PHYSICS–MATHEMATICS RELATIONSHIP
Historical and Epistemological Notes

Raffaele PISANO
Centre François Viète, Université de Nantes
pisanoraffaele@iol.it

ABSTRACT

Nowadays it is unthinkable to learn and teach the scientific sense of physical and mathematical sciences without deepening its intellectual and cultural background, e.g. history and its foundations. In my talk several case–studies on the relationship between physics and mathematics were presented. Here, for brevity's sake, I will only discuss some of them.

1 Notes on Archimedean science and its heritage in *Opera geometrica*

1.1 On Archimedes

Archimedes (fl. 287–212 B.C.) was a deeply influential author for Renaissance mathematicians according (we can say) to the two main traditions: the humanistic tradition, adhering strictly to philological aspects, followed by Willem van Moerbeke (1215–1286), Regiomontanus (1436–1476) and Federigo Commandinus (1509–1575) and the purely mathematical tradition followed by Francesco Maurolico (1694–1575), Luca Valerio (1552–1618), Galileo Galilei (1564–1642) and Evangelista Torricelli (1608–1647). Nevertheless, Archimedean tradition represents an historical and epistemological component which has apparently already been solved (Heath 2002, XXXIX). One might ask: *how did the various cultures, practical and theoretical environments interact with the recovery of the Archimedean roots for the birth of modern science?* The response is not commonplace at all and requires more space (Capecchi and Pisano 2010b). Nevertheless, an hypothesis can be made regarding a frame of reference of the roots and of the Archimedean tradition and its Renaissance physical–mathematical relationship can be divided (as an outline) into a mixture of four epistemological interpretations: 1) a strictly mathematical Tartaglia root 2) a humanistic Moerbeke–Regiomontanus–Commandino root, 3) a mathematical–geometrical Maurolico root and 4) a modern Valerio–Galilei–Torricelli's root. (Capecchi and Pisano 2007). Based on previous works (Pisano 2007; Pisano 2008; Capecchi and Pisano 2010a). Here I present some epistemological reflections on the fourth. Knowledge of Archimedes' contribution (Clagett 1964–1984) is truly fundamental for a historical study of Evangelista Torricelli's (1608–1647) mechanics. Archimedes set mathematical *rational criteria* for determining physically centres of gravity and his work contains physical concepts formalized on mathematical–geometrical foundations. In *Book I* of *On Plane Equilibrium* (Heath 2002, 189–202; Heiberg 1881) Archimedes, besides studying the rule governing the law of the lever also finds the centre of gravity of various geometrical plane figures (Heath 2002). Moreover Archimedes does not develop all mechanics axiomatically, but sometimes he uses an approach for problems (the problematic approach). In that sense the relationship between physics and mathematics in the theory could produce novelties in its historical foundations: e.g., the epistemological status of the *Suppositio* is different. In fact, Archimedean suppositions are not (all of them) self–evident as are those of the Euclidean (and Aristotelian) tradition and may have an empirical nature. The use of *ad absurdum* proofs, due to the lack of reference to the first suppositions, does not allow for the assumption of a strictly mathematical–axiomatic structure (Bailly and Longo), e.g., typically deductive.

1.2 On Torricelli

In regard to Torricelli's works, I focus on *Opera geometrica* (Torricelli 1644; Capecchi and Pisano 2007; Pisano 2009). In particular, I focused on his discourses on the theory of the centre of gravity dealing with his famous principle in *Liber primis De motu gravium naturaliter descendentium*.[1] Evangelista Torricelli, in his theory on the centre of gravity[2], followed Archimedes' physical–mathematical approach using and the proofs can change, too: a) *Reductio ad absurdum* as a particular instrument for mathematical proof; b) Geometrical forms implicit in weightless beams and indirect reference in geometrical forms to establish the law of the lever; c) Empirical results to establish principles. Torricelli, e.g., conceived twenty–one different ways of squaring (Heath 2002, *Quadrature of the parabola–Propositio* 17 and 24, 246; 251) the parabola, which had already been studied by Archimedes: eleven times with exhaustion, ten with the indivisibles method. The *reductio ad absurdum* proof is very often present. Some results obtained via the indivisible technique were always checked by using different methods.

Torricelli presents problems that remain unsolved, according to him, in Galileo's dynamical theory. His main concern is to prove Galileo's supposition according to which velocity degrees for a body are directly proportional to the inclination of the plane over which they move (also called *Galileo's theorem*): "The speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal" (Galilei 1890–1909, Vol., VIII, 205). It is an attempt to prove Galilei's *supposition*. Torricelli seems to suggest that this supposition may be *physically proved* beginning with a "theorem" according to which "the momentum of equal bodies on unequally inclined planes are to each other like the perpendicular lines of equal parts of the same planes" (Torricelli 1644, 98). Moreover, Torricelli also assumes that Galilei's theorem has not yet been *mathematically* (in Archimedean sense) *proved*.[3]

1.3 On Archimedean influence in Torricelli's mechanics

In many parts, Torricelli explicitly declares his Archimedean background[4]. Like Archimedes, in the case of *quadratura parabolae*, he first obtains results via the mechanical approach and then reconsiders the discourse with the classical methods of geometry to confirm in a *rigorous* way the correctness of his results. Thus, Torricelli, with the help of a driving idea of a duplicate procedure, devotes many pages to proving a

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[1]"*Praemittimus* [of equilibrium]. Duo gravia sumul coniucta ex se moveri non posse, nisi centrum commune gravitatis ipsorum discenda". (Torricelli 1644, 99). ("It is impossible for the centre of gravity of two bodies in a state of equilibrium to sink from any possible movement of the bodies" [my translation]).

