L´HOSPITAL'S CHALLANGE TO GEOMETERS FOR THE RECTIFICATION OF DE BEAUNE'S CURVE.

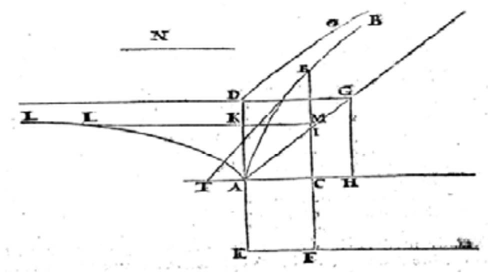
Eduardo SEBASTIANI FERREIRA
IMECC, UNICAMP, Campinas, Brazil
esebastiani@uol.com.br

ABSTRACT

In 1692 l'Hospital wrote in Journal des Sçavans about the De Baune curve and cast a challenge for the calculation of its length, which was solved by Varignon 11 years later in the same journal. (L´Hospital, 1692, pp. 401-403, Varignon 1703, pp.117-121)
In this text I present Varignon´s calculation of the length of the De Beaune curve and an analysis of this solution. I show here that it was Bernoulli who solved the challenge of l'Hospital. Varignon used Bernoulli's solutions to write his article, without mentioning his name once.
The text is a good example for undergraduate students on the one hand, of how integration can be made simpler with the help of computer software and on the other hand, it exemplifies the importance of letters, which are reasonably easy to be read, as sources of research in the History of Mathematics. Nowadays, this kind of documentation is no longer being produced.

1 INTRODUCTION

I'll present l´Hospital's appeal for the rectification of the curve suggesting that he already knew it, due to the 4 properties he attributed to it. The interesting fact is that he does not make use of the logarithmic function.



L´Hospital, 1692, p.401

It is clear that the nature of this curve *AbB* depends on the quadrature of the hyperbola and also that it is mechanical in the sense of Descartes. Descartes was the first to consider the expression of curves by equations. This idea, upon which the application of Algebra to Geometry is founded, is very successful and fertile. Curves are divided into algebraic parts, which we call, following his ideas, geometric curves, and transcendental parts, which are the mechanical ones for him. The mechanical curves cannot be expressed by equations between *dx* and *dy*. There are two types of this kind of curves; they are:

1 The exponential curves, where one of the unknown, or both of them occur in exponents.

2 The intertranscendental curves in which the equations are expressed by means of radicals.

Let us consider now some of the properties of De Beaune`s curve, for L´Hôpistal:

1º Elle a pour asymptote la ligne DO parallele à AI.

2º Si l´on nomme AC x , *BC* y, *l´espace ABC compris par les droites AC, et par la portion AB de la courbe,* $= xy - \frac{1}{2}y^2 + nx$. *La distance du centre de gravité de l´espace ABC de la droite AC,* $= n + \frac{3xy^2 - 2y^2}{6xy - 3y^2 + 6nx}$ *et de AK* $= \frac{1}{2}n + \frac{3x^2y - y^2}{6xy - 3y^2 + 6nx}$, *Et l´on a par conséquent les solides, demi solides, formés par la révolution de cet espace, tant autour de AC que de AK ou BC.*

4º Il est facile de déterminer les centres de gravité de ces demi-solides. Mais comme on a besoin d´une adresse particulière pour rectifier cette courbe, en supposant la quadrature de l´hyperbole, je propose ce problème aux Geometres, les assurant qu´il mérite leur recherche. (L´Hospital, p.403)

One of the facts that have deeply puzzled me, is the link between the logarithmic function and the quadrature of the hyperbola, because of the analysis of historical books and documents that I have been doing for some time. This fact, although known since the 17th century, was not clear for the mathematicians. I quote Bourbaki:

Quoi qu´il en soit, J. Gregory, en 1667, donne, sans citer qui que ce soit,..., une règle pour calculer les aires des segments hyperboliques au moyen des logarithmes (décimaux): ce qui implique à la fois la connaissance théorique du lien entre la quadrature de l´hyperbole et les logarithmes, et la connaissance numérique du lien entre logarithmes "naturels" et "décimaux". Est-ce à ce dernier point seulement que s´applique la revendication de Huygens, que conteste aussitôt la nouveauté du résultat de Gregory....? Ce n´est pas plus clair pour nous que pour les contemporains ; ceux-ci en tout cas ont eu l´impression nette que l´existence d´un lien entre logarithmes et quadrature de l´hyperbole était chose connue depuis longtemps, sans qu´ils pussent là-dessus se référer qu´à des allusions épistolaires ou bien au livre de Grégoire de Saint-Vincent. (Bourbaki, 1969, p.214)

