SOME MATHEMATICAL TOOLS FOR NUMERICAL METHODS FROM 1805 TO 1855

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ABSTRACT

This work covers the first half of the XIX^{th} century, and more specifically the period 1805 to 1855, which concerns our quest to find roots to numerical mathematics. We chose the date of 1805 with Legendre on least squares, and Gauss on the Fast Fourier Transform (FFT), also the closing date of 1855 with Chebychev and his discrete generalized Fourier series. During this period, at least four great mathematicians: Gauss, Cauchy, Jacobi and Dirichlet contributed greatly to the "approximation mathematics", and we shall present some of their motivations. In this present work, what concerns us mostly is what is called the "almost lost and found again". We have selected by chronological order, the 1833 Duhamel principle on convolution, and the 1841 Sarrus article about systems of non-linear equations, with the concept of robustness in numerical mathematics.

1 Introduction

This work covers the first half of the XIX^{th} century, and more specifically the period from 1805 to 1855, which concerns our quest to find roots to numerical mathematics. We selected the date of 1805 and the least squares method for the wave it produced in applied mathematics, and the closing date of 1855 with Chebychev's work on discrete generalized for its modern applications in signal processing. Our objective is to present some mathematical tools which were almost forgotten such as the Duhamel principle with its links with the convolution integral, and the solution of systems of non-linear equations with Sarrus' ideas and the evolution towards robustness in mathematics. Firstly, let us quote Odifreddi (p. 92, 2004) about pure and applied mathematics:

> "Mathematics like the Roman god Janus, has two faces. One is turned inward, towards the human world of ideas and abstraction, while the other looks outwards, at the physical world of objects and material things… The second face constitutes the applied side of mathematics, where the motives are interested, and the aim is to use those same creations for what they can do. "

Householder (p. v, 1970) argued that the adjectives "*pure*" and "*applied*" are meaningful only to describe mathematicians, but not branches of mathematics! The truth is that some mathematicians tried to build mathematical tools for practical applications and their usefulness. They were also motivated by the validity of their approach. This present work concerns a period of fifty years from 1805 to 1855, i.e., from the Napoleonian regime to the Second Empire in France. From this epoch, two men seem to dominate: C.F. Gauss for the fertility of his methods and ideas, and Cauchy for the validity of mathematical tools. If we take Cauchy as an example, he developed the existence theorems for ordinary differential equations, the solution of systems of linear differential equations, the interpolation theory, the spectral theorem for matrices, the convergence properties of the Newton method for finding roots, and the steepest descent for minimization methods, etc. Gauss' contributions on the solution of linear equations were a major breakthrough, and also his Gaussian quadrature for the numerical solution of integrals. These contributions are well presented in books on the history of numerical methods, such as Chabert et & al. *Histoire d'algorithmes* (1993). Let us list by chronological order some main mathematical tools that mathematicians developed for this period from 1805 to 1855:

1805: Legendre: least squares
1805: Gauss: Fast Fourier Transform (FFT)
1809- 1810 : Gauss: least squares
1809-1810: Gauss: gaussian elimination for systems of linear equations
1809 : Gauss: solution of systems of two non-linear equations by *Regula Falsi*
1816: Gauss: gaussian integration
1823: Gauss: iterative method for systems of linear equations
1819,1820, 1830: Horner and Holdred: root computation
1824-1835: Cauchy: on the existence of solutions of ordinary differential equations (ODES's)
Cauchy on systems of ODE
1826: Jacobi: on Gaussian quadrature
1829: Cauchy: convergence of Newton's method for root finding
1829: Lejeune Dirichlet: pointwise convergence on Fourier series
1831: Gauss: on numerical lattices
1833: Duhamel: Duhamel principle for inhomogeneous differential equations
1840: Cauchy: on interpolation
1841: Sarrus: on the solution of systems of non-linear equations
1845; Jacobi: iterative method for systems of linear equations
1846: Jacobi: algebraic eigenvalue problems
1847: Thomson-Dirichlet principle
1847: Cauchy: steepest descent (optimization theory)
1850: Lejeune-Dirichlet: on tessellations
1850: Sylvester: on matrices
1851: Shellbach: numerical solutions of partial differential equations (PDE's)
1855: Chebychev: generalized discrete Fourier series
1855: Cayley: on matrices