[2]The *Opera geometrica* is organized into four parts. Particularly, Part 1, 2, 3, are divided into *books* and Part 4 is composed of an Appendix (Torricelli 1644; Capecchi and Pisano 2010a).

[3]In the first edition of Galilei's *Discorsi e dimostrazioni matematiche* (Galilei, 41–458) in 1638, there is no proof of the "theorem". It was added to the edition of Galilei's Works, Bologna 1656. (See some letters from Torricelli to Galilei regarding the "theorem" (Torricelli 1919–1944, vol. III, 48, 51, 55, 58, 61).

[4] E.g.: "Inter omnia opera Mathematics disciplinas pertinentia, iure optimo Principem sibi locum vindicare videntur Archimedis; quae quidem ipso subtilitatis miraculo terrent animos" (Torricelli 1644, 7).

certain theorem on the "parabolic segment", first by following the geometry of ancients[5] and then by proving the validity of the thesis also by means of the first "indivisibilium" (Heath 2002, 55–84*), idem* for the "solido iperbolico acuto"[6]. He states that the ancient geometers moved according to a method other than that followed in "in invenzione" suitable "ad occultandum artis arcanum". In this respect, it is interesting to note that he underlines the "concordantia"[7] of methods of different "rigours"[8]. Torricelli also seems to hold on to the idea for which the method of mathematical demonstration of ancients, as in Archimedes' method, has intentionally been kept hidden. But the Archimedean tradition in Torricelli's work goes further. In *De sphaera et solidis sphaearalibus* (Heath 2002) he presents an enlargement of the Archimedean proofs in books I–II of *On the sphere and cylinder* (*Ivi*). Moreover, Torricelli faces problems not yet solved by Archimedes or by the other mathematicians of antiquity. Adopting the same style as Archimedes, he does not try to obtain the first principles of the theory and does not limit himself to a single way of demonstrating. A prominent example is the *Quadratura parabolae pluris modis*:

> […] In quibus Archimedis Doctrina de sphaera & cylindro denuo componitur, latiùs promovetur, et omni specie Solidorum, quae vel circa, velintra, Sphaeram, ex conversione poligonorum regularium gigni possint, universalius Propagatur.[9]
> Veritatem praecedentis Theorematis satis per se claram, et per exempla ad initium libelli proposita confirmatam satis superque puto. Tamen ut in hac parte satisfaciam lectori etiam Indivisibilium parum amico, iterabo hanc ipsam demonstrationis in calce operis, per solitam veterum Geometrarum viam demonstrandi, longiorem quidem, sed non ideo mihi certiorem.[10]

Let's note that exposition of the mechanical argumentation present in Archimedes' *Method* was not known during Torricelli's time, because the *Method* was discovered by Johan Ludvig Heiberg (1854–1928) only in 1906. Therefore, in Archimedes' writing there were lines of reasoning which, due to lack of justification, were labelled as mysterious by most scholars. Then, to prove both methods it was necessary to assure the reader not only of the validity of thesis, but mainly to convince him of the strictness of Archimedes' exhaustion reasoning and *reductio ad absurdum*, by proving his results with some other technique. Archimedes's himself did not attribute the same certainty to his method, as he did to classical mathematical proofs[11]. His reasoning on *Quadratura parabola* (Heath 2002, Proposition XXIV, 251) is exemplary. Addressing Eratosthenes (276–196 a.C.), he wrote at the beginning of his *Method* (Heath 1912, 13). One of the characteristics of Torricelli's proofs was the syntactic recall to demonstrating the approach followed by the

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[5]"Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta" (Torricelli 1644, 17–54). In *Opera geometrica* there are also some reference to Euclid's *Elements*, to Apollonius's *conic sections*, Archimedes, Galileo, Cavalieri's works.

[6]"De solido acuto hyperbolico problema alterum" (Torricelli 1644, 93–135). "De solido hyperbolico acuto problema secundum (*Ivi*, 112–135).

[7]"De solido acuto hyperbolico problema alterum" (Torricelli 1644, 103). "Concordantia praecedentis demonstrationis cum doctrina Archimedis" (*Ivi*).

[8]"Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta" (Torricelli 1644, 55).

[9]Torricelli 1644, *De sphaera et solidis sphaearalibus*, 2.

[10]Torricelli 1644, *De solido hyperbolico acuto problema secundum*, 116.

[11]It is well known that in the *Method* (Heath 1912; Id., 2002) Archimedes studied a given problem whose solution he anticipated by means of crucial propositions which were then proved by the *reductio ad absurdum* or exhaustion.

ancient Greeks with the explicit declaration of the technique of reasoning actually used. Besides the well known *Ad absurdum* there were also the *permutando* and *the ex aequo*. In *De proportionibus liber* he defines them explicitly:

> *Propositio* IX. Si quatuor magnitudines proportionales fuerint, et permutando proportionales erunt. Sint quatuor rectae lineae proportionales AB, BC, CD, DE. Nempe ut AB prima ad BC secundam, ita sit AD tertia ad DE quartam. Dico primam AB ad tertiam AD ita esse ut secunda BC ad quartam DE. Qui modus arguendi dicitur permutando.
>
> *Propositio* X. Si fuerint quotcumque, et aliae ipsis aequales numero, quae binae in eadem ratione sumantur, et ex aequo in eadem ratione erunt. Sint quotcumque magnitudines A, B, C, H, et aliae ipsis aequales numero D, E, F, I, quae in eadem ratione sint, si binae sumantur, nempe ut A ad B ita sit D ad E, et iterum ut B ad C, ita sit E ad F, et hoc modo procedatur semper. Dico ex equo ita esse primam A ad ultimam H, uti est prima D ad ultimam I. Qui modus arguendi dicitur ex aequo).[12]