I started my research with De Beaune's letter to Roberval dated 16th October 1638 (Waard, pp.139-150), where he tries to solve the curve, which was named after him, under conditions on the tangent at any of its points. It is one of the first inverse tangent problems. Descartes' reply, correcting his proof, comes in a letter to Roberval dated 15th November 1638. Eventually, Leibniz, in a letter to Oldenburg in August 1676, demonstrates, using his differential calculus and his notation, that:

$$\int \frac{dy}{y} = -\frac{x}{c}$$

which gives: $y = -c \log \frac{y}{c}$

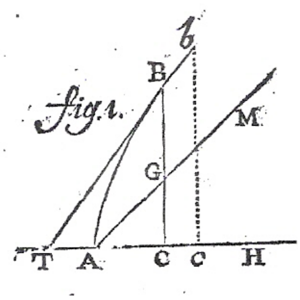
The elucidation came to me when l' Hospital wrote, in the *Journal des Sçavans*, about this curve and casted a challenge for the calculation of its length, which was solved by Varignon 11 years later in the same journal. (L´Hospital, 1692, p. 401-403 (Varignon. 1703, p. 111-121)

In this text, Varignon´s calculation of the length of the De Beaune's curve and an analysis of this solution will be presented.

2 Solution presented by Pierre Varignon in 1703

Varignon was a great friend to l'Hospital and defended him at the Academie des Sciences de Paris, against Rolle's attacks. It was he, who, eleven years later, accepted the challenge and demonstrated how to rectify DeBaune's curve, using differential calculus and knowledge of the rectification of the hyperbola. Such a solution was published in the Journal des Sçavans in 1703 (Varignon, 1703, pp.117-121):

SOLUTION DU PROBLÈME QUE M. LE MARQUIS DE L´HOSPITAL A PROPOSÉ DANS LE JOURNAL DES SÇAVANS DE 1692. PAG.598
(Journal de Sçavans, 1703, 2G, p.117) (Varignon)
Rectification of the curve from M. de Beaune
Considering *ABb* the curve from M. de Beaune, such that *BC* is divided at *G*, by the line *AGM*, which has a 45 degree angle with the axis *ACH*, *BC* is to the sub-tangent *CT* as given line *a* to *BG*, part of the ordinate comprehended between the line *AGM* and the curve



$$\frac{BC}{CT}=\frac{a}{BG}.$$

If *AC* is called *x* and *BC*, *y*, then we have *BG* = *y* – *x* (since the angle *GAC* is the same as the *AGC* angle) and this curve's property will be given by $a\,dx = y\,dy - x\,dy$ hence, $\frac{BC}{CT}=\frac{dy}{dx}=\frac{a}{y-x}$

It is known that the length of every curve is $\sqrt{dx^2+dy^2}$ (being *x* the abscissa, and *y* the ordinate).
Placing then, the value of dx determined by the equation of the curve in this expression, it is possible to find the small arc:

$$Bb = ds = \sqrt{\left(\frac{y\,dy - x\,dy}{a}\right)^2 + dy^2} = \sqrt{\frac{y^2dy^2 - 2xy\,dy^2 + x^2dy^2 + a^2dy^2}{a^2}} = \frac{dy}{a}\sqrt{y^2-2xy+x^2}$$

Now to reduce this amount to a single indeterminate, we put $z = y - x$; then, $dy = dz + dx$. Inserting these values of *y* and *dy* in the element and multiplying the numerator and denominator by $a - y + x$ we get:

$$\frac{(a-y+x)}{(a-y+x)}\frac{dy}{a}\sqrt{y^2+a^2-2xy+x^2} =$$

$$\frac{a(dz+dx)+(z+x)(dz+dx)+x(dz+dx)}{(a-z)(dz+dx)a(a-z)}\sqrt{(z+x)^2+a^2-2x(z+x)+x^2} =$$

