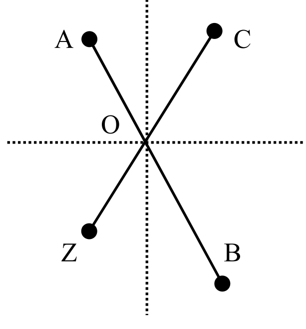


***2. Introductory activity on the terminology: A visitor's displacements**:*

A visitor stands on O. To visit A he is told to move: a) opposite to Z, b) opposite to B, c) in the same direction (or sense) with C.

Is this information exact?

This activity is suitable to distinguish between everyday language and mathematical language. In fact it helps students to make a distinction between path, sense, direction and orientation.



[2] A related situation is mentioned in a Cyprus mathematics school-book of the 2nd Grade of lower secondary education (Themistocleus & Anastasiadou 1992, p. 216)

3. Displacements between two towns: *Every morning a man travels from town I to town E:*

I ———————————— *E*

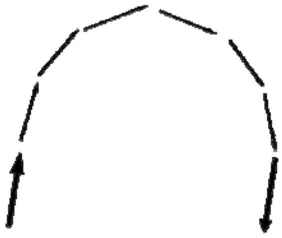
Every evening he travels back to his town I.

I ———————————— *E*

Is there any difference between these two itineraries?

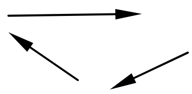
It is a simple and fruitful activity based on collinear vectors, privileged for introducing notation, geometric representation and opposite vectors.

4. Circular displacements: We present step-by-step the vectors of the figure asking: *"Do these arrows have the same direction?"* The student, who answers affirmatively, comes in cognitive conflict with his conception, when in the boundary positions the two vectors become opposite. It is a proposed activity, related to the circular conception of sense verified by our previous research, and contributes to the distinction between direction and orientation.

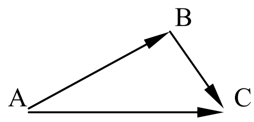


It seems that students' difficulties, due to the confusion between everyday language and mathematical language, lead them to confusion between vector sense (direction) and orientation of a motion (left handed vs. right handed):

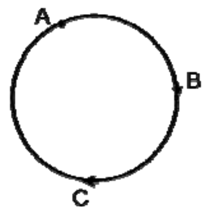
- *Direction of a continuous motion*:
  For some students these three vectors have the same sense.

  The same happened for successive vectors like $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

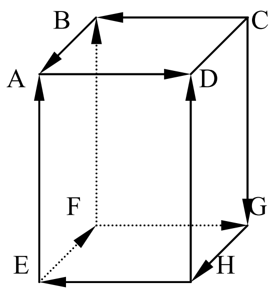
  

- *Circular motions in physics*
  Some students influenced by the representation of circular motions in physics, suggested these oriented arrows as examples of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same sense.

5. Activities in a geometrical context

5.a. *Among vectors indicated on the parallelepiped with square bases, distinguish those (i) of equal magnitude, (ii) of the same path, (iii) of the same sense to* $\overrightarrow{AB}$, *(iv) equal to* $\overrightarrow{AD}$, *(v) opposite to* $\overrightarrow{AD}$[3]

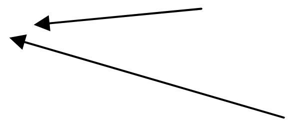


This is a purely (static) geometric situation, an activity offered for making practice. It is also connected to difficulties due to language, like the following:

- *Horizontal vs vertical directions means "opposite" directions*: For some students, all vertical vectors were opposite to the horizontal $\overrightarrow{AD}$. Similarly, vectors perpendicular to $\overrightarrow{AD}$, like $\overrightarrow{EF}$, were considered to be opposite to the horizontal $\overrightarrow{AD}$.
- *"Opposite" means "different"*: This is a strongly persisting difficulty, much stronger

[3]This exercise is included in a Greek mathematics textbook for the 3rd grade of high school (9th grade) that was in use for many years (Alibinishis et al 1998, p.247).

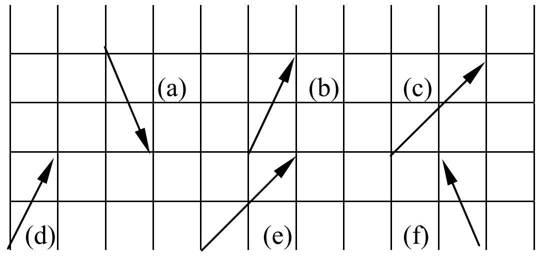
than the circular concept of sense.



