HISTORICAL METHODS FOR MULTIPLICATION

Bjorn SMESTAD
Oslo University College
Oslo, Norway
bjorn.smestad@hioa.no

Konstantinos NIKOLANTONAKIS
University of Western Macedonia
Florina, Greece
nikolantonakis@noesis.edu.gr

ABSTRACT

This paper summarizes the contents of our workshop. In this workshop, we presented and discussed the "Greek" multiplication, given by Eutokios of Ascalon in his commentary on The Measurement of a Circle. We discussed part of the text from the treatise of Eutokios. Our basic thesis is that we think that this historical method for multiplication is part of the algorithms friendly to the user (based on the ideas that the children use in their informal mental strategies). The important idea is that the place value of numbers is maintained and the students act with quantities and not with isolated symbols as it happens with the classic algorithm. This helps students to control their thought at every stage of calculation. We also discussed the Russian method and the method by the cross (basically the same as "Casting out nines") to control the execution of the operations.

1 Theoretical basis

During the University studies for the future teachers of mathematics of Primary (but also of Secondary) education it is very important to develop a multidimensional scientific-mathematical culture (Kaldrymidou & als, 1991). The dimensions to work on could be:

- The knowledge, which includes the global apprenticing of mathematical notions and theories, one approach of school mathematics and in parallel one pedagogical and psychological approach of mathematics education,
- The knowledge about the knowledge, the understanding of the role, of the dynamic and the nature of mathematics,
- The knowledge for the action, the theoretical context of the organization and the approach of school mathematics and via this the evaluation of the work in the classes,
- The action, the praxis and the experience of mathematics education.

The set of these dimensions determine the base, which can help and train the future teachers. For training but also for teaching Mathematics everything should be constructed. In the next pages we should try to make an approach for the following question: Which shall be the initial and the continued training of teachers of primary (but also of Secondary) Education for supporting the introduction of a cultural, historical and epistemological dimension for the teaching of mathematics?

The main methodological issues of our work with our students were based in the following thesis (Arcavi & als, 1982, 1987, 2000; Bruckheimer & al, 2000):

- Active participation (learning should be achieved by doing and communicating),
- Conceptual history (evolution of a concept, different mathematical traditions, difficulties, etc.),
- Relevance (with the curriculum),
- Primary sources and secondary sources using primary sources,
- Using of worksheets
- Implementation

The purposes of this introduction of the historical and cultural dimension in the training and teaching of mathematics are various and are fixed more or less in the following (Fauvel & van Maanen, 2000):

- to humanize mathematics,
- to put mathematical knowledge in the context of a culture,
- to give students the opportunity to change their beliefs for the subject,
- to find and analyze epistemological obstacles and notions that are not very well understood by the teachers and consequently by the students,
- to show that mathematics has a history and was influenced by cultural and social parameters,
- to be another interdisciplinary project that could be studied with the students,
- to develop and enrich mathematical knowledge included in the curriculum.

The ways and strategies used for this introduction in the training and teaching of the subject are: histories, construction of activities, construction of exercises, reproduction of manuscripts, portraits, biographies, interdisciplinary projects, use of primary sources, use of new technologies etc.

2 The “Greek” Multiplication

In the following we are going to discuss the way we have worked the multiplication in an historical perspective by discussing the “Greek” multiplication.

For Biographical elements and descriptions about the treatises of Eutokios you can see Nikolantonakis, K., (2009), History of Mathematics for Primary School Teachers’ Training, *Ganita Bharati*, Vol. 30, No 2, Page 181-194 but also Heath T. L., (1912). *History of Greek Mathematics*, Vol. I & II, Oxford.

To do the workshop you will need the following equivalences between the Greek alphabetical number system and our modern arithmetical system.

Units (1 to 9)	$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\epsilon}, \bar{\varsigma}, \bar{\zeta}, \bar{\eta}, \bar{\theta}$
Tens (10 to 90)	$\bar{\iota}, \bar{\kappa}, \bar{\lambda}, \bar{\mu}, \bar{\nu}, \bar{\xi}, \bar{o}, \bar{\pi}, \bar{ϙ}$
Hundreds (100 to 900)	$\bar{\rho}, \bar{\sigma}, \bar{\tau}, \bar{\upsilon}, \bar{\phi}, \bar{\chi}, \bar{\psi}, \bar{\omega}, \bar{ϡ}$
Thousands (1000 to 9000)	,α, ,β, ,γ, ,δ, ,ε, ,ς, ,ζ, ,η, ,θ

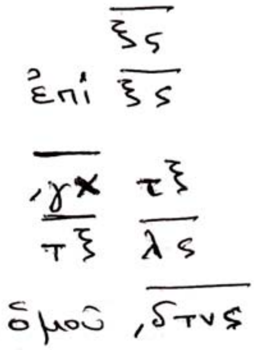
M = Myriad (10.000)

$\overset{\beta}{M}$ = 2 x 10.000 = 20.000, $\overset{\gamma}{M}$ = 3x10.000 = 30.000 etc.

