ICT AND HISTORY OF MATHEMATICS:
the case of the pedal curves from 17th-century to 19th-century

Olivier BRUNEAU
Laboratoire Histoire des sciences et Philosophie, Archives Poincaré
91 avenue de la Libération
54000 NANCY FRANCE
e-mail: bruneauolive@free.fr

ABSTRACT

Dynamic geometry softwares renew the teaching of geometry: geometrical construction becomes dynamic and it is possible to "visualize" the generation of curves. Historically this aspect of the movement (continuous or not) is natural and was well known to 17th-century mathematicians. Thus, during the 17th-century, the mechanical or organic description of curves was re-evaluated by scholars like Descartes or Newton.

In this article, we want to focus on a special class of curves: pedal curves. The definition of this kind of curves given, we will then briefly retrace their history between the 17th and the end of 19th-century. And finally, we present some activities which can be produced by a dynamic geometry software like geogebra.

1 Introduction

Before providing a short summary of the history of this kind of curve from the 17th to the 19th century, let's recall the standard definition of the pedal curve:

Definition 1. *(C) is a plane curve and O a point on the plane. We consider the foot P of the orthogonal straight line to the tangent from any point M on the curve. The pedal curve of (C) is the locus of P when M describes the curve.*

For instance, if we consider a circle (C), and O one point of the plane, then its pedal curve of centre O is the figure 1:

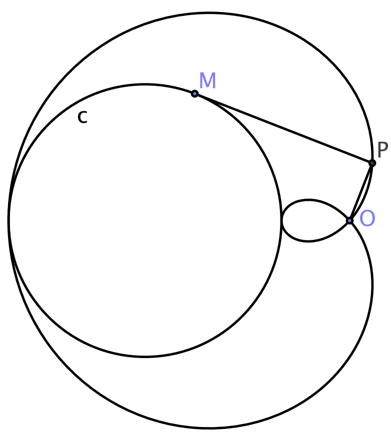


Figure 1: Circle pedal

2 The 17^{th}-century

2.1 Roberval

In fact, this curve (Figure 1) is the first pedal curve found in the 17th-century by Gilles Personne de Roberval (1602–1675). He spent his time as mathematics teacher in Paris. He was an active mathematician in Mersenne's circle. He was in relation with Blaise Pascal, Pierre de Fermat and Carcavi, but he was in conflict with René Descartes.

We find this notion in a text entilted *Observations sur la composition des mouvemens, et sur le moyen de trouver les touchantes des lignes courbes* included in a posthumous edition of *Divers ouvrages de Monsieur de Roberval* in 1693. It was reprinted in the journal *Mémoires de l'Académie Royale des Sciences de Paris* in 1730. In this text, one can find the parallelogrammical construction for composition of motion and several methods to construct tangents of some curves like conic sections and spirals.

In the part of the construction and description of the well-known Pascal spiral, he shows how to find this kind of curve with another method (Figure 2):

> Mais voici une des belles spéculations qui se puisse sur la description de cette ligne [the cochlea], et par le moïen de laquelle elle a été trouvée par le sieur de Roberval. Soit supposé le cercle CEB, & l'intervalle CD comme aux figures précédentes : du point C & de l'intervalle CD soit décrit le cercle DG* ; (...) aïant tiré des touchantes GF à ce cercle, & du point B tiré des lignes BF perpendiculaires à ces touchantes, que chacun des points F sera dans notre limaçon.

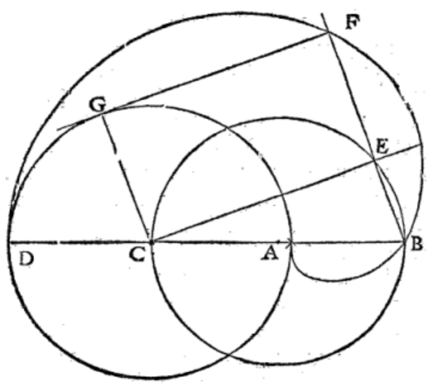


Figure 2: Roberval's construction

2.2 Newton

In the secondary literature, one finds that Newton knew pedal curves. According to B. Pourciau, in the 2nd edition of the *Principia Mathematica Philosophica Naturalis*, Newton uses pedal coordinates to show the relationship between the $\frac{1}{r^2}$ law and a trajectory. But, Newton is not explicit in this edition and Pourciau made the supposition that Newton has effectively considered this kind of coordinates. And, in the following editions this is the same thing. Moreover, in Newton's *Mathematical Papers* edited by Whiteside, there is no mention of this kind of pedal construction. But, at the end of nineteenth-century, it is usual to demonstrate this relationship with pedal coordinates.