- ✓ *Teacher:* Can you compare these vectors?
- ✓ *Student:* No, since these have opposite sense and path.

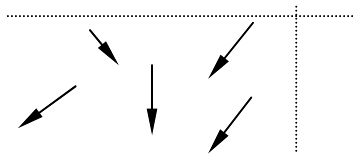
- ✓ *Teacher:* What's the relation between their magnitudes?
- ✓ *Student:* They are opposite.

***5.b.** Find vectors which are: i) of the same magnitude ii) of the same path iii) of the same sense iv) equal.*

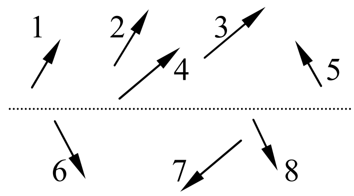


This activity is related to specific difficulties due to confusion between sense and orientation, which is a strong difficulty, still persisting after three months! Some students tended to separate the plane in semi-planes or quadrants, where vectors have the same sense, e.g.:

- Approximately parallel vectors were considered to have the same path, like c and d above.
- Sense & Quadrants: Vectors with the same orientation (SW or left down) were considered to have the same sense.



- Sense & Semi-planes: The sense of vectors 1, 2, 3, 4, 5 is upwards. The sense of vectors 6, 7, 8 is downwards.

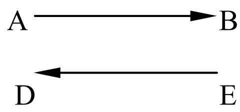


3.2 Introducing vector notation

Students' inventions negotiated in the class, for denoting the displacement between the two cities I and E (§3.1.3) were of two types: at the very beginning they suggested symbols like IE, AB, x or y, strongly related to a line segment. Later on they suggested $\overrightarrow{IE}$ and $I \longrightarrow E$. For practical reasons, they soon rejected the last one.

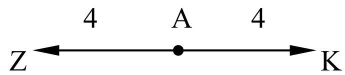
A very interesting invention was: $\overleftarrow{AB}$ for the vector: $A \longleftarrow B$, where students attempted to denote the sense of the vector, as well. This is an important result, indicating students' inventiveness. It led to a long discourse in the classroom and was finally rejected by the majority, presumably influenced by the didactical contract (*"Is it legal to use it in exams?"*). Only two students kept it until the end, but they failed to use it correctly in the final exam.

Students met difficulties with the symbol of the opposite vector in general. The confusion becomes obvious in their suggestions for the vector opposite to $\overrightarrow{AB}$: $-\overrightarrow{AB}$, $\overrightarrow{BA}$, $-\overrightarrow{BA}$, $A'B'$, $\overrightarrow{DE}$, $\overrightarrow{ED}$, $-\overrightarrow{ED}$, $\overleftarrow{AB}$.



Moreover, the opposite vector was also related to language problems, since for some students "different" means "opposite". Sometimes they used an idiosyncratic notation of the correct conception of opposite vectors (≠), e.g.:

- ✓ *Student A:* These are opposite.
- ✓ *Teacher:* How should we denote them?
- ✓ *Student B:* $\overrightarrow{AZ} \neq \overrightarrow{AK}$.



The classroom discourse on notation and the use of terms in everyday language raised the issue of the meaning of the arrow in vector's symbol. According to some students, it signifies the vector itself (indicating its vectorial nature), the terminal points, the path or the sense. They got confused on this point, when only one (small) letter was used to denote the vector.

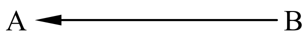
For a small percentage of students, scalar quantities were considered as vectors and vice versa, e.g: *"*$\vec{2}\,^0C - \vec{5}\,^0C = -\vec{3}\,^0C$*",* *"*$\overrightarrow{10}\ m$*",* *"*$\vec{\upsilon}$ *= 38 m/sec",* *"*$+\,2 - 5 = -\,\vec{3}$*".*

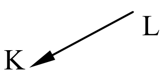
Students were also asked to suggest symbols for a vector's magnitude in the case of forces. Although the symbol $|\vec{\alpha}|$ was accepted for displacements by analogy to the absolute value of numbers (keeping the number line in mind), there was a confusion in the case of forces, because of the strong context implied by physics. In physics textbooks[4] occasionally vectors are denoted with arrows, or in boldface letters (a single capital letter)! The following symbols were used by the same student: *"*$\vec{F}$*=6N",* *"*$|\vec{F}|$*=20kg*",* *"*$\vec{F}$ *= 4cm".*

3.3 Introducing the geometrical representation of vectors

The following are students' proposals for the displacement between the cities I and E (§3.1.3). This is a privileged activity to combine notation with geometrical representation.