L' = 1/2, δ' = 1/4 etc

We have given to the participants one by one the following operations and we asked them to transcribe them from the Greek alphabetical number system to our modern one. We have proposed them three examples, one with two-digit numbers, one with four digit-numbers and one with fractional numbers. Once they have done the transcriptions we asked them to explain us how Eutokios arrives to the result and which is the property behind his method. We have closed our presentation by making a comparison between this

algorithm and our modern one and by stressing the need for the teachers and afterward for the pupils to work on the Greek multiplication before attacking the modern one.
The first example is the multiplication 66 x 66



The above calculations could be seen with modern symbolism

	66
x	66
3600	360
360	36
Total	4 356

The mathematical analysis of the above mentioned calculations is:
(6 Tenths + 6 Units) (6 Tenths + 6 Units)=
36 Hundreds + 36 Tenths +
36 Tenths+ 36 Units = 4 356
The second example is the multiplication 1351 x 1351.

With modern symbolism we have:

1351

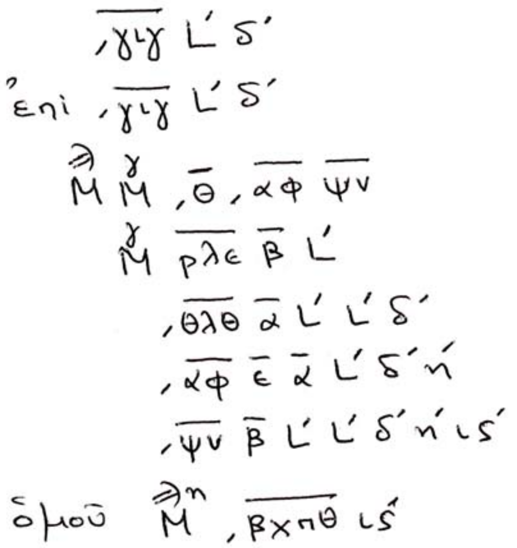
x 1351

1 000 000	300 000	50 000	1 000		
300 000	90 000	15 000	300		
	50 000	15 000	2 500	50	
		1 000	300	50	1

Total 1 825 201

The mathematical analysis of the above mentioned calculations is:
(1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit)(1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit) =
1 Million + 3 Hundred Thousands + 5 Myriads + 1 Thousand
3 Hundred Thousands + 9 Myriads + 15 Thousands + 3 Hundreds
5 Myriads + 15 Thousands + 25 Hundreds + 5 Tenths
1 Thousand + 3 Hundreds + 5 Tenths + 1 Unit = 1 825 201

The third example is with a fractional number.



With modern symbolism the multiplication is the following:

$3013\ \frac{1}{2}\,\frac{1}{4}$ [= 3013 ¾]

x $3013\ \frac{1}{2}\,\frac{1}{4}$

9 000 000	30 000	9 000	1 500	750
30 000	100	30	5	$2\frac{1}{2}$

9 000	30	9	$1\frac{1}{2}$	$\frac{1}{2}\ \frac{1}{4}$	
1 500	5	$1\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	
750	$2\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	1/16
Total	9 082 689 1/16				

The mathematical analysis of the above mentioned calculations is:

(3 Thousands + 1 Tenth + 3 Units $+\frac{1}{2}+\frac{1}{4}$)(3 Thousands + 1 Tenth + 3 Units + $\frac{1}{2}+\frac{1}{4}$)=

9 Hundreds Myriads + 30 Myriads + 9 Thousands + 15 Hundreds + 75 Tenths +

3 Myriads + 1 Hundred + 3 Tenths + 5 Units + 2 Units + $\frac{1}{2}$ +

9 Thousands + 3 Tenths + 9 Units + 1 Unit + $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}$

15 Hundreds + 5 Units + 1 Unit $+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

75 Tenths + 2 Units + $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$+1/16

By the use of the Greek multiplication we can give explanations to the typical algorithm.