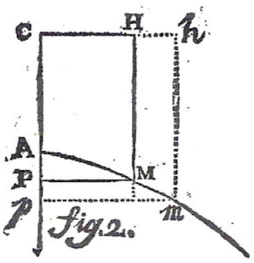
$$\frac{a\,dy - y\,dy + x\,dy}{a^2 - ay + ax}\sqrt{y^2+a^2-2xy+x^2}$$

Since $a\,dx = y\,dy - x\,dy = z\,dy$, then, $a\,dx - z\,dz - z\,dx = z\,dy - z(dy-dx) - z\,dx = z\,dy - z\,dy - z\,dx + z\,dx = 0$, which causes a change to this other quantity $\frac{dz}{a-z}\sqrt{a^2+z^2}$; and in order to have the radical sign in the denominator, we multiply the numerator and denominator by $\sqrt{a^2+z^2}$, which yields $\frac{dz}{a-z}\sqrt{a^2+z^2}=\frac{dz\left(a^2+z^2\right)}{(a-z)\sqrt{a^2+z^2}}=\frac{a^2dz+z^2dz}{(a-z)\sqrt{a^2+z^2}}$; adding and decreasing a^2dz from the numerator, we get: $\frac{2a^2dz-a^2dz+z^2dz}{(a-z)\sqrt{a^2+z^2}}=\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}-\frac{(a+z)dz}{\sqrt{a^2+z^2}}=\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}-\frac{a\,dz}{\sqrt{a^2+z^2}}-\frac{z\,dz}{\sqrt{a^2+z^2}}$

The integral of the last term is $-\int_0^z\frac{z\,dz}{\sqrt{a^2+z^2}}=-\sqrt{a^2+z^2}\,\Big|_0^z=-\sqrt{a^2+z^2}+a$

Since at that time the trigonometric integrals weren't known, Varignon uses the following 'trick' to get the integral of the second term: multiplying both the numerator and the denominator by az and adding and subtracting the term $\frac{2dz\sqrt{a^2+z^2}}{a}$ gives $-\frac{a\,dz}{\sqrt{a^2+z^2}}=-\frac{a^2z\;dz}{a\sqrt{a^2z^2+z^4}}+\frac{2dz\sqrt{a^2+z^2}}{a}-\frac{2dz\sqrt{a^2+z^2}}{a}$. The integral of this term is $-\int_0^z\frac{a\,dz}{\sqrt{a^2+z^2}}=\frac{z}{a}\sqrt{a^2+z^2}-\int_0^z\frac{2\sqrt{a^2+z^2}}{a}dz$. This latter term is the integral of a hyperbola, that is, the quadrature of a hyperbola multiplied by 2 and divided by *a*, supposedly known to l'Hospital. Then, the integral is equal to the area of the rectangle HMP divided by *a*, the equation of the equilateral hyperbola *AMm* is $x^2-z^2=a^2$, so $AP=\sqrt{a^2+z^2}=x$. Therefore, the solution of the integral of this second term is minus the sum of the areas of the rectangle $CHPM=z\cdot\sqrt{a^2+z^2}$ and the space $CHAM=\frac{2}{a}\int\sqrt{a^2+z^2}dz$.

The integral of the first term $\int_0^z\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}$ can be found by quadrature of an equilateral hyperbola whose apex is at the abscissa $\frac{a}{4}$.



[1]Varignon, 1703, p.120

Being the equilateral hyperbola *AMm*, whose semi-axis is *CA*, or be it $\frac{1}{4}a$, the abscissa *CP* will be $\frac{a\cdot\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}{a-z}$, the ordinate $PM=\frac{a^2+az}{4(a-z)}$.

$$CM=\frac{a\cdot\sqrt{\frac{3}{16}a^2+\frac{3}{16}z^2+\frac{1}{8}az}}{a-z}$$

Mapping the line *Cm* from point *C*, infinitely close to *CM* and from point *C*, as the center, describing a small arc *MN*, it is clear that the triangle *CMN* will be the hyperbolic triangle *CMA*. Then, in this case, Varignon considered the curve *MN*, as well as *Mm*, as being small segments.

To get the expression of this elementary triangle, the following are required: the analytical value of *CM*, finding *MN*, multiplying these two quantities for each other and dividing the result by 2. This means that the area of the right triangle *CMN*, right at *M*, is $\frac{CM\cdot MN}{2}$.