In some cases, it seems that they really saw a text when "reading":

- ✓ from left to right : $\overrightarrow{AB}$ for 

- ✓ or forward to backward : $\overrightarrow{KL}$ for 

3.4. Comparing vectors

Equal vectors are equivalent in magnitude, path and sense. In that sense, "equality" means "equivalence" (relation), not "identity". Students' main difficulties on equality are related

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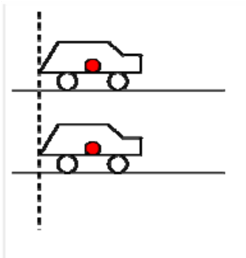
[4]cf. footnote 1.

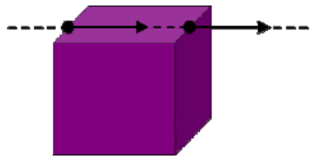
to the fact that it is conceived as equality of line segments and it is limited to equality of magnitudes.

We first organized a discussion on the conventional use of equality as equivalence, using examples with line segments or numbers (e.g. 3+5=8 or $\frac{2}{3}=\frac{4}{6}$), and concluding for example, that the **equal** vectors $\overrightarrow{BA}$ and $\overrightarrow{GH}$ in (§3.1.5) are **not identical**; they **differ**, since they are in different places; however they are **similar** with respect to **some** of their elements. Then, we used situations realizing vector equivalence. Here vectors with different initial point have **equivalent results** concerning the displacement. Vectors were considered **as operators**, not static objects (the same holds for numbers when teaching multiplicative structures). Physical and geometrical situations are privileged in this respect. The following are some related activities given to our students.



*Example 1***.** Situations with velocities of different initial points.

Two marbles lying in two cars are moving with the same velocity. The observer sees that the two marbles cover the same distance in the same time.

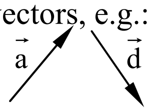


Example 2. Situations with forces (sliding along the same line). This is more difficult to understand, since physics is involved here in a more essential way.

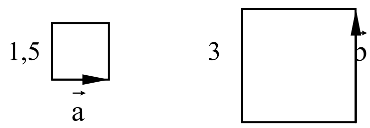
On a wooden compact cube, equal forces are applied, at different points, parallel to the edges.

Students' main difficulty when comparing vectors was their misconception that equality of magnitudes is sufficient for equality of vectors, e.g.:

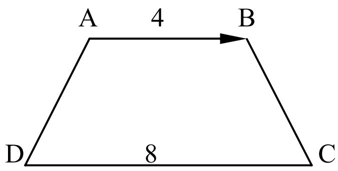
1. Equal magnitudes ≡ Equal vectors; e.g. $\vec{a} = \vec{d}$



2. Vectors with proportional magnitudes
   - Non-collinear: e.g. $\vec{b} = 2\vec{a}$



   - Collinear: Isosceles trapezoid: $\overrightarrow{DC} = 2\overrightarrow{AB}$



Equal magnitudes and different orientation lead to opposite vectors, e.g. $\overrightarrow{AD} = -\overrightarrow{CB}$

3.5 Adding non-collinear vectors

An important didactical comment: **Start teaching vectors in 2-dimension situations!** Our research indicated that 1-dimensional situations are often more confusing than n-dimensional ones because of lack of rich enough geometrical context.[5]

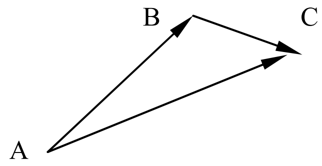
1. The triangle law**:** It is worth mentioning that vectors were used as **operators** on real objects (bodies). Thus operations between vectors were conceived as the **final effect** on

[5]E.g. think of matrix algebra in 1D (where it is trivial and reveals none of the virtues and subtleties that appear in two or more dimensions); or key concepts in differential geometry, like (intrinsic) curvature, which in one dimension is identically zero, but not so in two or more dimensions.

objects.

