USE OF THE HISTORY OF NEGATIVE NUMBERS IN EDUCATION

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ABSTRACT

It is well known that pupils have many conceptual difficulties with negative numbers. The notation of negative numbers used in ancient China employed two colors (red and black) to represent positive and negative numbers. In this way it was possible to give clear semantic interpretation of the various rules of calculation (especially the rules of multiplication). In the paper I describe the use of the notation of negative numbers by colored sticks to help children clarify the rules for calculation with negative numbers.

1 The problems with negative numbers

An important motivation for studying the history of negative numbers is the fact that many pupils have problems with them. There are common mistakes in calculations caused by formal learning of the rules of counting with negative numbers and such mistakes are very frequent not only at primary and secondary schools but also at colleges. Some of the possible reasons which leading to these problems are that pupils have just few opportunities to deal with negative numbers in their real life, teachers neglect the development of intuitive perception of negative numbers at primary schools, sometimes teachers tell the pupils who did not learn negative numbers that 2 minus 5 equals 0 and pupils learn the rules without comprehension of the concept of a negative number.

2 Literature survey

Our main question is how the use of the history of mathematics to avoid the problems pupils have with the understanding of negative numbers. Professor Milan Hejný in his Theory of the Mathematics Education (Hejný, 1989) describes formal knowledge as knowledge of rules learned by heart without real understanding. When we want to eliminate formalism we can search for inspiration in the history of mathematics. By an analysis of the history of mathematics we can receive useful ideas about the genesis of thinking which we can try to apply in teaching. (Hejný 1989, p.25)

Various answers to our question can be found in the book: *Mathematics Education, The ICMI Study,* edited by John Fauvel and Jan van Maanen (Fauel and van Maanen, 2000). Here we can find the results of experience with the use of history of mathematics in learning and teaching; different ways in which the history of mathematics might be useful; scientific studies of its effectiveness and many other information. We can find also passages on the use of the history of negative numbers.

Using the history of mathematics brings several other advantages. These advantages are mentioned by Tzanakis & Arcavi in the seventh chapter of that book:

> "An advantage of implementing history in the presentation and learning of mathematics is the opportunity it presents to appreciate and make explicit use of the constructive role of (i) errors, (ii) alternative conceptions, (iii) changes of perspective concerning a subject, (iv) paradoxes, controversies and revision of

implicit assumptions and notions, (v) intuitive arguments, that appeared historically and may be put to beneficial use in the teaching and learning of mathematics, either directly or didactically reconstructed (Tzanakis & Arcavi and 2000, p.238)

Relations between epistemology, history of mathematics, and teaching and learning of mathematics we can find in several other papers as for instance: Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics written by Luis Radford in 1997, Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics written by Fulvia Furinghetti and Luis Radford in 2002 and Historical conceptual developments and the teaching of mathematics: from phylogenesis and ontogenesis theory to classroom practice written by Fulvia Furinghetti and Luis Radford in 2008. The authors place the relation between historical development of mathematics and its education into a broad evolutionary and psychological perspective.

3 The significant points from the history of negative numbers

Negative numbers in ancient China

The oldest notations of numbers from China can be found on magical cubes from the 14th to the 11th century BC and on ceramic or bronze objects and coins from the 10th to the 3rd century BC. From the 4th century BC the Chinese used rods in their calculations.
In notation they expressed the value by means of real rods.

	1	2	3	4	5	6	7	8	9
ones, hundreds, ten thousands …	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
tens, thousands, hundred thousands…	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

Figure 1 Chinese numerals

The counting with these numbers had a positional character and this way of counting - counting with rods – is the oldest positional decimal system.

A very important mathematical work in China was Mathematics in Nine Chapters (see Hudeček 2008, and Kangshen et al., 1999). It is a collection of 246 problems with solutions which was prepared from older sources between the years 206-255 BC. In the third century AD Liu Hui added his commentary to this work. The eighth book of the Mathematics in Nine Chapters deals with solving systems of equations. For example:

"Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?”