From this we get the value of *CM*. To find the value of $MN=\sqrt{Mm^2-mN^2}$. Since *Nm* is the differential of *CM*, then $d(CM)=\frac{a^3+a^2z}{8(a-z)^2\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}dz$.

The differential of *MP* will be $d(MP)=\frac{a^2}{2(a-z)^2}dz$.

Raising each of the expressions to the square, adding and extracting the square root and applying Pythagorean theorem: $Mm=\sqrt{\frac{\left(a^2z+a^3\right)^2}{64(a-z)^4\left(\frac{1}{8}a^2+\frac{1}{8}z^2\right)}+\frac{a^4}{4\left(a-z\right)^4}}dz$ Then, if we have the expression of the small lines *mM* and *mN*, we can find the segment $MN=\sqrt{Mm^2+Nm^2}$, so $MN=\frac{a^2dz}{\sqrt{8a^2+8z^2}\sqrt{3a^2+2az+3z^2}}$.

Calculating the area of the right triangle *CMN*, we get: $\text{area}\,CMN=\frac{CM\cdot MN}{2}=$ $\frac{a}{32\sqrt{2}}\frac{2a^2dz}{(a-z)\sqrt{a^2+z^2}}$, whose integral is the same as the hyperbolic domain *ACM*.

Then, the first integral is the hyperbolical space *ACM* divided by *a* and multiplied by $32\sqrt{2}$.

Then, the entire integral of the three first terms will be the same as 32 times the *ACM* space, multiplied by $\frac{\sqrt{2}}{a}$, minus twice the sum of the hyperbolical space *ACM* and the rectangle *CHPM*, also divided by *a* and minus the line given by the quantity $\sqrt{a^2+z^2}$ plus *a*, *which is what we looked for.*

3 Johann Bernoulli: His participation in this challenge

It is interesting to understand what happened in these eleven years between the challenge of l'Hospital and Varignon's response. Researching the letters Varignon and Bernoulli exchanged, we found the whole process of rectification of the De Beaune's curve.

I'll start with the letter Varignon sent to Bernoulli on May 24th, 1696, 4 years after l'Hospital's article had been published. In this letter Varignon writes:

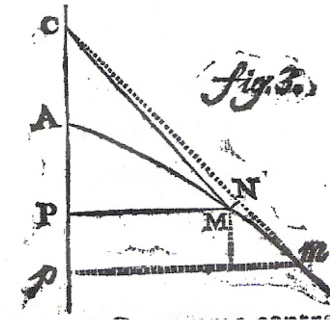
> *De grâce, Monsieur, dites moi comment vous trouvez les longueurs des logarithmiques. Je m´en suis démontré tout ce que M. Huguens en a dit*; mais je n´en saurais trouver les longueurs que par des progressions infinies, quoiqu´elles se puissent trouver autrement. Je suis dans la même peine par rapport aux longueurs de la courbe de M. de Beaune: je m´en suis aussi démontré l´une et l´autre des solutions que vous en avez données dans les Journaux de France et de Leipzig; mais pour les longueurs je ne les vois point encore; aussi dites vous dans le Journal de France qu´il faut une adresse particulière pour le trouver: c´est cette adresse que je vous prie aussi de me montrer. (*) (in Discours de la cause de la pesanteur,* which was attached to Traité de la Lumière, published in 1690; Costabel & Peiffer, 1988, p.98)

Bernoulli responds to Varignon on June 5th, 1696, but doesn't talk about the substitutions needed to solve the problem, which are again requested by Varignon in a letter on 18th June of the same year. In this letter Varignon writes:

> *Je savais bien que* $\frac{dy\sqrt{a^2+y^2}}{y}$ *est la différentielle de la courbe logarithmique; mais je ne vois pas, et je ne vois pas encore que* $\frac{dy\sqrt{a^2+y^2}}{y}$ *soit la différentielle d´un espace hyperbolique.* (Costabel & Peiffer, 1988, p.102)

On 11th August, 1696 Bernoulli writes back, showing Varignon how the integral of this differential can be calculated as a hyperbolic area, which is the area ABF, using the infinitesimal triangle as shown in the article by Varignon. First, Bernoulli demonstrates that: $\frac{dy\sqrt{a^2+y^2}}{y} = \frac{y^2+a^2}{y\sqrt{y^2+a^2}}dy = \frac{y\,dy}{\sqrt{y^2+a^2}} + \frac{a^2\,dy}{y\sqrt{y^2+a^2}}$.