Torricelli seems to neglect the algebra of his time and remains glued to the language of proportions. He dedicated a book to this language, *De Proportionibus liber* (Torricelli 1919–1944, 295–327), where he only deals with the theory of proportions to be used in geometry. In this way he avoids using the *plus* or *minus*, in place of which he utilizes the *composing* (*Ivi*, 316) and *dividing* (*Ivi*, 313). Such an approach allows him to always move with the ratio of segments. By following the ancients to sum up segments he imagines them as aligned and than translated and connected, making use of terms like “simul” “et” or “cum” (*Ivi*, Prop. XV, 318). In the following section, we present a table which summarizes the most interesting part of *Proportionibus liber* where Torricelli again proves theorems by referring to Archimedean style reasoning.

Table 1 The Archimedean mathematical–physical approaches–proofs in Torricelli’s *Quadratura parabolae*

Contents	**Kind of proofs**	**References (Torricelli 1644)**
		QUADRATURA PARABOLÆ PLURIS MODIS […] ANTIQUORUM
Lemma II,V,VI, X–XI,XII–XIII XVII – Propositio IV	*Ad absurdum* proof	“Quadratura parabolæ pluris modis per duplicem positionem more antiquorum absoluta”, 17–54. (Torricelli 1644, *Opera geometrica*, op. cit.)
Lemma XIV	*Ex æquo* et *dividendo* et *permutando*	(*Ibidem*)
Lemma XVI, XVIII	*Ex aequo*	(*Ibidem*)
Lemma XIX	*Ex æquo* et *Ad absurdum* proof	(*Ibidem*)
Propositio III	*Componendo*	(*Ibidem*)
Propositio V	*Ad absurdum* proof and *Componendo*	(*Ibidem*)
Propositio IX[13]	*Ex æquo* et *Ad absurdum* proof	(*Ibidem*)
		DE SPHAERA ET SOLIDIS SPHAEARALIBUS
Propositio XVIII	*Ad absurdum* proof	“De sphæra et solidis sphæralibus”, Libro Primo, 28–29 (*Ivi*)

[12]Torricelli 1919–1944, Proportionibus liber, pp. 313–314.

[13] *Lemma* e *Propositio* are proved by means of *componendo* and *conversione.*

Propositio XIX	*Ad absurdum* proof	"De sphæra et solidis sphæralibus", Libro Primo, 30–33 (*Ivi*)
Propositio XXII–XXIII	*Ex æquo*	"De sphæra et solidis sphæralibus", Libro Primo, 35–36 (*Ivi*)
Propositio IV	*Idem* proof powered by Archimedes	"De sphæra et solidis sphaearalibus", Libro secondo, 51–52 (*Ivi*)
		QUADRATURA PARABOLÆ PER NOVAM INDIVISIBILIUM
Lemma XXII,XXIX	*Ex æquo*	"Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 61–72 (*Ivi*)
Lemma XXX Lemma XXXI[14]	Idem proof powered by Archimedes *Equiponderant*	"Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 74–77 (*Ivi*)
		DE SOLIDO ACUTO HYPERBOLICO
		"De solido acuto hyperbolico problema alterum", 93–112 (*Ivi*)
Exemplum I–II, IV–V, X,XII–XIV Altri *Lemma* e *corollari*[15]	Proof *in concordantia præcedentis demonstrationis cum doctrina Archimedis* *Ex æquo* (Exemplum X)	"De solido acuto hyperbolico problema alterum", 95–108; (*Ivi*) (Ivi, 113–135)
Exemplum III,XI	Proof *in concordantia cum theoremata Euclidis*	"De solido acuto hyperbolico problema alterum", 97; 104–105 (*Ivi*)
		DE MOTU PROIECTORUM[16]
Lemma (Follow from Prop. XXXIII)	Proof of Archimedes' proposition on *Conoids and spheroids*[17]	"De motu proiectorum", Libro secondo, 187 (*Ivi*)
		DE PROPORTIONIBUS LIBER[18]
Propositio[19] III Propositio VIII Propositio IX Propositio X Propositio XI Propositio XII Propositio XIII–XIV Propositio XV Propositio XVI	*Ad absurdum* proof *Dividendo* *Permutando* *Permutando* et *Ex æquo* *Ex æquo* *Componendo* et *Conversone* *Permutando* *Permutando* et *Componendo* *Permutando* et *Ad absurdum* proof	"De Proportionibus liber" (Torricelli E. 1919–1944. *Opere di Evangelista Torricelli*, by Loria G. et Vassura G., Vol. I, Parte prima, Montanari, Faenza, 308; 313–317).

We notice that only proofs by means of indivisibles are not *reductio ad absurdum*. This is because these proofs are algebraic. Instead, in nearly all other proofs he uses the

[14] Torricelli cites "Antonio Roccha praestandi Geometra" (Giovannantonio Rocca (1607–?) as author of) *Lemma XXXI* in the style of the "Schola Cavaleriana" which he attempts to demonstrate. (Torricelli 1644, "Quadratura parabolæ per novam indivisibilium Geometriam pluribus modis absoluta", 76). Caverni also cites Rocca. (Caverni 1895, Vol. IV, 136, line 8).
[15] The others are not *ad absurdum* proofs. They are proved by geometrical construction.
[16] Torricelli 1644, "De motu proiectorum", Libro secondo, 154–190, line 15.
[17] Heath 2002, 99–150
[18] At the end of the book he wrote in *concordantia* to show the reader that he was in agreement with Euclid's demonstration (e.g., propositions XVII and XVIII, XXIV and XXV of *Elements, Book V.*).
[19] The others are not *ad absurdum* proofs. They are proved by geometrical construction.