And below we can find the answer and the method of solution:

“Cattle price 1200, sheep price 500, pig price 300.
Method: Use the Array Rule rule: lay down2 cattle, 5 sheep, positive; 13 pigs, negative; surplus coins positive; next 3 cattle, positive; 8 pigs, positive. Deficit couns, negative. Calculate using the Sign Rule.” (Kangshen et al., 1999, p.405)

Here, in the eighth book of the *Mathematics in Nine Chapters* we come across negative numbers for the first time in the history of mathematics. Very important is *the Sign Rule* which describes the rules for adding and subtracting of positive and negative numbers. For these numbers red and black rods were used. Thus the black rods meant negative and the red rods positive numbers. Positive numbers were called “cheng” and negative numbers “fu”. Liu Hui adds his commentary to this method:

“(c) Liu: Subtracting one column from another depends on appropriate entries with the same signs. Opposite signs (entries) are from different classes. If from different classes they cannot be merged but are subtracted. So merging red by black is to subtract black, merging black by red is to subtract red. Red and black merge to the original colour this is “adding”; they are mutually eliminating. This is by eliminating (the top entries) using addition and subtraction to achieve the bottom constant (shi).

The prime purpose of the rule is to eliminate the first entry: magnitudes of entries in other position of no concern, either subtract or merge them. The reasoning is the same, not different.
(d) “Without extra” is “without merging”. When nothing can be subtracted put the subtrahend in its place (with colour changed), subtracting the resulting constant (lu shi) from the bottom constant. This rule is also applicable to columns whose entries are mixed signs. In the rule, entries with same signs subtract their constant terms, entries with opposite signs add their constant terms. This is positive without extra, make it negative; negative without extra, make it positive.” (Kangshen 1999, p.405)

Diophantos, 3rd century AD

Mathematicians encountered many difficulties in dealing with negative numbers. This can be seen for example in the case of Diophantos, for whom a negative number was just an operator. He chose coefficients in equations only in the way that he could get rational positive solutions. When an equation had two positive roots, he chose just one of them – the higher one and when there were two irrational or negative roots he declared that the equation had no solution.

India, 7th century AD

Indians dealt with negative numbers and they did not restrict themselves just to adding and subtracting. In their works we can find rules also for multiplication, division and second power. And so they moved debts to a more abstract one perception.

A quotation from ***Brahmagupta***´s work gives evidence of this:
“Negative, taken from cipher, becomes positive; and affirmative, becomes negative. Negative, less cipher is negative; positive, is positive; cipher nought. When

affirmative is to be subtracted from negative, and negative from affirmative, they must be thrown together" (Murray 1817, p. 340).

Cardano, 16th century AD

Cardano accepted also negative numbers as roots of equations but he called them fictious while positive solutions were called true.

"It will be remembered also that 9 is derivable [by squaring] equally from 3 and -3, since a minus times minus produces plus. But in the case the odd powers, each keeps its own nature: it is not plus unless it derives from a true number, and a cube whose value is minus or we call debitum, cannot be produced by any expansion of a true number."

"...For example: If

$x^2 + 4x = 21$

the true solution is 3 and the fictitious one -7" (Cardano 1545, p. 9-10)

Descartes, 17th ctr. AD

Descartes accepted negative numbers partially. He called negative roots false and showed that equations with negative roots can be transformed into equations with positive roots. Based on this fact he was willing to accept negative numbers.

Newton 17th and 18th century AD

A work of Issac Newton (1643-1727) *"Arithmetica universalis, sive de compositione et resolutione arithmetika liber"*- ***Universal Arithmetic or a Treatise of Arithmetical composition and resolution*** makes the issue of negative numbers clear. Newton says that quantities are either positive, or greater than nothing, or negative, or less than nothing. He took zero for nothing. He interpreted negative numbers as negative motion.

„IV. Quantities are either Affirmative, or grater than nothing, or Negative, or less than nothing. Thus in human Affairs, Professions or Stock may be called affirmative Goods, a Debts negative ones. And so in local motion, progression may be called affirmative motion, and Regression negative motion, because the first augments, and the other diminishes the Length of the Way made. And after the fame Manner in Geometry, if a line draw any certain Way be reckoned for affirmative then." (Newton 1720, p.3)

As we could see, negative numbers were only fictitious or false for mathematicians until the seventeenth century and many of them did not accept a negative number as a solution of an equations. This information could be helpful for teachers to understand that pupils can have problems with negative numbers despite the fact that they know the rules for counting with them. To help them to overcome these problems we can use some tasks from the original mathematical works in a class.

4 CHINESE NUMERALS IN A CLASS

I used the counting with Chinese rods as an activity in a class. Until the last year the pupils in Slovakia learnt all of the operations with negative numbers in the sixth grade of primary school. The new reform moved this topic to the beginning of the eighth grade.