1.a Situations with displacements

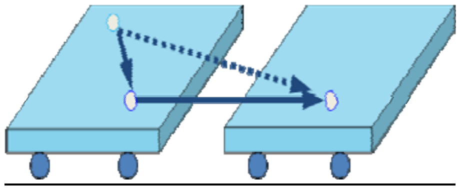
Hercules travels from town A to town C, via B. Indicate his trajectory. Hercules' brother travels directly to C. Compare the final displacement of Hercules to that of his brother.



1.b Situations with velocities

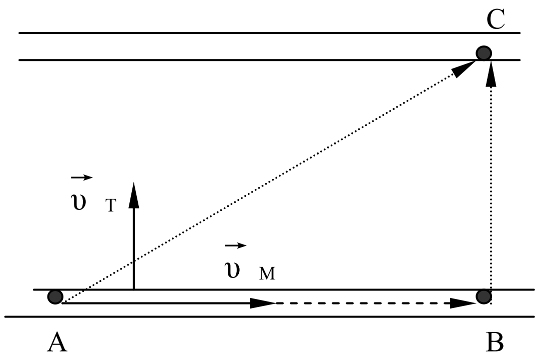
The situations used were thought experiments, based on reconstruction of time, where velocities were treated as successive displacements in a unit time interval. These thought experiments were inspired by Galileo's great achievement, the appreciation of the independence of motions, that was raised to a main epistemological principle of what came to be understood as physics since then[6]. We preferred this presentation, since it is known that many students face difficulties in understanding the concept of velocity in high-school physics. They mostly know only the formula for (constant) speed, and velocity is conceived as a scalar quantity.

1. *The board problem*: A marble is moving on a board, 1m per sec, while the board is moving 5m per sec to the right. The motion is analyzed in two *successive motions:* the marble moves first for 1sec, then the board moves for 1 sec too (this is really deep physics!).



2. *The tube problem*:[7] A marble is moving along a tube with 4cm/sec, while the tube is moving upwards by 3cm/sec. In 2sec:

1. If the tube is not moving, marble's displacement is $\overrightarrow{AB}$.
2. If only the tube is moving, marble's displacement is $\overrightarrow{BC}$.
3. In simultaneous motions, marble's displacement is $\overrightarrow{AC}$.



Teaching improved students' understanding of the triangle law. They escaped from using rules of addition depending on the context in each case (see Booth, 1981).

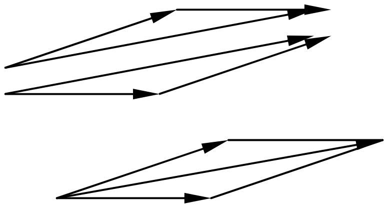
***2. The parallelogram law*:** Simultaneous events with forces were used. However, a preparatory work had been done in the classroom, to introduce the commutativity of vector addition and the parallelism of equivalent vectors.

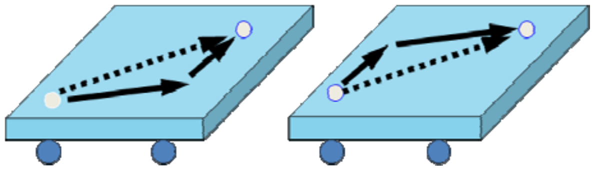
2a. The commutativity of vector addition follows from the independence of motions/displacements. It leads to the parallelogram rule as law **equivalent** to the triangle law, in the sense that it leads to the **same** result (vectors being conceived as operators). A

[6]Motion, which was a *property* of bodies in Aristotelian physics, in Galileo's conception of nature became a *state* of bodies, for which bodies were "*indifferent*". This is intimately related to the "principle of independence of motions", which, seen in a modern mathematical context, is nothing less than the vectorial character of velocity.

[7]The idea for this activity is based on a similar one, included in a Cypriot physics schoolbook for the 11th grade (2nd year of upper secondary education; Gavriilidis & Papadopoulos 1992, p. 66-67).

word of caution however: The triangle and parallelogram laws suit better in different situations each! The former is more suitable for velocities and displacements, whereas, the latter is more suitable for forces.