This period of time from 1805 to 1855 saw a blossoming of mathematical tools for numerical problems. This genetic approach for the work could be used for second year students in applied mathematics, who have some knowledge on partial differential equations and the method of separation of variables, while the second part of this research could be used for students having some interest on numerical analysis or optimization theory.

2 Duhamel principle and convolutions

Unfortunately, if the convolution integral is a classical modern tool in mathematical physics and signal processing, its long period of genesis is generally ignored by educators.
In its modern definition, a convolution corresponds to the mathematization of a memory problem, smoothing processes, or more precisely, it involves an operator with translational invariance, and the famous principle of causality being that the future cannot influence the present (Sirovich, pp. 80-82, 1988).

During the $XVII^{th}$ century, physical properties of the convolution appeared in the Huygens *Treatise on Light* (1690). Figure 1 is an illustration of Huygens' principles about the propagation and the superposition of spherical waves. He wrote:

> "But we must consider still more particularly the origin of these waves, and the manner in which they spread. And, first, it follows from what has been said on the production of Light, that each little region of a luminous body, such as the Sun, a candle, or a burning coal, generates its own waves of which that region is the centre. Thus in the flame of a candle, having distinguished the points A, B, C, concentric circles described about each of these points represent the waves which come from them"

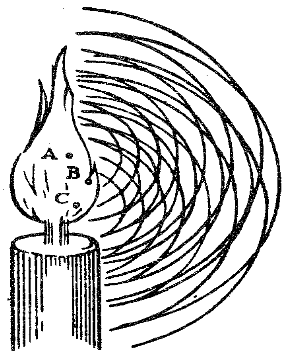

Figure 1: Huygens' candle and the propagation and superposition of spherical waves

Convolution integrals appeared at the beginning of the XIX^{th} century in the potential theory, the heat conduction equation, and the wave equation, with Cauchy, Fourier and Poisson. They appeared from trigonometric operations as a convenient way to represent analytical results. For example, Poisson (1823, p. 434) wrote his Euler-Fourier series as:

$$fx = \frac{1}{2\ell}\int_{-\ell}^{+\ell} fx'dx' + \frac{1}{\ell}\int_{-\ell}^{+\ell}\sum \cos\frac{n\pi(x-x')}{\ell} fx'dx' \quad (1)$$

Indeed the history of the convolution integral is also linked to integral equations and Abel's work (1826) who presented a convolution integral for the time of descent of a particle starting at a point *P* sliding down a smooth curve.

From these works emerge the 1815 Cauchy contribution *Wave propagation in deep water* (Cauchy 1815-1827; Dahan Dalmedico, 1989) and the Duhamel principles (1833) on radiating heat processes, with complex radiating boundary conditions. These Duhamel principles are occasionally mentioned in books on mathematical physics such as Courant and Hilbert *Methods of Mathematical Physics* (vol. 2, 1953-1962), or Hildebrand *Advanced calculus for Applications* (pp.464-465, 1976). Jean-Marie Constant Duhamel was born in Saint-Malo, France in 1797, and died in Paris in 1872. He became a professor at the École polytechnique in Paris. His third theorem is stated as follows:

> "If a part of the surface of a body is maintained, along with certain points within the interior, at temperatures that vary with time in some way, and the rest of the surface radiates in a medium where all points have some variable temperatures with time, we will obtain in the following way, temperatures of the points of the system, and we will calculate the temperatures of the different points of the body, while supposing that no change takes place in temperatures of the medium, and of all the points of the surface and its interior, of which the law of temperatures is given: we shall consider next the increases produced, at some instant in the temperatures of these last points and of the medium, and we shall calculate the temperatures of various points of the body at the end of time *t*, by giving them zero as initial temperature: we will do the sum of the temperatures of homologous points of these systems, and we will that way obtain temperatures of the same points in the proposed system."