Greek Multiplication	Modern Algorithm
25	25
x17	x17
35	175
140	25
50	425
200	
425	

The goal is that pupils can understand the way of production of the partial products and the place-value of numerals of the factors of multiplication.

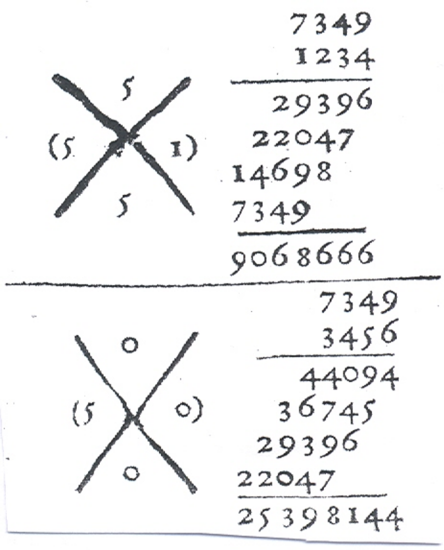
2 Multiplication in an old Norwegian textbook

The first textbook in what we today call mathematics in Norwegian was Tyge Hansøn's *Arithmetica Danica* from 1645. Geir Botten has recently written a book about it, based on the sole known copy. *Arithmetica Danica* shows how to use the numeral system, how to add, subtract, multiply, divide, use regula de tri, regula falsi, calculate square roots and do lots of "practical" calculations. It is believed that Hansøn partly based his book on earlier books in Nordic languages, although these connections have not been investigated. (Botten, 2009)

We are going to look at how multiplication is presented in this book, but first I'll point to some other aspects of the book that Botten finds interesting. I would love comments on them:

- The use of poems throughout. Example: "O Ungdom haff din Tid I act/ For Leddigang du dig vel vact/ Viltu I Regenkunst bestaa/ Ei nogn dag forgæffs lad gaa." ("O Youth, take notice of your time/ For idleness you must avoid / If you want to learn the art of calculation/ No day must pass in vain.") Also, some of the tasks are written as poems.
- The use of particular numbers with connection to monarchs etc: "I want to calculate the root of 2 486 929: On the basis of that I will learn in which year His Majesty my most gracious Lord and King Christianus 4th is born." (1577)
- Unrealistic answers. One task asks the age of a man, and the answer is 120.
- A special, explicit concern for female readers. Four pages are devoted to that, with topics such as buying fabrics, weaving etc.
- The prominence of alcohol in some exercises: "A man earns 15 shillings a day at the harbour when he is working, and drinks 9 when he is not. As the year passed, everything was spent drinking and he also owed 7 marks 8 shillings. How many days had he worked and how many had he not. (Facit 112 days worked, 200 days held sacred") (16 shilling = 1 mark)

Arithmetica Danica features this calculation:



Understanding the calculation is straightforward for modern students. But what are the crosses to the left? This was unknown notation for me when I first saw it, although it was not too hard to figure out. To my surprise, some Greek teacher students knew this when I asked them last year, as it is apparently still mentioned in some Greek classrooms.

The idea is basically the same as "casting out nines". We find the *repeated digit sum* of the first factor and put it to the left. Then we find the repeated digit sum of the other factor and put it to the right. The product of the digit sums we put on top. This should be equal to the digit sum of the product, and we put this below.

It should be noted that the term "casting out nines" suggests a process where we throw away nines as we go along, while "repeated digit sum" suggests that we calculate the full digit sum at first. In the workshop, we noted that both ways of doing it is still represented. We have no way of knowing what process Hansøn utilized.

Of course, the correctness of the method can be proved by use of *modular arithmetic***.** We could show this to students by looking at

$(a+9n)\cdot(b+9m) = ab + 9bn + 9am + 81nm = ab + 9(bn + am + 9nm) = ab + 9l.$

It can also be used for addition and subtractions, and even divisions.