mathematical technique typical of proportions, *Dividendo, Permutando* ed *Ex aequo*. In the end, Torricelli adopts the Archimedean method with some variations:

a) The use of Bonaventura Cavaliere (1598–1647)'s *method of the ancients* and *method of the indivisibles*[20] (Torricelli 1644), that is to say, differently from his predecessors, Cavaliere dealt with physical–geometrical matters using both methods, offering the reader judgment on the quality of the methods used.
b) This makes the analysis and interpretation of scientific thought and of the Torricelli *corpus* even more complex and therefore more interesting compared to the type of geometry applied, for example, to statics by his predecessors.
c) The later time period suggests a greater attention for those who, in 1644 on the threshold of the birth of the Leibneizian–Newtonian infinitesimal analysis and with theoretical knowledge of both the geometry of the Cartesian coordinates and of the indivisibles, sensed the necessity for a re–evaluation of the *method of the ancients* also attempting, for themselves, the comparison as can be seen clearly from Torricelli's correspondence with Cavalieri (Torricelli 1919–1944)[21].

2 Notes on the historical epistemology of science

2.1 The role played by logic and language in physics–mathematics contents

Crucial features in scientific language have to do with the paradox of the formalization of logic (Carnap 1943). In order to express the axioms and to construct a *meta–discourse* about them, we should use the natural language, which is not formalized; we cannot formalize it in advance, because we risk producing a regression to *infinitum*. Moreover, it is not natural to state the axioms of logics and then to consequently deduce all the rest from them (*Ibidem*).

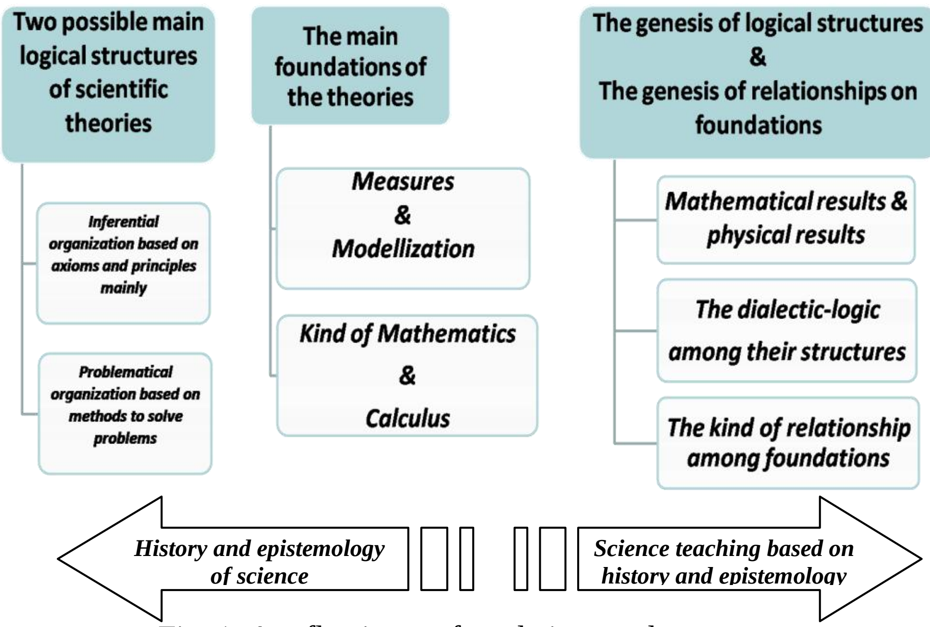

Fig. 1 A reflection on foundations and structures

In mathematical–classical logic, so–called *well–formed–statements* are assumed to be

[20]E.g.: *Quadratura parabola* (Torricelli 1644, 17–84).
[21]One could see letters between Cavalieri and Torricelli: "Racconto d'alcuni problemi, carteggio scientifico", by Giuseppe Vassura, Vol. III. See also: Capecchi and Pisano 2007.

either true or false, even if we do not have proof of either. In fact, from an inferential and classical logic system (e.g. a list of inferential propositions) one can only obtain a scientific dichotomy of hypothesis–these *free–from–self–contradiction* and among them, and to be *scientific*, a theory must be testable, e.g., subject to *falsification* (Popper 1963). Let's note that in that kind of system of reasoning, it is not possible to obtain *undecidable contents*, e.g. like (apparently) those generated by scientific DNSs belong to non classical logics. Particularly, if *undecidable* contents belong to a given principle of the theory, then we have an *out of the ordinary principle*.[22]

In the historical epistemology of science in relation to the logical structure of scientific theories, one can encounter both *Axiomatically Organised* theories (AO–theories) from which a few self–evident principles (or axioms) the whole theory is derived, as well as theories whose organization is based on solving given problems contained in the theories, which are thus *Problematically organized* (PO–theories). The assumed principles are often not as self–evident as the axioms are non–axiomatic principles. An AO theory is generally developed by means of advanced mathematics (e.g., mathematics–physics theory by differential equations, et al..) which starts its derivations directly from the axioms. A PO theory uses less advanced mathematics with the principles that only indicate a direction for the development of the theory and they may be «methodological» in nature (Kieseppä 2000). Moreover, it is characterized by the use of DNSs and most of the results are expressed by *reductio ad absurdum* statements. In previous studies it has been noted[23] that when in a scientific theory DNSs are largely used, a use of sophisticated mathematics is lacking, and the theory is based on declared problems to solve and without stating axioms or principles typically of an Aristotelian approach. In this sense, a formal characteristic of PO appears to be the occurrence of some DNS's that cannot be turned into equivalent positive sentences because the operative tools for proving them do not exist. In other words, as previously mentioned, in this kind of theoretical organization, the scientific contents of DNS «$\neg\neg A$» cannot be converted into an affirmative sentence corresponding to «A» because of the latter lack scientific proof. Therefore, DNSs, within the scientific theory, characterize a particular approach to science. Following that point of view, a borderline between classical logic and most non–classical logics is represented, not by the law of the excluded middle, but the double negation law. Generally speaking, when the double negation law fails, we are arguing outside classical logic and, *in–first–*