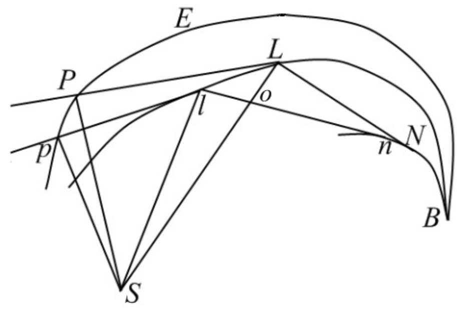


Figure 3: Maclaurin's construction

2.3 Which status for pedal curves?

One can say that there is no status for pedal curves in Roberval's works because he considers these *by the way*. They are not in a central place but are linked with his work on tangents, and he does not make any theory about it. The first major step about this curve is made by a Scottish mathematician, Colin Maclaurin.

3 The 18^{th}-century: the case of Maclaurin

Colin Maclaurin (1698–1746) become professor of mathematics in Marishall College (Aberdeen, Scotland) at the age of nineteen (in 1717). His two first papers sent at the Royal Society of London in 1719 are the first step of the recognition of his mathematical skills. One of them, entitled *Tractatus de curvarum constructione et mensura ; ubi plurimae series curvarum infinitae vel rectis mensurantur vel ad simpliciores curvas reducuntur*, is the first exposition of pedal curves. These two papers are the bases of his first main work, the *Geometria Organica: sive Descriptio Linearum Curvarum Universalis* (1720). All that follows is extracted from this one.

His definition is the same that we have seen before (Figure 3). But, in the *Geometria Organica*, he gives some properties of this kind of curve and proposes to classify the pedal curves for a family of special curves.

To do so, he introduces an orthogonal frame with the pole S as center. The coordinates of L are x and y. For any x and y, the quotient $\frac{\dot{x}}{\dot{y}} = \frac{m}{n}$ exists with m and n as finite quantities and $\dot{x}$ is the fluxion (*i.e.* the derivative) of x. So he can give two quantities, SP and SL as : $SP = \frac{my-nx}{\sqrt{m^2+n^2}}$ and $SL = \sqrt{x^2+y^2}$.

His main tool is what we call now the pedal equation, the *Æquatione radiali*, it is the quotient $\frac{SP}{SL}$.

Since a pedal curve of a given curve is also a curve, it is possible to draw the pedal curve of the pedal curve. So, he gives a construction of successive pedal curves, it is the sequence of positive pedal curves. But the main difficulty is to find the inductional relationship that enables us to define some pedal curves.

Nevertheless, from the radial equation of the first pedal curve, it is easy to deduce the radial equation to the second one by the same kind of quotient.

He introduces the construction of an antipedal as follows. The antipedal of a curve is a curve for which its pedal curve is the initial curve. He gives as we see below a relationship between a curve, its pedal and antipedal curves.

Let's give Maclaurin's property:

Proposition 3.1. *Given C' a pedal curve of centre S of the curve C. A simple geometrical construction produces the pedal curve of C of centre F from the pedal curve C'. To do so, it is sufficient from any point P of C' to have the perpendicular of PS and to F the perpendicular to PN. So the intersection point of these two perpendiculars, N, describes the pedal of F.*

After that, he proposes some examples. The first one is the pedal curve of a circle, he finds Pascal's spiral and announces that it is an epicycloid (found by Nicole in 1707) and it is a conchoid (circular base) too (by De la Hire in 1708). According to Loria, Cramer was the first one to link these three kind of curves, but the real one is Maclaurin. In fact, Cramer perused Maclaurin's book. The second examples are the pedal curve of conic sections (see below).

Maclaurin's main interest is curves whose radial equation is of the form $\frac{p}{r} = \frac{r^n}{a^n}$ where $p = SP$ and $r = SL$.
He proves the proposition:

Proposition 3.2. *The radial equation of its pedal (with same centre) is $\frac{p}{r} = \frac{r^{n/(n+1)}}{a^{n/(n+1)}}$, one of the m^{th} pedals is $\frac{p}{r} = \frac{r^{n/(mn+1)}}{a^{n/(mn+1)}}$ and its m^{th} negative pedal is $\frac{p}{r} = \frac{r^{n/(-mn+1)}}{a^{n/(-mn+1)}}$*

So, its possible to classify curves according to the radial equations:

Circle of radius a and the centre in the circonference	$p/r = r/2a$	$n = 1$
Right line with a distance a of the centre	$p/r = a/r$	$n = -1$
Parabola (the 1st antipedal of circle)	$p/r = (r/a)^{-1/2}$	$n = -1/2$
Equilateral Hyperbola (parameter a, centre is the origin)	$p/r = (a/r)^2$	$n = -2$
Cardioid (first pedal of circle)	$p/r = (r/2a)^{1/2}$	
Lemniscate (The first pedal of hyperbola)	$p/r = (r/a)^2$	