The first integral is $\sqrt{a^2+y^2}$. Then, he works on the second, where he makes the coordinate changes, that is $y = \frac{a^2}{m}$, where $m \neq 0$, he then has the differential $\frac{-a\,dm}{\sqrt{a^2+m^2}}$.
It is this integral which he claims to be the area of the region previously mentioned.



There is no need of such an artificial construction to show this; it is enough to calculate the area between the hyperbola and the line joining the point F to the origin *A*.

Taking the coordinates of *F* as *(x,y)*, which is also a point in the hyperbola $x = \sqrt{a^2 + y^2}$, the line *AF* is given by the equation $x = \frac{y^2}{\sqrt{y^2 + a^2}}$. As a result of the differential, we must integrate to find that the hyperbolic area will be: $\sqrt{y^2 + a^2}\,dy - \frac{y^2}{\sqrt{y^2 + a^2}}\,dy = \frac{a^2}{\sqrt{y^2 + a^2}}\,dy$. Since the line is shown with the *y* squared, we have two lines, *AF* and *AF'*, therefore we will consider only half of this region.

Varignon responds Bernoulli's last letter thanking him for such a long answer on how to calculate the length of the logarithmic curve, but he still does not know how to deal with De Beaune's curve, for he writes:

> *…ayant cherché de même celle de la Courbe de Beaune, j´en suis demeuré à une différentielle que ne n´ai pu intégrée: la voici. Appelant à ordinaire les abscisses x, les ordonnés y, et le lieu de cette courbe étant* $= y\,dy - x\,dy$ *; je trouve son Elément* $\left(\sqrt{dx^2 + dy^2}\right) = \frac{dy}{n}\sqrt{yy - 2xy + xx + nn}$ *(en faisant* $y - x = z$*)* $= \frac{dz\sqrt{zz + nn}}{n - z}$
>
> *c´est de cette dernière différentielle que je ne savais trouver l´Intégrale: Je vous la demande en Grâce, et tout du long, s´il vous plaît* (Costabel & Peiffer, 1988, p.109)

Bernoulli's answer comes on July 27th, 1697. In this letter he shows Varignon how to solve the problem of the rectification of De Beaune's curve; Bernoulli writes:

> *Je suis bien aise que ma méthode de rectifier la logarithmique ait eu le Bonheur de vous plaire; mais j´en aurais eu plus de joie si elle vous avait donné assez de lumière pour faire l´imitation das la courbe de Beaune, comme je l´esperais: voici pour achever le calcul que vous avez seulement commencé, comment il s´y faut prendre: vous avez réduit l´élément de la courbe à cette quantité differentielle* $\frac{dz\sqrt{zz + nn}}{n - z}$ *dont vous ne sauriez trouver l´intégrale, que je trouve pourtant fort aisément, car* $\frac{dz\sqrt{zz + nn}}{n - z} = \frac{zz\,dz + nn\,dz}{(n - z)\sqrt{zz + nn}} = \frac{zz\,dz - nn\,dz}{(n - z)\sqrt{zz + nn}}\left(\frac{-z\,dz - n\,dz}{\sqrt{zz + nn}}\right) + \frac{2nn\,dz}{(n - z)\sqrt{zz + nn}}$ *,*
> *ou l´intégrale de* $-\frac{z\,dz}{\sqrt{zz + nn}}$ *est* $-\sqrt{zz + nn}$ *; et l´intégrale de* $-\frac{n\,dz}{\sqrt{zz + nn}}$ *est comme je vous ai fait voir dans ma précédente un secteur hyperbolique; il reste*