For this activity – counting with Chinese rods – I chose two classes of the sixth grade at a primary school (12 years old children). These pupils had not dealt with negative numbers before.

I used the following aids:

- *Red and black rods*

They were made of hard paper or soft plastic and prepared by the teacher. If there is enough time for that, pupils can make them themselves.

- *Working sheet:*

The working sheet contained a brief history of Chinese mathematics with information needed for the work with rods together with a well-arranged table with Chinese numerals and some space for notes.

The phases of the activity were:

1. ***Narration about the history of Chinese mathematics***

– A motivational narration about how Chinese used to count.

2. ***Explanation how numerals looked like in China.***

– I wrote the Chinese numerals on the blackboard and, together with the pupils we wrote several numbers using these numerals. The pupils participated actively.

3. ***Create several numbers by means of the "Chinese rods".***

– The pupils put together numbers by means of rods which they had on their desks.

4. ***Discussion about debts.***

– I discussed with the pupils how to write down a debt, which colour symbolises a debt and I compared it with China.

5. ***Counting with red and black rods***

– I will show specific assignments and their solutions in a while

6. ***Competition between four-member groups.***

– After the pupils understood the principle of counting with coloured rods there was a competition between groups. These groups assigned tasks to each other. If a group found the right solution, it got a point. If the group with an assigned task did not find the right solution, the group which assigned the task could get the point if it found the right solution.

7. ***Feedback***

Creating several numbers by means of the "Chinese rods":

We tried to express the number four by means of coloured rods in various ways. These are some of them:

4 red rods:

I I I I

5 red rods and 1 black rod:

I I I I I I

6 red rods and 2 black ones:

I I I I I I I I

We also dealt with the number minus 2

How can we express the number -2?

I I

I I I I

I I I I I I

I I I I I I I I I I I I

Then we have here an example of several assignments:

- 7 red rods plus 4 black rods 7+(-4)

- 7 black plus 13 black (-7)-(-13)

- 4 red minus 12 black 4-(-12)

At last I showed the pupils how the counting with a counting board look like:

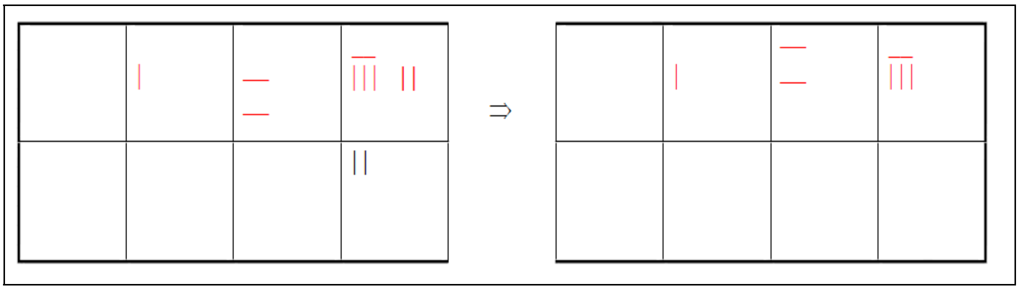

Figure 2

130+(- 2) = 128

Some facts from the feedback:

a. Motivational factor – the pupils were interested in the brief history, they asked for some more facts.

They were surprised that the Chinese could count already such a long time ago.

The pupils noticed that the zero was missing among the numerals.

b. Cultural context – the pupils associated the red colour with a debt but it was no problem for them to deal with the black colour, which symbolised a debt in China. It caused bigger problems for the teacher.

c. During the class the rules counting with negative numbers was not disclosed by the teacher of.

The pupils were able to get the right solutions quite fast also in the assignments which they had never seen before, for instance subtracting negative number from a positive one.

5 CONCLUSION

Looking at the historical examples we can see that for the understanding of negative numbers it was not enough to know rules for counting with them. I think this is one of the problems of today´s education. We teach pupil rules and then when they come across negative numbers for the next time, we assume that they already know how to count with them. It is questionable if it is a solution to move the whole thematic unit into a higher class as it was done in Slovakia. It would be better to focus on propaedeutic and use the fact that today except the debt we also know the thermometer, the elevator, the altitude above sea level and so on and this is our advantage over the mathematicians of the past.

Cooperation between subjects is important as well. A discussion of the connection between Newton´s decelerated motion and negative numbers in physics classes could bring more light into the ideas of pupils. It would be interesting to discuss with pupils what meaning the negative roots have according to them or whether they have thought about it and compare their views with the historical ideas.

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