Two results for curves which are their radial equation as $p/r = (r/a)^n$:

Theorem 3.1. (The rectification)
If L describes the curve and B a (start) point of this curve, and given P and N respectively points corresponding in the pedal and antipedal curves. Then $\mathrm{Arc(BP)} = \mathrm{(n+1)(Arc(BN) + LN)}$

And

Theorem 3.2. (Curvature radius)
The curvature is $\frac{a^n}{n+1} \times \frac{1}{r^{n-1}}$.

3.1 Which status for pedal curves?

These curves are central to his project. He proposes a systematic study and some properties : rectification, curvature,..., he tries to classify curves and applies them to a mechanical problem : Pourciau's indication is more for Maclaurin than Newton.

4 The 19^{th}-century

With the development of new geometries and several approaches for mathematical artefacts description, the pedal curves are used in many ways. And a lot of papers are entirely or partly devoted to pedal curves. During the first half of this century, this kind of curve is re-discovered by some German mathematicians, especially Jakob Steiner (1796–1863) who gave the German name : *Fusspuncktecurve*. Mostly during the second half of century, more than 400 articles concern pedal curves in different branches of mathematics : synthetic, differential geometry. And these curves are used as problems in the classroom and in entrance examinations for prestigious schools like the *École Polytechnique*.

4.1 In synthetic geometry

At the beginning of the nineteenth-century, following Poncelet's and Chasles' works, some mathematicians reconsidered the nature of the pedal curve. According to them, the pedal curve is the result of the composition of two transformations : After Poncelet's works, some mathematicians have found that the construction of pedal curve is equivalent to the composition of an inversion and a polar transformation.

During the first half of the nineteenth Century, some mathematicians like Jakob Steiner with his *Theorie der Kegelschnitte* and his two articles in Crelle's journal or Quételet in relationship with the caustics problem, work on pedal curves as a composition of transformations. But, the last one does not see it as a specific curve and does not extend it theoretically.

4.2 In differential geometry

Mostly during the second half of the nineteenth-Century, one can find some papers devoted to pedal curves. Some of them make the link with Maclaurin's construction like Haton de la Goupillière when he writes on curves with polar equation $\rho^n = A \sin n\omega$ (Haton, 1876). There are a lot of articles from Germans like R. Sturm with his paper, *Über Fusspunkts-Curven und -Flächen, Normalen und Normalebenen*, published in *Mathematische Annalen*.

At the end of nineteenth century, some mathematicians view pedal curves not only as curves but mostly as transformations. For instance, Sophus Lie considers pedal transformations as a part of his *Geometrie der Berührungstransformationen* (1896). Some years after, Gino Loria (1907) writes about the pedal transformation.

4.3 In teaching

Articles about this kind of curves are linked with teaching, and they are taught in France, Germany, Italy and Great-Britain as exercises. In the *Nouvelles Annales de Mathématiques* or *Educational Times*, one can see some exercises and problems focused specifically on pedal curves. For instance, the seventh problem of the 1847 entrance exam of École Polytechnique is "trouver le lieu des projections d'un sommet d'une section conique sur ses tangentes". It is in the commentary of the resolution of this problem given in the *Nouvelles annales de Mathématiques* that Terquem proposes to name pedal curve into french "podaire".

In textbooks devoted on geometry, one can find some problems in which pedal curves are important. For instance, at the end of the nineteenth-cetury, one can cite the fourth problem "Le sommet d'un angle constant c se meut sur une courbe directrice s pendant qu'un de ses côtés enveloppe une courbe donnée σ_1; trouver la courbe σ_2 enveloppe de l'autre côté." (Aoust, 1873, p. 171.). Aoust gives a relationship between caustic and pedal curves.

To conclude this part, one can say that pedal curves are present in all type of geometries without any central role, much like other tranformations. To summarize, pedal curves are significant enough to merit a specific field in the *Répertoire bibliographique des sciences mathématiques* (1894-1912): 02qα Podaires et podaires négatives.

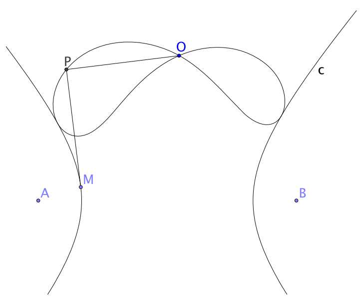
5 Pedal curves and Geogebra

We propose two short examples to show how to concile history of mathematics, old sources and dynamical geometrical software. The main interest of this kind of software is to integrate dynamic views with old problems and to open new interpretations in the creation process from mathematicians like Maclaurin or Newton.