donc encore $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}$*, dont il faut trouver l´intégrale, ce que je fais ainsi: je pose* $n-z=s$*, ce qui donne* $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}=$ *(en faisant* $s=\frac{nn}{t}$*)* $\frac{2nn\,dz}{(n-z)\sqrt{zz+nn}}=\frac{-2nn\,ds}{s\sqrt{2nn-2ns+ss}}=$ *(en mettant* $t=u+\frac{1}{2}n$*)* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}$*, ou l´intégrale de* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}$ *est aussi un secteur hyperbolique; ou si vous aimez mieux retenir une même formule, vous n´avez qu´à mettre* $\sqrt{uu-\frac{1}{4}nn}=t$*, et vous trouverez* $\frac{n\,du\sqrt{2}}{\sqrt{uu-\frac{1}{4}nn}}=\frac{r\,dr\sqrt{2}}{\sqrt{rr+\frac{1}{4}nn}}$ *Voilà donc l´intégrale de* $\frac{dz\sqrt{zz+nn}}{n-z}$ *réduite à* $-\sqrt{zz+nn}$ + *la différence de deux secteurs hyperboliques, que l´on peut par conséquent construire par la logarithmique; or puisque la courbe de Beaune se construit aussi par la logarithmique, il est évident que l´intégrale de* $\frac{dz\sqrt{zz+nn}}{n-z}$ *c´est-à-dire la rectification de la courbe de Beaune dépend de sa propre construction; et qu´ainsi la courbe étant une fois décrite, elle se mesure par des lignes droites.* (Costabel & Peiffer, 1988 pp.114-115)

Comparing the solution Varignon explores in his article, we can see which steps he took to come to a way to integrate this last differential $\frac{2n^2\,dz}{(n-z)\sqrt{z^2+n^2}}$. He considers the equilateral hyperbola *AMm* whose semi-axis *CA* is $\frac{1}{4}a$. The abscissa *CP* will be $\frac{a\cdot\sqrt{\frac{1}{8}a^2+\frac{1}{8}z^2}}{a-z}$. This abscissa is the result of all the substitutions Bernoulli executed. As it was put by Varignon, is seems to be a fixed value, or at best the function of a function, which changes the character of the curve.
.

4 Conclusion

The challenge of the rectification of the De Beaune’s curve was first cast by l’Hospital, in 1692, in the Journal des Sçavans, as a problem for geometers to solve. The solution came to light eleven years later, in the same journal, under the credit of Varignon. Little was known about how Varignon had solved the problem, since the solution he presented was achieved through the use of a variable substitution that was completely incomprehensible, which seemed to be a ‘forced’ substitution used to achieve a result he already expected to find. This was a dilemma for mathematicians, who could not find a sensible resolution for the problem. It was only after the publication of Johann Bernoulli’s letter, in the 1940s, that we could have access to the correspondence between Varignon and Bernoulli, through which it was possible to understand Varignon's solution for the rectification of the De Beaune's curve, by way of the quadrature of the hyperbola that was known at the time, and which was solved by Bernoulli after a long correspondence between the two mathematicians.

All the steps found in the letters were here demonstrated, as well as Varignon's appeal and Bernoulli's respective responses, presenting the desired solution. Although Varignon was an important mathematician of his time and had a definite role in the history of the development of Differential and Integral Calculus, Bernoulli's merit in this field was far greater and his role in the support of important problems are both undeniable facts. I don't know why Varignon, on the occasion of the publication of his article solving l'Hospital's challenge, doesn't acknowledge Bernoulli, since it was him who in fact solved the challenge.

APPENDIX: Rectification of a curve

It seems that the first mathematician who rectified a curve was William Neil (1637-1670). His procedure to calculate the length of the segment of a "semi-cubical parabola" $(y^2 = x^3)$ over the interval $0 \le x \le a$,

> *was to subdivide this interval into an indefinitely large number n of infinitesimal subintervals, the ith one being* $\left[x_{i-1} - x_i\right]$. *If* S_i *denotes the length of the (almost straight) piece of the curve* $y = x^{\frac{3}{2}}$ *joining the corresponding points* (x_{i-1}, y_{i-1}) *and* (x_i, y_i), *then* $S_i = \left[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right]^{1/2}$, *and the length of the curve will be given by* $S \cong \sum_{i=1}^{n} S_i$. (Edwards, 1973, p.118).