Duhamel introduced into mathematics the Huygens physical concepts about memory, time-delay, and superposition of events. For example, let us assume that $\varphi(t)$ is the variable temperature of the surrounding medium of a one-dimensional heat conducting rod, which is radiating into this medium, at its extremities. Then, $\varphi(t)$ has a complex action on the temperature of the rod. Duhamel assumed that he started from an initial condition $\varphi(0)$ with an action $A(x,t)$. Then, at time θ_1 and the condition $\varphi(\theta_1)$, the action would be delayed as $A(t-\theta_1)$ updated by an additional step $(\varphi(\theta_1)-\varphi(0))$, and so on. It was a discretisation of a complex integration, a technique of quadrature, and the final temperature corresponded to the summation of the action of all rectangular panels. The result would be the classical convolution integral. The summation process and the convolution integral are illustrated in the following equations:

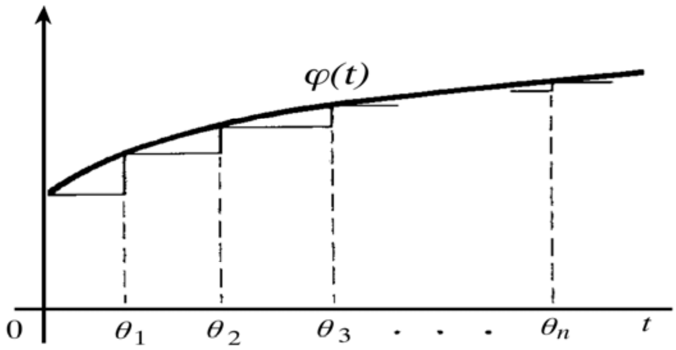

Figure 2: Approximation of the function $\varphi(t)$ by rectangular panels.

$$V(x,t)=\varphi(0)A(x,t)+[\varphi(\theta_1)-\varphi(0)]A(x,t-\theta_1)+[\varphi(\theta_2)-\varphi(\theta_1)]A(x,t-\theta_2)]+...$$
$$+[\varphi(\theta_n)-\varphi(\theta_{n-1})]A(x,t-\theta_n)]$$
$$V(x,t)=\varphi(0)A(x,t)+\sum_{k=0}^{n-1}A(x,t-\theta_{k+1})\left(\frac{\Delta\varphi}{\Delta\theta}\right)_k\Delta\theta_k$$

We obtain the convolution integral by taking the limit version of the above expression:

$$V(x,t)=\varphi(0)A(x,t)+\int_0^t A(x,t-\theta)\varphi'(\theta)d\theta \qquad (2)$$

Duhamel illustrated his technique by applying it to the general solution of the heat conduction equation of a one-dimensional problem with three arbitrary conditions,

$$\frac{\partial V}{\partial t}=a\frac{\partial^2 V}{\partial x^2};\ D=\{(x,t)\,|\,0<x<\ell,t>0\}, \qquad (3)$$

where a is the thermal diffusivity coefficient, ℓ is the length of the rod, and D is the domain of integration. The equation was coupled with radiating boundary conditions at both ends:

$$A\frac{\partial V}{\partial x}(0,t)-h(V-\varphi(t))=0$$
$$A\frac{\partial V}{\partial x}(\ell,t)+k(V-\psi(t))=0 \qquad (4)$$

and the initial condition: $V(x,0)=F(x)$, where $\varphi(x),\psi(x),F(x)$ are arbitrary functions. The radiating boundary conditions corresponded to the Newton law of cooling, and both ends radiated into mediums of thermal conductivities *h* and *k,* while the lateral surface is isolated.
Duhamel proceeded by steps:

