THE ALGORITHMS OF POINCARÉ, BRUN, AND SELMER

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ABSTRACT

The idea of continued fractions in several dimensions has at least two roots. One is the idea of generalizing J. L. Lagrange's characterization of quadratic irrational numbers as periodic continued fractions. This path was followed by C. G. Jacobi. The other idea is to provide approximations to an n-tuple of numbers by rational numbers with a common denominator. This problem is deeply rooted in history and related to musical theory. Several proposals related to the names of Jacobi, Poincaré, Brun, Selmer, and others have been made. Due to the elementary nature of posing the problem one can use some original publications even in school.

1 Jacobi's attempt

One of the oldest algorithms is the Euclidean algorithm. Most probably its subtractive form is the original version (Euclid uses the verb $\alpha\nu\theta\upsilon\phi\alpha\iota\rho\epsilon\iota\nu$ 'to subtract reciprocally' to describe this operation; see Fowler 1987). One starts with two real numbers $a_0 > 0$ and $a_1 > 0$ with $a_0 \geq a_1$. Then we form $\sigma(a_0, a_1) = (a_0 - a_1, a_1)$ and if necessary we reorder to obtain a new pair. If $a_0 - a_1 > a_1$, then we take $a_0' = a_0 - a_1$ and $a_1' = a_1$. We take $a_0' = a_1$ and $a_1' = a_0 - a_1$ in the other case. It is possible to speed up this algorithm by replacing subtraction with division. This means that we form the pair $\delta(a_0, a_1) = (a_0 - ka_1, a_1)$. Here we put $k \geq 1$ the greatest integer such that the equation $a_0 - ka_1 \geq 0$ holds. Then in all cases $a_1 \geq a_0 - ka_1$ and we have a cyclic reordering. From the present viewpoint it is preferable to use *matrices* and to describe the algorithm in the form

$$\begin{pmatrix} a_0' \\ a_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

All important recursion relations can be derived easily by using the product of matrices. Rational numbers $\frac{p_n}{q_n}$ which are called the convergents of the algorithm will be obtained as follows.

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$$

It is easy to see that the algorithm stops if and only if $a : b$ is rational.
The algorithm is homogeneous in the following sense. The pair (x_0, x_1) and the pair $(\lambda x_0, \lambda x_1)$, $\lambda \neq 0$ lead to the same algorithm. This property suggests using the inhomogeneous version

$$Tx := \frac{1}{x} - k,\ x = \frac{a_1}{a_0},\ Tx = \frac{a_1'}{a_0'},\ k = k(x) = [\frac{1}{x}].$$

If $a : b$ is not rational then the rational numbers $\frac{p_n}{q_n}$ are good approximations to the irrational value $\frac{a}{b}$. It is possible to make a short digression to the relation between matrices and fractional linear maps. In ancient Greece it was well known that geometric ratios can lead to periodic algorithms. The most common examples are the Golden Ratio and the square root of 2. Translated into present day language we put $\lambda = \frac{-1+\sqrt{5}}{2}$ and we find the relation

$$\lambda \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

The idea of periodicity therefore is related to eigenvalues and eigenvectors. In inhomogeneous notation we find

$$\lambda = \frac{1}{1+\lambda}.$$

The connection to the Farey sequence $1, 1, 2, 3, 5, ...$ is immediate. If $F_{n+1} = F_n + F_{n-1}$ then one sees

$$\frac{F_n}{F_{n+1}} = \frac{F_n}{F_n + F_{n-1}}$$

which shows

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lambda.$$

There are two ways to explore periodicity. One may start with given irrational numbers like $\sqrt{2}, \sqrt{3}, ...$ or one starts at the other end. The periodic expansions

$$x = \frac{1}{k+x}$$

or more generally

$$x = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}$$

lead to quadratic irrational numbers. Last but not least one can state the famous result of Lagrange.

There are at least two roots for multidimensional continued fractions (see e. g. Schweiger 2006). One root is the attempt to extend Lagrange's theorem to n-tuples of irrational numbers (this was the main point in Jacobi 1868). The other problem is the question about approximation of an n-tuple of real numbers by rational numbers with a common denominator. This question is related to musical theory (a good introduction is Wright 2009; a broader discussion can be found in Assayag et al. 2002). Jacobi, Poincaré, Brun, Selmer, and others made various proposals for multidimensional continued fractions. In what follows we will shortly describe their ideas.