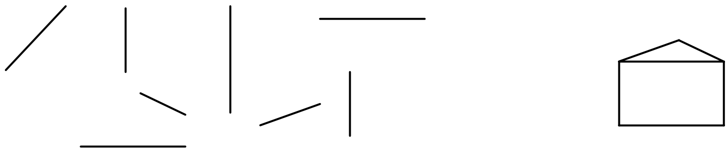




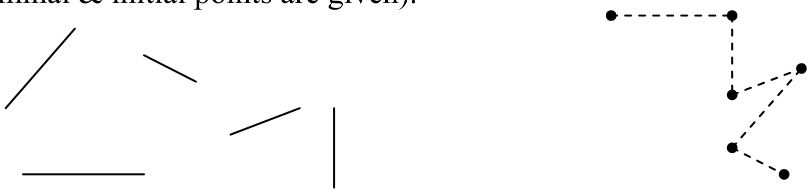
The final displacement of the marble is the same, no matter how we observe/analyze the order of motions. The activity is suitable for the study of the commutativity of addition (triangle law) and as a preparatory one to introduce the parallelogram law.

2b. Preserving parallelism: Situations where parallelism should be preserved.

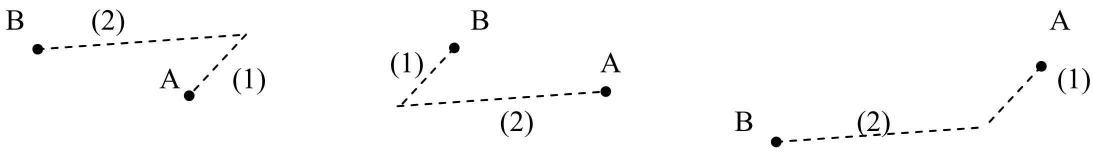
- Line segments moving parallel to form a concrete figure.



- Situations with line segments where students cannot avoid preserving parallelism (terminal & initial points are given).



- Trajectories between points A & B by moving 2 given segments, while parallelism is preserved.



- Trajectories between two given points, by moving two vectors. Preserving only magnitude and path is not sufficient; sense should be preserved too.

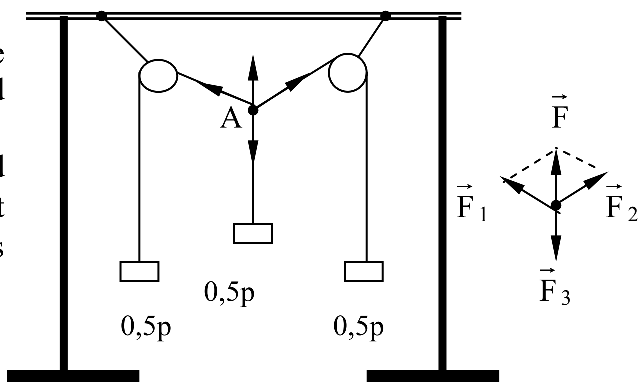
After these preparatory activities, a real experimental activity with forces was performed in the classroom:

What weight should be hung from A, to ensure equilibrium?
Students invented/discovered the parallelogram law through trial and measurements.

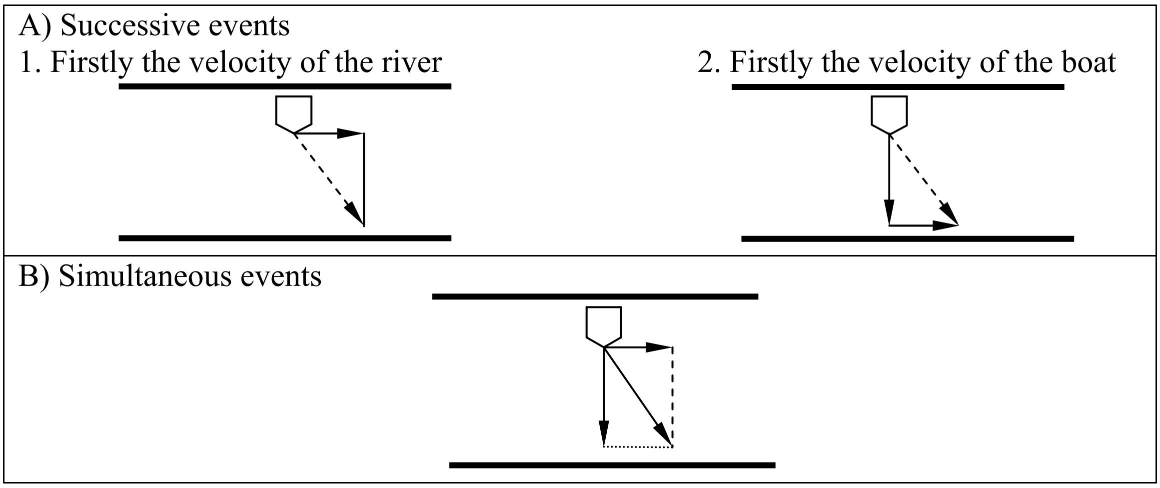
The experiment may also lead some students to a cognitive conflict with their misconceptions. If it is considered that:

$\vec{F}_1+\vec{F}_2= \vec{F}$ and $|\vec{F}_1|+|\vec{F}_2| = |\vec{F}|$ for all directions, then:

$$0{,}5 + 0{,}5 = 0{,}5 \ !$$



3. Equivalence between triangle and parallelogram laws: Students' faced a difficulty of didactical origin: The parallelogram law was connected to the «resultant» force in the physics context, while the triangle law was connected to vector «addition» in the geometrical context. We used didactical activities with successive events. Students realized that the result was the same, either for simultaneous, or for successive motions. The privileged concept for such situations is velocity (recalling Galileo's principle of the independence of motions) in problems like those with the board or the tube (§3.5.1). The problem with a boat moving on a river was also used in the classroom, based implicitly on the independence of motion, by "reconstructing" time.

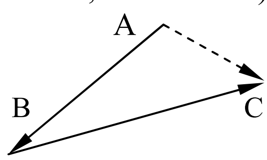


The final conclusion drawn by the students was that the two laws do not replace each other, since they are useful / applicable in different situations: the triangle law fits better to successive vectors, while the parallelogram law to vectors of common initial point. However, their common characteristic is that both lead to the same physical result.

4. Difficulties associated with the triangle and parallelogram laws: We next present some of the difficulties we identified both in connection with the triangle and parallelogram laws. Our research indicated that students managed better the triangle law than the parallelogram one.

(1) *Similarities in form*, *led to intuitive strategies for addition of non-collinear vectors*, which differ from the typical models of the two laws (see Donaldson 1995, Booth 1981). We give some examples:

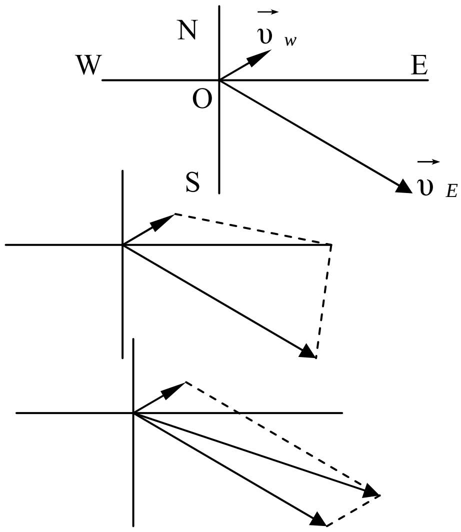
(1a) *Models similar to the triangle law*: According to this law,



"*the sum of 2 successive vectors has initial point the initial point of the 1*[st] *and terminal point the terminal point of the 2*[nd]*":* $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Some students applied the law, even in cases without all the required conditions, e.g.:

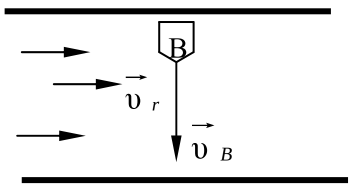
- *Successive but not ordered* (according to the law) *vectors lead to zero or opposite vector:* $\overrightarrow{BC}+\overrightarrow{AB}=\overrightarrow{CA}$ *or* $\vec{0}$
- *Vectors with a common initial point lead to a vector with edge points their edge points:* $\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{BC}$



(1b) *Incomplete parallelograms*: Two vectors with common initial point and a half-line between them, e.g.:

- ✓ *Teacher:"What is a plane's velocity for an observer when NE winds blows 30 miles/h and the engine's velocity is 150 miles /h, in SE direction?"*
- ✓ *John:* E (East!)
- ✓ *Hercules:* By the parallelogram law.