a) He first looked for a permanent simplified solution in the limit $t\to+\infty$, with $\varphi(t)=V_1=const\ \ \psi(t)=V_1+\xi=const$.
b) He then solved the time dependant problem for homogenous boundary conditions with $V_1=\xi=0$, and then did a transformation of variables in the case $V_1\neq\xi\neq0$.
c) Finally, he replaced V_1 and $V_1+\xi$ by the initial values $\varphi(0)$ and $\psi(0)$,
d) At time $t-\theta$, he superposed a solution $V_1 \cong \varphi'(\theta)\,\Delta\theta,\ \xi\cong(\psi'(\theta)-\varphi'(\theta))\,\Delta\theta$

Duhamel ended up with an integral of the type: $\int_0^t e^{-am^2(t-\theta)}\{M\varphi'(\theta)+N\psi'(\theta)\}d\theta$, which has the same behavior as the convolution integral given by Eq. 2. The solution appeared as a time-convolution integral in terms of the derivatives of the boundary conditions.
Convolution integrals also, slowly, appeared in linear non-homogenous ordinary or partial differential equations. An example of this was the Liouville *second memoir* (1837) on differential equations, where the convolution integral appeared in the solution of the differential equation:

$$\frac{d^2U}{dx^2}+\rho^2U=\lambda U$$
$$U(x)=A\cos\rho x+B\sin\rho x+\frac{1}{\rho}\int_0^x\lambda(x')U(x')\sin\rho(x-x')dx' \qquad (5)$$

The famous 1882 Kirchhoff formula for the solution of the three-dimensional wave equation can also be understood as a time-convolution integral (Kline, vol.2, p. 694, 1972).

3 Zeros of a vector equation: towards robustness

The second tool concerns the 1841 Sarrus contribution to the solutions of non-linear vector equations. It is an enlightened anticipation towards the modern concept of robustness in mathematics (Box, 1953), because finding the roots of a vector equation is a very difficult task. A method is said to be robust, if it is reliable and efficient. For example, the classical least squares method is not, because it is too sensitive to outliers. Firstly, we shall review the state of knowledge for finding roots of a single non-linear equation $f(x) = 0$. This is one of the most commonly occurring problems of applied mathematics. At the beginning of the XIX^{th} century, available tools were the Newton-Raphson method, the *regula-falsi* (also known as the method of the false position) and the method of continued fractions. These numerical methods for root finding are iterative methods. The ongoing interest in continued fractions was reflected in Gergonne and Liouville journals, where we found at least 7 articles concerning this topic. In England, important progress was made in computation of a real root of a polynomial equation with the Horner-Holdred rule. In 1818, Fourier (Ostrowski, 1966) emphasized that the Newton-Raphson method was one of the most useful tools in all analysis. This is why it was important to complete it and to overcome its deficiencies, i.e. the divergence problem. Indeed, for the period of time between 1805 to 1855, the 1829 Cauchy contribution about convergence properties of the Newton method dominated. At this epoch, Lagrange book "*Traité de la résolution des equations numériques*" (1798) became a corner stone in this domain. Again, the two volume Legendre book "*Théorie des nombres*" (1830) contained an appendix about numerical root findings and a voluminous chapter on continued fractions. Given this state of knowledge for the roots of a non-linear equation, we examine the problems for the zeros of a vector equation. From Gauss (1809), Fourier (1818), and Sarrus (1841), we were struck by the concern of geometers about the robustness of their methods. This robustness is directly linked to the consequences of the intermediate value theorem for continuous functions, which states that:

> "*between every two values of the unknown quantity, which give results of opposite sign, there must always lie at least one real root of the equation*."

Before Bolzano (1817), proofs of this theorem were based on geometrical propositions. Furthermore, Bolzano utilized a bisection technique on the proof of one of his theorem.

These robustness concepts appeared in the 1809 Gauss article who selected the analogy of the *regula falsi* for two equations with two unknowns. The *regula falsi* utilizes the intermediate value theorem and root bracketing. The concept of the *regula falsi* was to progressively decrease the interval of uncertainty for root findings.