[22]Generally speaking, in non classical (or *constructive* or *intuitionistic*) logics, a statement is only true if there is proof that it is logically true, and only false if there is proof that it is logically false. In some previous papers (Drago and Pisano 2000; Id., 2004) on the history and foundations of physics, it has been shown that the discursive part of Sadi Carnot's *Réflexions sur la Puissance Motrice du Feu* of 1824 (Carnot S 1978; Id., 1986; Pisano 2001) does not include any principles (e.g, like in Newton's theory) but it presents more than 60 Doubly Negated Sentences (DNSs). A DNS (where $\neg\neg A$ = *A fails*) does not depend on an inferential scientific structure based on a classical logical dichotomy of theses, e.g., obtained by listed–deductive theorems (Popper 1959). Since scientific DNSs are not equivalent to the corresponding affirmative sentences, they belong to non–classical logic; typically the law of double negation, $\neg\neg A$=A. I also remark that DNSs as used by scientists cannot be dealt with by using classical first–order logic (Hodges 1983). In this sense, logical elements are considered as special categories (Pisano and Gaudiello 2009; Id., 2010) and not in the sense of a theory or of a new theory. Thus, one could think that *undecidable* contents are not logically adequate within an inferential system. Nevertheless, *what kind of logical organization can support DNSs in a scientific theory?* By means of that point of view, a discussion on the role played by investigation and methodology is science is possible.

[23]In previous papers of mine the reader can find more details on the use of logic and the organization of the theory in historical investigations (Drago and Pisano 2000; Pisano and Gaudiello 2009, 2010; Capecchi and Pisano 2010a: Id., 2010b).

approximation, we are arguing, within intuitionist logic.
Finally, two general ways to organize a scientific theory can be claimed: (1) a former one, e.g., based on Aristotle's argument, is organized through axioms (AO) and its logic is classical, (2) a *Problematic Organization* (PO), belongs to non–classical where a result could be also «fuzzy».

2.2 Logic in the history of mathematics

It could be said that if an object is shown by means of an *absurdum* proof, its existence is not soundly proved mathematically. While, in classical logic: «*A is true when* $\neg\neg A$ *is true*» because, within classical logic, one should only verify that a contradiction does not emerge. In *multi–valued logics*, however, more than two truth–values can exist. Let's see an example. Let *a* be a number in decimal form:

$$0,a_1a_2a_3...a_n$$

$$a_n = \begin{cases} 0 \Leftrightarrow 2n = p_1 + p_2 & \text{where } \{p_1, p_2\} \textit{ primes numbers} \\ 9 \end{cases}$$

The property « $2n = p_1 + p_2$ » is valid but we do not know if it is valid for every integer. In fact, in the XVIII century the mathematician Christian Goldbach (1690–1764) conjectured[24] that every even integer greater than 2 (*Goldbach number*) can be expressed as the sum of two primes numbers[25]. In this sense we can write, e.g, 10=7+3, 14=13+1, 18=13+5. As often occurs with conjectures in mathematics, one can read a large number of supposed proofs of the *Goldbach conjecture*, but they are not currently accepted by the mathematical community. To be brief, it is not possible to prove truth content in proofs of the *Goldbach conjecture*. Moreover, a counter–example is also impossible. Thus, generally speaking, one can write that *a = 0 (for all of its figures) cannot be claimed because scientific proof is lacking.* Let's note that its negated ($\neg(a = 0)$) sentence cannot be claimed. In fact, if the latter could be proved then that would mean that one was able to claim $a \neq 0$; but, it means that we should also be able to present a counter–example (e.g., negated of negated) of the *Goldbach conjecture*. Thus, one should conclude that: «*a* $\neg\neg$ (*a = 0*) *fails*».

2.3 Logic in the history of physics–mathematics

In the *Preface* (and in *Rules of Reasoning in Philosophy*) of *Philosophiae naturalis principia mathematica* Newton (Newton 1803) assumed his idea on relationship between physics and mathematics separating mechanics into two parts: practical and rational.

> Since the ancients (as we are told by *Pappus*) made great account of science of mechanics in the investigation of natural things; and the moderns lying aside substantial form and occult qualities, and endeavoured to subject to phaenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics so far as it regards philosophy. The ancients considered mechanics in a twofold respect: as rational which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name.[26] [...] rational Mechanics will be the science

[24] In 1742, the *Goldbach conjecture* was proposed in a letter addressed to the Swiss mathematician Leonhard Euler (1707–1783).
[25] Let's remark that it is a curious property because prime numbers cannot be deduced by division, while odd numbers and the sum of two odd numbers concern another operation.
[26] Newton 1803, "IX, line 4. (*Italic style* by the author*)*.

of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated [...] And therefore we offer this work as mathematical principles of philosophy. For all the difficulty of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of Nature, and then from these forces to demonstrate the other phenomena [...][27].