In the year 1868 E. Heine published the paper „Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird" which he found in the legacy of G. G. J. Jacobi (1804-1851). Due to its complexity this paper is hardly suitable for use in a classroom. Jacobi considers three real numbers („unbestimmte Zahlen") a, a_1, a_2 and a sequence of given quantities („gegebene Grössen") $l, m, l_1, m_1, l_2, m_2, \ldots$. Then he defines

$$\begin{array}{lcl} a_3 & = & a + la_1 + ma_2 \\ a_4 & = & a_1 + l_1 a_2 + m_1 a_3 \\ a_5 & = & a_2 + l_2 a_3 + m_2 a_4 \\ \dots & \dots & \dots\dots\dots \end{array}$$

Translated into matrix theory his algorithm is much more comprehensible in the form

$$(a_1, a_2, a_3) = (a_0, a_1, a_2) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & l \\ 0 & 1 & m \end{pmatrix}.$$

As before one sees that the triplets (a_0, a_1, a_2) and $\lambda(a_0, a_1, a_2)$ determine the same algorithmic course. Later in the paper Jacobi chooses a different approach. Let u_0, v_0, w_0 be three positive numbers and define $l_0 = [\frac{v_0}{u_0}], m_0 = [\frac{w_0}{u_0}]$. Then the recursion starts with $u_1 = v_0 - l_0 u_0, v_1 = w_0 - m_0 u_0, w_1 = u_0$. With the help of matrices one sees

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -l_0 & 1 & 0 \\ -m_0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}.$$

Naturally one proceeds by iteration.

In his examples Jacobi puts the first coordinates $u_0, u_1, u_2, \dots$ equal to 1. Three examples are given which lead to periodic expansions.

i)

$$\begin{array}{lcl} (u_0, v_0, w_0) & = & (1, \sqrt[3]{2}, \sqrt[3]{4}) \\ (u_1, v_1, w_1) & = & (1, \sqrt[3]{2} + 1, \sqrt[3]{4} + \sqrt[3]{2} + 1) \\ (u_2, v_2, w_2) & = & (1, \sqrt[3]{2} + 2, \sqrt[3]{4} + \sqrt[3]{2} + 1) = (u_3, v_3, w_3) \end{array}$$

ii)

$$(u_0, v_0, w_0) = (1, \sqrt[3]{3}, \sqrt[3]{9})$$

$$(u_2, v_2, w_2) = (u_4, v_4, w_4).$$

iii)

$$(u_0, v_0, w_0) = (1, \sqrt[3]{5}, \sqrt[3]{25})$$

This is a very awkward example because after some lengthy calculations one finds $(u_7, v_7, w_7) = (u_{13}, v_{13}, w_{13})$. Note that the original publication contains two printing errors at this point.

But now we are confronted with a difficult problem. Up to now it is even not known if the triplet $(\sqrt[3]{2}, \sqrt[3]{4}, 1)$ leads to a periodic expansion. The generalization of Lagrange's theorem remains an open problem.

Matrices provide a good representation for convergents by using the equation

$$\left(\begin{array}{ccc} P_{i+1} & P_{i+2} & P_{i+3} \\ R_{i+1} & R_{i+2} & R_{i+3} \\ Q_{i+1} & Q_{i+2} & Q_{i+3} \end{array}\right) = \left(\begin{array}{ccc} P_i & P_{i+1} & P_{i+2} \\ R_i & R_{i+1} & R_{i+2} \\ Q_i & Q_{i+1} & Q_{i+2} \end{array}\right) \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & l_i \\ 0 & 1 & m_i \end{array}\right).$$

The initial values are given by the unity matrix for $i = 0$. If we consider the first example then we get $l_0 = 1, m_0 = 1$, $l_1 = 2, m_1 = 3$, $l_2 = 3, m_2 = 3$ and then the algorithm continues periodically with $l_3 = 3, m_3 = 3$, $l_4 = 3, m_4 = 3$. In this way we obtain a sequence of approximating fractions with common denominator, the convergents, for the pair $(\sqrt[3]{2}, \sqrt[3]{4})$, namely the sequence $(1,1), (\frac{4}{3}, \frac{5}{3}), (\frac{15}{12}, \frac{19}{12}), (\frac{58}{46}, \frac{73}{46}), \ldots$.

2 Brun, Selmer and Poincaré

Much more easy to read are the papers by Viggo Brun “Algorithmes euclidiens pour trois et quatre nombres” (Brun 1957) and “Euclidean algorithms and musical theory” (Brun 1964). A revealing epilogue on Brun’s contributions has been written by Scriba 1985. Clearly Brun’s paper from 1957 requires some knowledge of French. In fact the oldest papers by Brun on this subject appeared in Norwegian. The starting point is a triple (a_0, a_1, a_2) with $a_0 \geq a_1 \geq a_2 > 0$. Then we form $\sigma(a_0, a_1, a_2) = (a_0 - a_1, a_1, a_2)$ and reorder. There are three possibilities.

$$a_0' = a_0 - a_1, a_1' = a_1, a_2' = a_2$$

$$a_0' = a_0, a_1' = a_0 - a_1, a_2' = a_2$$

$$a_0' = a_0, a_1' = a_1, a_2' = a_0 - a_1.$$

Again the use of matrices is recommended. For practical purposes the inverse matrices which correspond to the three types of reorder are helpful.

$$B(0) = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), B(1) = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right), B(2) = \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

But now it is time to reveal the connections with musical theory which are also described in Scriba 1985. In the just intonation scale the ratio of the frequencies of a a tone to its octave is $1:2$. We further find the ratio $2:3$ for the (just) fifth and $3:4$ for the (just) fourth. Now you want to construct a musical instrument which has a fixed number of strings within an octave. If you think of the keyboard of a pianoforte these should be twelve strings which will become shorter related to the inverse ratio of the pitch. This could be a good opportunity to make a digression to the history of musical scales or to the music of the various cultures. Now we request that the ratio of two subsequent pitches is a fixed number. We look for a number λ such that

$$\lambda^x \approx 2,\ \lambda^y \approx \frac{3}{2},\ \lambda^z \approx \frac{4}{3}.$$