In "Methods of Series and Fluxions", Newton applies the basic fluxional technique for the computation of an arc length, which he describes as follows:

> *Problem 12: To determine the lengths of curves: In the previous problem we showed that the fluxion of a curve line is equal to the square root of the sum of the squares of fluxions of the base and the perpendicular ordinate. Consequently if we consider the fluxion of the base as a uniform, determinate measure (namely unity) to which the other fluxions shall be referred, and on top of this seek out the fluxion of the ordinate by means of the equation defining the curve, the fluxion of the curve line will be hand and from that its length should be elicited by Problem 2.* [(716)]
>
> [(716)] *Newton sketches the standard procedure for rectifying a curve: namely the arc-length t is the 'fluent' (taken between appropriate integration bounds) of the fluxion* $\dot{t} = \sqrt{\left[\dot{y}^2 + \dot{z}^2\right]}$ *, where the curve is defined with regard to perpendicular Cartesian coordinates* $AB = z, BD = y$. *The analogous procedure for the other types of coordinate systems is no discussed in any of the nine following examples, but we may conjecture that the simple polar at least would have been dealt with if Newton had ever completed his present tract.* (Whiteside 1967, p.315)

This is the known formula: $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

For Leibniz,

given a curve whose arc length is sought, letting denote the length of the tangent line intercepted between the x- axis and a vertical ordinate of (constant) length a. Then, from the similarity the triangle, it follows that $\frac{ds}{t} = \frac{dy}{a}$ *or* $a\,ds = t\,dy$ *. Hence* $\int a\,ds = \int t\,dy$ *, so the rectification of the given curve reduces to a quadrature problem the calculation of the area of the region between the y-axis and a second curve whose abscissa x is the tangent t to the given curve.* (Edwards, 1973, pp.242-243).

REFERENCES

– Bernoulli, J.1988, *Der Briefwechsel Von Johann I Bernoulli*, Band 2, Birkhäuser Verlag, Basel.
– Bourbaki, N., 1969, *Éléments d´Histoire des mathématiques*, Hermann, Paris.
– Costabel P., Peiffer, J., 1988, *Der Briefwechsel Von Johann I Bernoulli*, Band 2, Bierkhäuser, Berlin.
– Descartes, R. 1897, *Oeuvres, ed. Adam et P. Tannery*, 13 vol. Paris (L. Cerf), t.II, p.514–517,
– Edwards, C. 1937, *The Historical Development of the Calculus*, Springer-Verlag, N.Y.
– Hugenii, C., 1751, Zulichemii Philosophi vere magni, Dum viveret Zelemii Toparchae, Opera... 4 tomes em 1 vol., , Lugd. Batav.
– Huygens, C., 1888-1950, *Oeuvres complètes*, 22 vol. t. VI, La Haya (M. Nijhoff).
– Gregorii, J. A Sancto Vicentio, 1647, *Opus Geometricum Quadraturae Circuli et Sectionum Coni*,... 2 vol, Antverpiae.
– Gregory, J., 1647, Vera Circuli et Hyperbolae Quadraturae... Pataviae 1667-, reproduzido em [Chirstiani Hugenii, Zulichemii Philosophi vere magni, Dum viveret Zelemii Toparchae, Opera... 4 tomes em 1 vol. , Lugd. Batav. 1751]
– L´Hospital, G., 1692, "Solution du problème que M. De Beaune propsa autrefois à M. Descartes, et que l´on trouve dans la 79 de ses lettres", tome 3, *Le Journal des Sçavants* (p. 401-403), Chez Jean Cusson, Paris.
– Loria, G., 1900, "Le ricerche inedite di Evangelista Torricelli sopra la curve logarítmica", *Bibl. Math*. (3) t.1, p.75-89.
– Torricelli, E., 1919, *Opere*, 4 vol. t. I, G. Loria & G. Vassura (eds), Faenza (Montanari).
– Varignon, P., 1703, Solution du problème que M. le Marquis de l´Hospital a proposé Geometres dans le Journal des Sçavans de 1692, p.59, *Le Journal des Sçavants*, p.117-121), Chez Jean Cusson, Paris.
– Waard, C., 1963, *Correspondance du P. Marin Mersenne*, v. VIII, Éditions du Centre National de la Recherche Scientifique, Paris.
– Whiteside, D. T. (ed), 1969, *The Mathematical Papers of Isaac Newton*, Vol. III (1670-1673), with the assistance in publication of M.A. Hoskin and A. Prag. Cambridge University Press, London.