Let's see the Newtonian principle of inertia (NPI):

> DEFINITION III. *The* vis insita, *or innate force of matter is a power of resisting, by which every body, as much as in it lies, endeavours to preserve in its present state, whether it be of rest, or of moving uniformly forward in a right line.*[28] (Newton 1803, I, 2; *Italic style* and capital letters belong to the author).
>
> Axioms; or Laws of Motion. Law I. *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*[29]

At present, one can read: *Every body will persist in its state of rest or of uniform motion (constant velocity) in a straight line unless it is compelled to change that state by forces impressed on it*. It has been remarked (Nagel 1961) that all physical laws can be expressed by means of a proposition preceded by two *universal* and *existential* quantifications[30]: ($\forall$) ("for all") and ($\exists$) ("there exists" or "for some"). A possible logical expression of the Newtonian principle of inertia can be:

$$A = \forall x \exists y : P(x, y) \tag{4}$$

A	= the proposition of the principle of inertia.
X	= a body.
y	= a complex system constituted by an inertial system, a closed system, and a clock.
P(x,y)	= a predicate concerning *x* and *y*: «if *x* is in *y*, than it is of its state of rest or of uniform motion».

Nevertheless, the aforementioned discussion belongs to classical mathematical logic which – by nature – is not interested in the *effective calculability* of its functions with respect to, operative–experimental–data. In this sense, the content of the first principle is lacking in experimental and *calculable evidence*. In particular:

The Newtonian principle of inertia claims that *y exists*, but it does not claim how one can find it.

The "whether" (or commonly "unless") contained in the proposition is not an *operative situation*. It only explains *a posteriori* the changes of state of motion occurred to the body.

A precise distinction when $\vec{v} = 0$, and when $\vec{v} \neq 0$, is required by NPI.

A precise knowledge when for $\vec{v} =$ *constant* in orientation (direction and versus) and in scalar–magnitude for the entire path is required by NPI.

A precise knowledge of absence–forces or of a non–zero net force is required by NPI.

The Newtonian principle of inertia is valid subordinately to validation of $\sum F_i = 0$

[27]Newton 1803, X, line 3.

[28]Newton 1803, I, 2. (*Italic style* and capital letters belong to the author).

[29]Newton 1803, I, 19. (*Italic style* belong to the author).

[30]At beginning of the last century, Thoralf Albert Skolem (1887–1963) suggested a technique to formalize the existential quantification on *y*–variable of a given predicate into a constructive mathematical function (Skolem [1920] 1967).

(for material–point and on the entire path).

Every physical variable should be subjected to its measurement. If the measurement cannot apply, the scientific content generates *uncertainties* in scientific knowledge. For that reason, the content of NPI as mentioned above, can be expressed by a DNS, «$\neg\neg A$: *It is not true that* $\vec{v}=0$ *is not equal to* $\vec{v}\neq 0$». Thus, all of the examined experimental–logical–*ambiguities* reported can be found in the Newtonian principle of inertia within a non–classical logic investigation.

2.4 A case study: principle of Inertia in Newton and in Lazare Carnot

Based on the previous section, if we consider an operative physics, to translate each *x* body in an effective procedure it is necessary to obtain an inertial system: an isolated system and a clock. Two centuries of unprofitable research demonstrate convincingly that with the sole knowledge of the body *x* it is not possible to operatively obtain many bodies. In order to make a physical–mathematical equation like (4) relatively operative, it might be obtained by forcing the predicate, which is by means of one of these three approaches.

1) Substitute the quantifier $\exists$ in (4) with a constant value y_0. This affirmation results in:

$$\forall x\ A(x, y_0) \tag{5}$$

That is, each *x* body at rest or in rectilinear uniform motion if placed in y_0 (that is if its motion is measured, with respect to a given inertial system, in a given closed system provided of a precise clock). Equation (5) corresponds to defining the clock and the reference system in two very different ways:

a. In the way followed by the physicists since the age of Galilei, that is with an empirical clock as a reference, with the closed system verified empirically and with the earth reference system; except changing those on a case by case basis in accordance with the following definitions).
b. In the idealistic way suggested by Newton (that is idealising this experimental method to the limit, transcending the same experience: introducing the idealised concepts of absolute space and time, that fix once and for all the clock and the inertial system and then implicitly suggest that we are always capable, as a matter of principle, of verifying if $\vec{F}=0$ or not and then knowing when a system is isolated or not).

2) To accept the fact that in general we ignore the generic function $\alpha(x)$ but to annul the problem of the existential quantifier saying: in specific circumstances experimental Physics can define: «$\forall x\ A(x, y)$», without further explanation as to what the experimental physicists should do. 3) In order to deny the physical importance of these problems,[31] qualifying them as metaphysical ones. We can only affirm that we can make experimental observations on an «*x*» body: the impossibility for a single body under observation to change on its own its status of motion when at rest or in a rectilinear uniform motion. No quantifiers, nor «$\exists$», nor then «$\forall$». This is what Lazare Carnot (1753–1823) did in *Principes fondamentaux de l'équilibre et du mouvement.* (Carnot L. 1803). In *Principes fondamentaux de l'équilibre et du movement,* Lazare Carnot offers his version of the principles of a PO type mechanics by the formulation of seven fundamental hypotheses (Gillispie and Pisano). I am only interested in the first one for this article:

[31]It has been demonstrated that it is possible to apply this method to the third principle of dynamics «for every action there is an equal and opposite reaction: that is the actions of two bodies are always equal to one another and directed towards opposite directions.