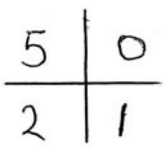
The method is not infallible – the main problem is that it is unable to spot a simple switching of two digits, for instance if you are calculating (successfully) that 9x4 = 36, and then are thinking "we put 3 down and carry the 6" instead of the other way around. There are also other common mistakes it does not detect, for instance in addition:

$$\begin{array}{rl} & 34 \\ + & 27 \\ = & 511 \end{array}$$

The history of "Casting out nines" is a bit uncertain. Florian Cajori (Cajori, 1991, p. 91) claims that it was known to the Roman bishop Hippolytos as early as the third century, but although the process of repeated digit sum (*pythmenes*) was known at that time (Iamblichus, 4^{th} c AD) (Heath, 1981, pp. 113-117), there seems to be consensus that the test was established in India or the Arab world. Avicenna (978-1036) supposedly referred to it as "the method of the Hindus". (Swetz & Smith, 1987, p. 189) and it is said to have appeared in the *Mahasiddhanta* of Aryabhata (II) (probably 10^{th} century). I have not checked this. However, it is simple to establish that it was included in Liber abaci (Fibonacci & Sigler, 2002) and in Maximus Planudes (1255-1305)'s *The Great Calculation According to the Indians* in late 1200s (Brown, 2006). Later, it was included in the Treviso Arithmetic in 1478 (Swetz & Smith, 1987).

In Liber Abaci, the problems of using 9 was discussed, and 7 and 11 were also used. Using 11 is almost as simple, and is slightly better.

The particular way of writing the check was used in several medieval schools (Flegg, 2002; Tattersall, 2005). Tattersall mentions that it was called the "cross bones check". This phrase is almost unknown by Google, for instance, so it doesn't seem to be widely known today. In the Treviso Arithmetic, this notation was used for division:



(Swetz & Smith, 1987, p. 87)[1]

The division in question is 7624:2 = 3812 rest 0. "If you wish to prove this by the best proof, multiply the quotient by the divisor, and if the result is the number divided the work is correct. // If you wish to prove it by casting out 9s, put the excess in the divisor, which is 2, in a little cross, underneath the left; then put the excess in the quotient, which is 5, above this 2; then place the excess of the remainder, which is 0, after the 5 on the other side. Then do as follows: multiply the excess of the divisor by that of the quotient, 2 times 5 making 10; add the 0 remainder, leaving 10; cancel the 0, cleaving [sic!] 1 for the

1 If anyone knows more about different notations used, we would be interested.

principal excess, and write this in the cross under the excess of the remainder. Then see if the excess of the number divided also equals 1, in which case the result is correct.”

”Casting out nines” (”nierprøven”) was included in Norwegian textbooks at least until 1985 (Viken, Karlsen, & Seeberg, 1985, p. 45). Here, only the method for multiplication was shown, and there was no proof.

Why is it no longer included in textbooks in Norway? Maybe because even this proof was considered too complex, and that a method without justification is unwanted. Moreover, there is anecdotal evidence that even teachers didn’t understand that the method didn’t find all errors. (“I received a note from an elementary teacher who asked why the method had been objected to in the texts above if it was a check that was taught in schools today. She was not aware that sometimes the method would confirm a false result. In particular if a digit reversal occurs in the answer, the method of casting out nines will not catch the error.” (Ballew, 2010))

3 Egyptian-Russian method

In Norwegian textbooks for teacher education, you can find the following algorithm, called “Russian Peasant Multiplication”:

Halfs	**Doubles**
49	183
~~24~~	~~366~~
~~12~~	~~732~~
~~6~~	~~1464~~
3	2928
1	5856
	8967

The students need a little time and more than one example just to realize how the algorithm works. They need to see that what we do is to halve the numbers in the left column while doubling the numbers in the right column (all the time leaving any fractions out) until we get to 1 in the left column. Then we cross out the lines having an even number in the left column, and find the product we wanted by adding the numbers that are not crossed out in the right column.

After that, they need quite a bit of time and help to understand *why* it works. We have found it helpful to let the students work on 16x23 and 17x23 as steps towards finding a general explanation. This makes it possible for them to see that while 16x23 = 8x46 = 4x92 = 2x184 = 1 x 368 = 368 is simply a matter of halving the one factor and doubling the other factor all the time, with 17x23 you get an additional 23 that you have to keep in mind while you go on with 8x46. Of course, such numbers that have to be “kept in mind” turn up whenever there is an odd number in the left column.

Of course, a similar – but not fully equal – way of doing it is the much older Egyptian one (as seen in the Ahmes Papyrus (aka Rhind Papyrus)):

X	1	183
	2	366
	4	732
	8	1464
X	16	2928
X	32	5256
	49	8967

The method also works the other way around. What is 8967:183?

X	1	X	183
	2		366
	4		732
	8		1464
X	16	X	2928
X	32	X	5256
	49		8967

It is exactly the same numbers, but the thought process is different.