John draws a "parallelogram" with OE as its diagonal. Only after we remind him that the opposite sides should be parallel, he draws the right model.

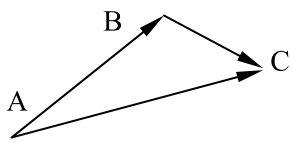


(2) *Difficulties related to the initial point*

Some students had problems with parallel the displacement of vectors for applying both laws. It was not clear for them how to make the vectors successive or with the same initial point.

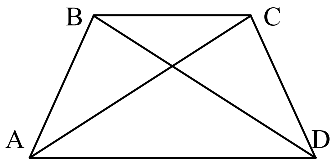
(3) *Difficulties related to the magnitude of the sum of two vectors*; *the symbols* "+" & "=": Students used them as if they were identical to those in arithmetic or algebraic operations they were familiar with. This is an epistemological obstacle. The same difficulty is often encountered by students of mathematics, or even mathematicians, when the same symbols are used while treating isomorphic algebraic structures!

A lot of work had been done to make clear the distinction between vector addition and addition of line segments or numbers. For example:



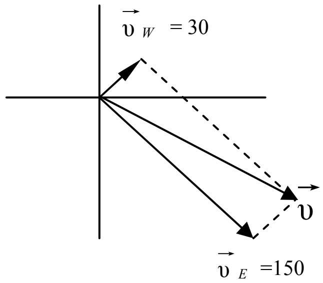
- In the case of the triangle law, for example, students measured: AB= 32, BC=24, AC=45, 32+24 = 56 ≠ 45, and concluded that the symbol "=" means that the two displacements have the same terminal point, the same result. Hence, even though they are equivalent, they are not equal.
- Similarly, for the parallelogram law, when using the experimental setting of §3.5.2, with balance weights 10g, we wrote: $\vec{F}_1+\vec{F}_2=\vec{F}$, 10 + 10 = 10, that is 20=10, and asked for comments:
  - ✓ *Ares:* This is not a simple addition of line segments.
  - ✓ *Alexis:* Vectors are not collinear. Only for collinear vectors of the same sense, the relation $\vec{F}_1+\vec{F}_2=\vec{F}$ and $|\vec{F}_1|+|\vec{F}_2|=|\vec{F}|$ holds.

However for some students, addition of non-collinear vectors was treated as addition of scalars, e.g.:



- ✓ *Teacher:* Compare the sums: $\overrightarrow{AB}+\overrightarrow{BD}$, $\overrightarrow{BD}+\overrightarrow{AB}$, $\overrightarrow{AC}+\overrightarrow{CD}$, $\overrightarrow{CD}+\overrightarrow{AC}$.
- ✓ *Alexander:* All are equal, since these are associated to equal triangles.

Also when estimating the resultant's magnitude:



- ✓ *John:* (trying to find the magnitude of the resultant velocity)
  $\vec{\upsilon}_E + \vec{\upsilon}_W$ = 150 miles + 30 miles = 180 miles.

(4) *Difficulties with composite motions*: In the tube problem (§3.5.1), one of the two motions was ignored. Also students expressed misconceptions based on common sense:

- ✓ The tube will swing!
- ✓ The marble cannot go up, because there is the glass of the tube!
- ✓ The marble will roll a little!
- ✓ The marble remains at rest!

In the airplane problem (§3.5.3), they were unable to distinguish between the two systems of reference: "*The motion as motion! No matter who observes, the motion is vertical!*". For some students the passenger sees both movements, while for others the observer at rest sees only the downfall. Here physics may introduce genuine difficulties, which are known historically. As far as the introduction of vector addition is concerned, and following our experimental teaching, we would suggest: (a) to put emphasis on the independence of motions, (b) to analyze composite motion in successive motions (as displacements per unit of time) and (c) to avoid the use of moving frames.

4 Some concluding remarks

Vectorial concepts exhibit many different aspects. Because of this epistemological characteristic and the appearance of vectors in different parts of the secondary education curriculum, students conceive vectors in the following contexts:

Algebraic: As scalars, characterized solely by their magnitude.

Geometric: As linear segments, characterized by their length.

Physical: Physical terminology, symbols and concepts are used to characterize vectors in their abstract form.