In 1841, Pierre–Frédéric Sarrus published his article in Liouville Journal: *Sur la résolution des équations numériques à une ou plusieurs inconnues et de forme quelconque*. Sarrus was born in 1798 in a small town in France. In 1829, he became professor of mathematics at the University of Strasbourg. He died in 1861. He published regularly in Gergonne's journal, where he had 23 publications from 1820 to 1828.

In his 1841 article, Sarrus proposed three safe methods for root findings. In the first method, he searched for upper and lower bounds for all variables, which include the roots. Then, he proposed a subdivision of the system of bounds for root bracketing. Unfortunately, Sarrus was not explicit enough on his subdivision; probably, it corresponded to a technique of bisection. This technique of division of an interval by a factor 2 was well known as a binary search during the $XVIII^{th}$ century. For example, it was the algorithm for finding a word in a dictionary. We can consider that Sarrus' first method is an ancestor for the modern "cubic" algorithm where roots are included in n-dimensional parallelepiped with sides parallel to the coordinate axes (Hansen, 1980; Galperin, 1988). Sarrus' style was very direct, efficient and modern with almost an algorithmic approach, as we will see in the following sentences:

> "In view of one or more equations of any form:
>
> $L = 0, M = 0, N = 0, \ldots,$
>
> In one or several unknowns, which number can be different of that of the equations; in view of, besides a system of limits of values of unknowns, find all values of unknowns which can be comprised between the given limits, and satisfy, at the same time the equations $L = 0, M = 0, N = 0, \ldots$.
>
> *First method*: We will start by looking at inferior limits of values which can be received by the functions $L, M, N, \ldots$ when we vary $x, y, z, \ldots$ between the given limits.
>
> But when all calculated inferior limits will be negative, we will proceed to calculate the superior limits of values of the same functions $L, M, N, \ldots$."

We then have to reduce the intervals of uncertainty for each variable:

> "Accordingly to that, we will subdivide the system of given limits of values $x, y, z, \ldots$ in several systems of limits more closely, which the entirety will contain the same extent as that of the systems of primitive limits"

Sarrus'second method was also based on interval analysis, but this time, Sarrus transformed a system of non-linear equations into a minimisation problem. In 1847, Cauchy will do the same for his steepest descent algorithm.

> *Second method*. We will take some positive numbers α, β, γ, and doing so, to abridge,
>
> $V = \alpha L^2 + \beta M^2 + \gamma N^2 + \ldots$
>
> We will then only have to resolve the sole equation $V = 0$. Then we will treat this latter equation by the process of the first method, but with this difference, that it is entirely useless to calculate the upper limits of values of the auxiliary function V. Indeed, according to the compositions of this function, it can never become negative; consequently, his upper limits will always be positive, and that's everything we need to know.

The third method corresponded to a linearization of the equations and a modified generalized Newton method for a system of non-linear equations. Again, Sarrus, being a Fourier follower, was searching upper and lower bounds for all variables. He said to stop calculations, if the system had tendency to diverge.

> *Third method.* It is easy to modify Newton's method, in such a way, that it is never at fault, at least, till the number of unknowns will not surpass that of the equations."

Unfortunately, Sarrus didn't illustrate his methods with examples. Curiously, in 1847, Joseph Liouville, presented a new method for the solution of vector equations, but he hadn't mention Sarrus' contribution!

4 Conclusion

For this work, we could have chosen as well the 1846 Jacobi method on the eigenvalue problem. This method is well covered in any textbook on numerical analysis, but the history of the numerical eigenvalue problem is not. Among other fruitful mathematical tools which were developed during the period of time from 1805 to 1855 are the 1850 Dirichlet work on tessellations, the 1851 Shellbach work on numerical solutions of partial differential equations, and the 1855 Chebychev article on generalized discrete Fourier series.

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