What about 8970 : 183? We would end up with a remainder of 3, and the answer would be 49 3/183 or 49 1/61. (Here, of course, the Egyptians would only use unit fractions, while we could use any fraction)

4 Discussion

What is the use of the different methods of multiplication in teacher training? In the Department of Primary Education of the University of Western Macedonia in Greece we propose an optional course in the contents of which we are discussing the above mentioned methods but also many others and we are trying to give to the students – future primary school teachers – many alternative historical origin ways to have a deeper understanding of the modern multiplication algorithm. We would like also to stress the fact that the above "Greek multiplication" is part of the curriculum proposed to the third class of the Greek primary school (pupils of 9 years old) before the typical modern algorithm as a preparatory stage. In Norway, "Greek multiplication" is not a standard part of the textbooks, but it is discussed in teacher education.

For teachers, it is important to be able to understand different algorithms (for instance their students' own efforts) and looking at historical algorithms can be a good way. This can be seen in connection with what Ball and others write about Special Content Knowledge for teaching (Loewenberg Ball, Thames, & Phelps, 2008). This entails an enrichment of their knowledge of multiplication.

For teachers, it is also important to see that the multiplication table is not necessary to do multiplications. These methods also show that history of mathematics has developed.

We hope that teachers will also become curious as to why ”our” algorithm works and why it has prevailed in school and society.

REFERENCES

- Arcavi, A., Bruckeheimer, M., Ben-zvi, R., 1982, “Maybe a mathematics teacher can profit from the study of mathematics”, *For the Learning of mathematics,* n° 3.1, 30-37.
- Arcavi, A., Bruckheimer, M. Ben-zvi, R., 1987, “History of mathematics for teachers: the case of irrational numbers”, *For the Learning of mathematics*, n° 7.2, 18-33.
- Arcavi, A., Bruckheimer, M., 2000, “Didactical uses of primary sources from the history of mathematics”, *Themes in Education*, **1** (1), 55-74.
- Ballew, P., 2010, “Casting out nines”, retrieved on 14/7/2010, from http://www.pballew.net/arithm17.html#castnine
- Botten, G.., 2009, *Min lidle norske regnebog: noen dypdykk i ei lærebok i matematikk fra 1645*. Oslo: Universitetsforl.
- Brown, P. G.., 2006, “The Great Calculation According to the Indians, of Maximus Planudes”, *Convergence,* **3**.
- Bruckheimer, M., Arcavi, A., 2000, “Mathematics and its history: An educational partnership”, in Katz, V., (ed.), *Using history to teach mathematics: an international perspective,* Washington DC: The Mathematical Association of America pp.135-146.
- Cajori, F., 1991, *A history of mathematics*. ,ew York, N.Y.: Chelsea Publ. Co.
- Fauvel, J.G., van Maanen, J., 2000, *History in mathematics Education: the ICMI study.* Dordrecht: Kluwer Academic Publishers.
- Fibonacci, L. P., & Sigler, L. E., 2002, *Fibonacci's Liber abaci: a translation into modern English of Leonardo Pisano's Book of calculation*, New York: Springer.
- Flegg, G.,.(2002, *Numbers: their history and meaning*, Mineola, New York: Dover.
- Heath T. L., 1912, *History of Greek Mathematics*, Vol. I & II, Oxford.
- Heath, T. L., 1981, *A history of Greek mathematics*. New York: Dover Publications.
- Kaldrymidou, M., Kastanis, N., Oikonomou, A., 1991, “Education and Professional Training of Mathematics teachers”, *Diastasi, 1-2*, pp.18-28. [in Greek]
- Loewenberg Ball, D., Thames, M. H.,, Phelps, G.. 2008, Content Knowledge for Teaching. *Journal of Teacher Education*, 389-407.
- Nikolantonakis, K., 2009, “History of Mathematics for Primary School Teachers’ Training”, *Ganita Bharati*, Vol. **30**, No 2, pp.181-194.
- Swetz, F., Smith, D. E., 1987, *Capitalism and arithmetic: the new math of the 15th century, including the full text of the Treviso arithmetic of 1478, translated by David Eugene Smith*.,La Salle, Ill.: Open Court.
- Tattersall, J. J. 2005, *Elementary number theory in nine chapters*.,Cambridge [u.a.]: Cambridge Univ. Press.
- Viken, E., Karlsen, H., Seeberg, T. C., 1985, *Formel 7: årsbok*. [Oslo]: Tano.