This leads to a problem of Diophantine approximation, namely to approximate the triple $(\log 2, \log \frac{3}{2}, \log \frac{4}{3})$ by three whole numbers. Using some decimal approximations Brun algorithm leads to the sequence 122010100001.... The wanted triples of numbers (x, y, z) can be found as the columns of the matrices.The expansion 1220101 gives

$(12, 7, 5)$ and the longer expansion 12201010000 the values $(53, 31, 22)$. In a similar way we can also start with the ratios for the octave, the fifth and the major third $(5 : 4)$. This means to approximate $(\log 2, \log \frac{3}{2}, \log \frac{5}{4})$. One finds the triple $(12, 7, 4)$.

In the Western musical practice this approximation is of considerable importance. It corresponds to a partition of the octave into twelve steps such that subsequent pitches change by the factor $\lambda = \sqrt[12]{2}$. Therefore, a fifth is given in the equally tempered scale by $\lambda^7 \approx 1,498$ instead of by $\frac{3}{2}$. In the equally tempered scale the fifth is approximately two *cents* flat. On the other hand the fourth $\frac{4}{3}$ is replaced by $\lambda^5 \approx 1,335$ which is approximately two *cents* sharp. A *cent* corresponds to the division of an octave into 1200 microtones which gives 1 cent $\approx 1,0005778$. This factor can be derived in a quite different way. One requires that seven steps of an octave are the same as 12 steps of a fifth. In fact this is not true in the just scale, since 2^7 is different from $(\frac{3}{2})^{12}$. We find $2^{19} = 524288$ and $3^{12} = 531441$. The ratio $3^{12} : 2^{19} \approx 1,01365$ is called the Pythagorean comma. If we put the fifth equal to λ^7, then we obtain $2 = \lambda^{12}$.

Ernst Selmer's paper also appeared in Norwegian. He makes a very simple change in Brun's ideas. Let $a_0 \geq a_1 \geq a_2 \geq 0$. Then we put $\sigma(a_0, a_1, a_2) = (a_0 - a_2, a_1, a_2)$ and reorder. The corresponding (inverse) matrices are given as

$$D(0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

However, if $a_0 > a_1 > a_2 > 0$ then the appearance of $D(1)$ or $D(2)$ does not permit a following $D(0)$. A genuine reordering of $\sigma(a_0, a_1, a_2)$ is necessary if $a_0 - a_2 < a_1$ which is equivalent to $a_0 < a_1 + a_2$. The new triple is either $(a'_0, a'_1, a'_2) = (a_1, a_0 - a_2, a_2)$ or $(a'_0, a'_1, a'_2) = (a_1, a_2, a_0 - a_2)$. However, in both cases we obtain $a_1 = a'_0 < a'_1 + a'_2 = a_0$. Selmer had the hope that the question of periodicity would be easier to solve but his hope was in vain! He calculates the expansion of $(\log 2, \log \frac{3}{2}, \log \frac{5}{4})$ but the important denominator 12 does not appear.

In the year 1884 Henri Poincaré proposed a different approach. His paper is geometrically inspired and densely written but Arnaldo Nogueira (1995) has given a broad exposition including some figures. The arithmetic description is very simple. Let (a_0, a_1, a_2) be a triple of non-negative real numbers. Then there is a permutation π such that the condition $a_{\pi 0} \leq a_{\pi 1} \leq a_{\pi 2}$ is satisfied and we form

$$(a'_0, a'_1, a'_2) = P(a_0, a_1, a_2) = (a_{\pi 0}, a_{\pi 1} - a_{\pi 0}, a_{\pi 2} - a_{\pi 1}).$$

This algorithm looks very innocent but in fact it leads to difficult problems. Contrary to Poincaré's hope it is not very useful for Diophantine approximation. After some calculations the triple $(\log 2, \log \frac{3}{2}, \log \frac{4}{3})$ leads to the approximation $(11, 5, 4)$ but again the important approximation $(12, 7, 5)$ is left out.

3 Conclusions

Multidimensional continued fractions could be a good topic for instruction at undergraduate level or even in school. The approach is elementary but leads to interesting problems including the up to now unsolved problem of finding a simple algorithm to generalize Lagrange's theorem. Clearly, some concepts of this paper preferably are thought for the background knowledge of the teacher. It would be also helpful to work with continued fractions before starting with their generalizations to higher dimensions. Hand-held calculators and computers can be used to calculate the very first approximations of given triples. It is possible to use matrices and if one goes into direction of periodicity to consider their eigenvalues. It is possible to use at least some parts of the original publications and to discuss mathematical language as expressed in different languages (German, English, French, and - very ambitious - Norwegian). This can connect independent working with the development of mathematics. Last but not least older papers can show that a good notation is very helpful.

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