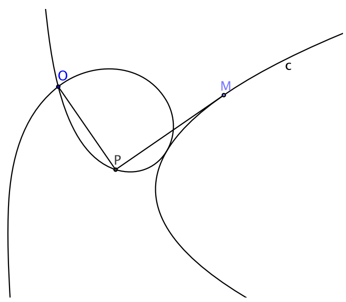
The first is to exploit an extract of Maclaurin's *Geometria Organica* in which he gives different cases of the pedal curve of conic sections. And the second one is from the *Nouvelles Annales* in which a method is given to use pedal curves to trisect angles.

5.1 Pedal curve of conic section

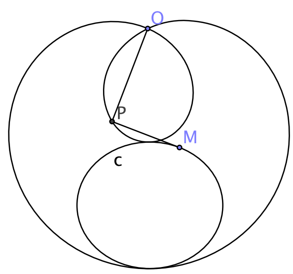
Let us follow Maclaurin in his examination of pedal curves of conic sections:



Pedal curve of hyperbola



Pedal curve of parabola



Pedal curve of ellipse

The three examples come from *Geometria Organica*, which Maclaurin split into three cases. In each case, he determines the situation with respect to the pole. With Geogebra, it is extremely easy to rediscover each case with a simple movement. For instance, for the pedal curve of the parabola, Maclaurin makes a relationship with the Newtonian description of some third order curves.

5.2 Angle trisection

The trisection of angles is an old problem: is it possible to trisect any angle only with compass and straightedge? From ancient times, mathematicians tried to resolve this one, but with only compass and straightedge, it is an impossible problem. This was demonstrated in the 19th-century with the works of Abel or Galois.

Here is an interesting use of pedal curve :

> M. le docteur Toscani, professeur de physique au lycée de Sienne (Toscane), fonde la trisection sur le lieu géométrique d'une *podaire* du cercle. Soit C le centre et V un point fixe, extrémité de l'arc à trisecter. Prolongeons le rayon CV d'une longueur $VP = CV$; projetons le point P sur toutes les tangentes au cercle; T étant le point de contact et P' la projection correspondante de P, lorsqu'on aura $VP' = VT$, alors $\widehat{VP'P} = \frac{1}{3}\widehat{CVP'}$. (Poudra, 1856)

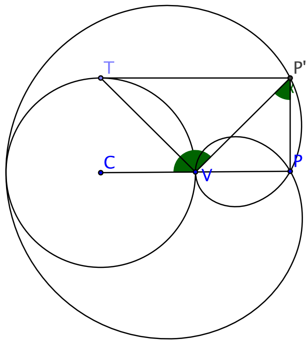


Figure 4: Trisection of angle

REFERENCES

- Aoust, L.-S., 1873, *Analyse infinitésimale des courbes planes...*, Paris: Gauthiers-Villars.
- Haton de la Goupillière, J.-N., 1866, "De la courbe qui est elle-même sa propre podaire", Journal de Mathématiques pures et appliquées S. 2 T. 11, 329-336
- Laguerre, E., 1875, "Sur les polaires d'une droite relativement aux courbes et aux surfaces algébriques", Bulletin de la S. M. F. 3, 174-181
- Lie, S., 1896, *Geometrie der Berührungstransformationen*, Leipzig: Druck und Verlag von B. G. Teubner.
- Loria, G., 1907, "Le Transformazioni pedali ed antipedali nel piano e nello spazio", Perio di mat XXII, 214-224
- Maclaurin, C., 1720, *Geometria Organica*, London: William & John Innys.
- Poudra, M., 1856, "Trisection de l'angle", Nouvelles annales de mathématiques S.1, t. 15, 381-383
- Quételet, A., 1825, "Mémoire sur une nouvelle manière de considérer les caustiques", *Nouveaux Mémoires de l'Académie Royale des Sciences et des Belles-Lettres de Bruxelles* 3, 89-158
- Roberval, G., 1693, *Divers ouvrages de M. de Roberval dont Observations sur la composition des mouvemens, et sur le moyen de trouver les touchantes des lignes courbes*, Paris.

- Schoenflies, A., 1893, *La géométrie du mouvement : exposé synthétique...*, Paris: Gauthiers-Villars
- Steiner, J., 1840, "Von dem Krümmungs-Schwerpuncte ebener Curven", Journal für die reine und angewandte Mathematik 21, 33-63, 101-133
- Sturm, R., 1873, "Über Fusspunkts-Curven und -Flächen, Normalen und Normalebenen", *Math. Ann* 6, 241-263