Notions préliminaires. Hypothèses admises comme lois générales de Équilibre et du mouvement. Conséquences déduites de ces hypothèses. 1° Hypothèse. Un corps une fois mis en repos, ne suroît en sortir de lui–même, et une fois mis en mouvement, il ne suroît de lui– même changer ni sa vitesse, ni la direction de cette vitesse.[32]

One of the main differences between Lazare Carnot's mechanics and Newton's mechanics lies in the fact that the first speaks of every body in every time and in every place, while L. Carnot speaks of a restricted whole of situations: those where it is possible to affirm that a body is at rest or in motion. These situations are indicated by an intentional generic introduction "[…] once […]". It is thanks to these generic terms that Lazare Carnot's version avoids the problem implicit in Newton's terms, when we talk about rectilinear and uniform motion "[…] until [...]" that is. Lazare Carnot avoids the problem of deciding when it is $\vec{F} \neq 0$ along the course (potentially infinite.) Therefore, in his principle of inertia L. Carnot correctly does not name the forces and does not ask for a verification of their absence $\sum_i \vec{F}_i = 0$ along the entire course of the body. He says that it is not possible to evaluate in a definite way: *If a motion is absolute, or if there is a motion or a dragging force, […] and it has been very difficult to correct this error.* There is no verification of the absence of forces. Lazare Carnot, then, says deliberately "[…] once it is […]" then, in the condition where we can decide, due to specific circumstances, that a body is static or a rectilinear uniform motion. Therefore for the French scientist it is up to our judgment, empirical and occasional, to decide if a body is static or a rectilinear uniform motion. A problem equivalent to the previous one (establishing if $\vec{F} = 0$ is exact) is the following: Newton would claim to establish exactly when a body is in a resting status as different from the motion status; this means deciding if $\vec{v} = 0$ is exact (but not if $\vec{v} < \varepsilon$). Lazare Carnot's definition "[…] once […]" avoids this problem. The definition of Lazare Carnot's first hypothesis does not claim to provide rules to verify the status of rest or motion. Generally these are impossible since they would be circular by the definition of an inertial reference system.

In the end, the principle of inertia states that rest and rectilinear uniform motion are equivalent. But what does *equivalent* mean? Newton's statement treats the two cases as if they were the same thing (…"at rest or in motion…"). Carnot's statement, however, is more cautious; it is broken up into two parallel but distinct affirmations: it does not take the passage from statics to dynamics for granted. So after this initial hypothesis, his other hypotheses articulate this equivalency in gradual passages. In fact, while his second hypothesis still concerns static situations, the third and the fourth hypotheses include dynamics. So we conclude that Carnot's hypotheses (after the first) are a precise strategy of passage from statics to dynamics. In addition we note that all of the aforementioned hypotheses are constructive since they are essentially experimental, except for the fourth which is considered by Carnot as a mathematical convention.

3 What is the role played by history in sciences teaching?

By focusing on mathematics and physics, the previously quoted aspects move towards a

[32]Carnot 1803, 49, line 3. (*Italic style* from the author). First hypotheses: "A body once at rest would not be able to move on its own, and when put in motion could not change its speed or direction by itself" (My synthetic translation).

larger base of analysis which includes not only disciplinary matters but also interdisciplinary issues in philosophy, epistemology, logic and the foundations of physical and mathematical sciences. We need a strong effort for an interdisciplinary approach to *teach* and *learn* the relationship physics–mathematics as a discipline of study (Martinez; Meltzoff et al.). It has been noted that teachers regularly have great difficulty teaching historical and philosophical knowledge about science in ways that their students find meaningful and motivating. Thus, *how is it possible to keep on teaching sciences being unaware of their origins, cultural reasons and eventual conflicts and values?* And *how is it possible to teach and comment on the contents and certainties of physics and mathematics as sciences without having first introduced sensible doubt about the inadequacy and fluidity of such sciences in particular contexts?* Education needs to revaluate scientific reasoning as an integral part of human (humanistic and scientific mixed) culture that could build up an autonomous scientific cultural trend in schools (Pisano 2009b). *In this sense, what about the importance of introducing the history of science as an integral part of the culture of teaching education to the extent of considering such a discipline – in its turn – as an indissoluble pedagogical element of history and culture?* (Pisano and Guerriero) "To foresee the future of mathematics, the true method is to study its history and present state".[33] It would be useful to pay particular attention to the elaboration of the *teaching–learning* process based on the reality observed by students (inductively), by a continuing critical reflection, e.g. by means of studying the historical foundations of modern physical and mathematical sciences. Therefore, turning from *teaching based on principles* to *teaching (also) based on broad and cultural themes* would be crucial. It would mean teaching scientific education as well, which is a kind of education that poses *problems* and as far as physics is concerned, introducing it through historical and philosophical criticism as well. It would be helpful to practically support processes on a multidisciplinary or even on co operational level, a kind of pedagogy able to re–consider, from this point of view, the relationship between theory and experience, history and foundations. Let's think about (1) the lack of a relationship between physics and logic (Pisano 2005)…. the organization of a scientific theory (axiomatic or problematic) and its pedagogical aspect based on planned and calculated processes, (2) when we use the term *mechanical* associated with a problem, model, law et al...; (3) the problems of foundations, for example in the teaching phase of the passage from mechanics to thermodynamics, is not yet completely solved; (4) teaching of the non–Euclidean geometries or of the planetary model as an introduction to the study of quantum mechanics, was born, as a matter of fact, only thanks to the fact that the old concept of trajectory was abandoned in favour of the probabilistic one; (5) the concept of infinite and infinitesimal in limits compared to measures in a laboratory… (6) et al… Through an educational offer enriched with the study of the foundations of physical and mathematical sciences, complete with the intelligent use of pedagogical computing technologies, a kind of teaching might be accomplished with the model of the prevailing method of the teaching–learning process mainly related to and coming from reality. It would be an attempt necessary to show how paths usually chosen have not been unique in the history of science but very often an alternative possibility has existed. For example: the statics in Jordanus de Nemore (ca. XIII century) and in Tartaglia (Tartaglia 1554, books VI–VIII), the physics–chemistry of Newton and Antoine–Laurent de Lavoisier (1743–1794), the