Experiential: Vectors are understood via "spontaneously" generated concepts offered by the everyday life social environment.

Our teaching approach attempted to reveal the multifarious nature of vectorial concepts as much as possible, emphasizing their relation to geometrical and physical situations, without ignoring their more abstract algebraic aspects. In this connection, our approach was historically inspired, profiting **indirectly** from the complicated historical development of the basic vectorial concepts and operations, in the sense described in Tzanakis & Arcavi 2000 (§7.3.2, p.210) for an **implicit** integration of historical elements into teaching. More specifically, this development clearly shows that the prototypical (and generic) examples referred to both physical situations (forces and uniform motions, analyzed to displacements per unit time) and geometrical ones (displacements). It took quite a long time to establish these notions following a complicated path, based on both disciplines. One main point of our approach is that this fact cannot be ignored; instead, it permeated our teaching, somehow setting the agenda for the order and the way the various topics were presented, at the same time helping the teacher to get a deeper awareness of the (epistemological) difficulties inherent in the subject (this is close to Jankvist's concept of a *history-based approach*; Jankvist 2009, §6.3).

It was a difficult task to lead students to overcome the partial conception of vectors that they had gained in the four different contexts above and develop a deeper conception in which aspects coming and/or prevailing in each context are integrated into a coherent whole. This was the main positive result of our experimental teaching that was verified by the statistical analysis of correlations between the answers to tests given before, immediately after and three months after the teaching to both the experimental and control group. The experimental group, both in its own development and in comparison with the control group exhibited a coherent understanding of vectorial concepts, in the sense that students succeeded to put together the different aspects of vectorial concepts into coherent conceptual objects. Nevertheless, students' difficulties and misconceptions greatly varied in character, depth and intensity and it has not been possible to overcome all of them. Below is a short summary of those mentioned in section 3.

Students encountered difficulties to understand the *sense* of vectors: Almost parallel vectors were compared with respect to their sense (§3.1.4), a misconception persisting 3 months after teaching. On the other hand the confusion between *sense* and *orientation* (§3.1.4) and *opposite* and *different* vectors (§3.1.5) that are due to the everyday life use of language were almost completely overcome after teaching.

There were lengthy classroom discussions and debates on the most appropriate and convenient vector notation and students exhibited great inventiveness (§3.2). This active involvement of students in classroom activities and discourse helped them overcome to a large extent the difficulties they faced in connection with vector notation. The analysis of the data from the teaching experiment indicates that the experimental group understood better vectorial notations, in the sense that they used them more consistently. Despite this fact, some students failed to distinguish vectorial from scalar quantities (§3.5.4), which is at least partly due to the use of the familiar symbol of numerical/algebraic equality, "=", for vector equality as well (§3.5.4.3). An associated difficulty is the addition of the magnitudes of non-collinear vectors to get their sum, presumably related to the fact that both equality and addition of vectors are denoted by the same symbols used for scalar (numerical) quantities "=" and "+".

Concerning the triangle law, our teaching helped students to escape from using intuitive, context and case-dependent rules. Using such rules was maintained to a moderate level, in contrast to the control group, where this phenomenon remained strong.

Showing the equivalence of the triangle and parallelogram laws of vector addition was not easy, mainly because of a didactically originated difficulty: The triangle law is primarily used in geometry to find the "sum" of two vectors, whereas the parallelogram law is used in physics to find the "resultant" of two vectors. Given the different conceptual framework and the different terminology and notation, some students faced difficulties to understand the equivalence of the two laws, which became possible, however, with the use of appropriate physical examples (§3.5.4).

Finally, vector addition in the context of situations based completely on composite motions and/or moving frames, did not help much our students, especially in the case of non-collinear vectors, mainly because of interference with experiential conceptions about motion of a pre-Galilean nature (a phenomenon already know in the literature) and the difficulties inherent to physical situations involving moving frames/observers (§3.5.4.4).

However, our teaching approach indicates that leaning upon situations from **both** physics and geometry increased students' ability to interconnect different aspects of vectorial notions, hence, to reach a more coherent understanding and view; not an easy task for such multifarious concepts. We do believe that as far as vector operations are concerned, it is advisable to benefit from situations involving displacements, whereas, if velocities are used, it is better to "translate" them into successive displacements.

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