[33] Julius–Henri Poincaré (1854–19121) quoted in: Klein 1980, 3.

mechanics of Lazare–Nicolas–Marguerite Carnot (1753–1823) etc. More specifically: the second Newtonian principle is not strictly a physical law and it has just a little in common with physical laws by Galilei rather than showing that the historical foundations of thermodynamics which are based on five (Pisano 2010) epistemological principles, more than the classical ones read in a textbook. From cognitive–epistemological point of view (George and Velleman), people do not naturally and scientifically reason by means of deductive or inductive processes only. In this regard, scientific reasoning (Lakoff and Nunez) is not a part of our common knowledge reasoning), although we often intuitively compare events, tables etc. Instead, it was remarked that we reason mainly by the association of ideas and sometimes concepts are far from the scientific ones, e.g. heat and temperature, mass, weight and force–weights, the solar system and atomic orbital system in quantum mechanics, the kinetic model of gases and thermodynamics, parallel straight, material points et al. Thus, the current scientific teaching system paradoxically changes the logical basis of reasoning. An hypothetical proposal, of course not the only one possible, could be the introduction within the educational plan of reading passages *ad hoc* centred on mathematics and physics to be analysed in the classroom, main books by Aristotle's mechanics (*mechanical problems*), Euclid (*Elements*), Archimedes (*On equilibrium of planes*), Tartaglia (*Quesiti*), Galilei (*Discorsi*), Torricelli (*Opera*), Lazare Carnot (*Essai*) Lavoiser (*Traité*) Sadi Carnot (*Réflexions*), Faraday (*Experimental Researches*) et al. Reading such passages, together with pre–arranged and effective work shared by several subjects, 1) *the student is placed before a problematic situation and driven to realise the inadequacy of his/her basic knowledge with regard to problem solving.* 2) *When the build up of scientific education begins, in order to overcome such difficulties.* The result will be pedagogy according to which science education (Osborne and Collins; Debru 1997; Id., 1999) essentially means *setting and solving problems* and teaching means re–evaluating the relationship between theory and experience and between history and foundations. They could come together with well–structured and practical interdisciplinary work by means of the history of science. International debate should take into account pedagogical research on foundations for history and learning–teaching science, discovering science teaching and informal learning activities as well. In this way, a student is the *protagonist*, both formally and informally (*hands–on*), of his *learning*. I feel the same about schools training experts, as these also should provide a setting that favours teaching research aimed at the critical re–construction of scientific meanings along with ideas, opinions and proper contents. In the end, this briefly proposed reflection should convey that it is urgent to establish the basis for a debate that ethically appears correct and professionally necessary. Maybe, operating in a different way, we could also contribute to building a school (or university) linked to the new perspectives of science, its image and teaching without limitations on specializations, pushing past disciplinary competences.

4 Final remarks

To sum up, one could think of:

Appealing to students for a scientific culture through the culture of history and philosophy, regardless of the sterile dichotomy between human and scientific disciplines.

Physics in the 20th century changed either the fundamentals of classical physics (and of science as well), or lifestyle (for better and for worse). A reflection based on a program, according to the spirit of research and inter–discipline, and pedagogically–oriented, is always to be regarded as a topic of interest, never obvious.

Inviting a motivated and interested study of physics and mathematics through a wider historical and philosophical knowledge of epistemological criticism.

Trying to re–build the educational link between philosophy and physics–mathematics. For ex., philosophy, from the end of the XIX cent., seems to have no longer found a steady link with physics whose interpretation of a phenomenon is sometimes based on the involvement of an advanced and elaborated mathematics.

Dissemination and sharing of difficult theoretical and experiential works.

Make the students understand that the history of scientific ideas is closely related to history of techniques and of technologies; that is why they are different from one another.

Make the others understand that scientists were once people studying in poor conditions.

Show the real breakthrough of scientific discoveries through the study of the history of fundamentals, not yet influenced by the (modern) pedagogical requirements. For ex: understanding the historical turnover of the principles of classical thermodynamics into the usual teaching of physics.

Let the students experiment discoveries with enthusiastic astonishment through a guided iter reflection on the fundamental stages of progress and scientific thought

Also, a provocative hypothesis: *generally speaking, we should not lose the certainty of a critical thought on science... but if we do not do anything, then nothing changes... but if we do something (a few crucial things), maybe something could be improved.*

> The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians [and scientists] to abandon science. With the “plague on all your horses” they have retreated to specialties in areas of mathematics [and physics] where the methods of proof seem to be safe. They also find problems concocted by humans more appealing and manageable than those posed by nature. The crises and conflicts over what sound mathematics is have discouraged the application of mathematical methodology to many areas of our culture such as philosophy, political science, ethics, and aesthetics. The hope of finding objective, infallible laws and standards has faded. The Age of Reason is gone. With the loss of truth, man lost his intellectual center, his frame of reference, the established authority for all thought. The “pride of human reason” suffered a fall which brought down with it the house of truth. The lesson of history is that our firmest convictions are not to be asserted dogmatically; in fact they should be most suspect; they mark not our conquest but our limitations and our bounds.[34]

“Revolution in Science Education[?]: Put Physics First” (Lederman, 44). All of us put a professional teacher first: teachers that teach, research and publish...

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[34]Klein 1980, 7, line 12; 99, line 14. The quotation belongs to the author.

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