THE TEACHING OF MATHEMATICS MEDIATED BY THE HISTORY OF MATHEMATICS

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ABSTRACT

The aim of this paper is to present issues related to a teaching exploitation methodology based on the History of Mathematics. These issues are related to the apprehension of mathematical concepts involved in a teaching sequence designed from the History of Mathematics. We explored the teaching sequence with in-service teachers and future mathematics teachers. They were challenged to find a general formula for quadratic equation solution using the related sequence. Upon short courses completion we verified participants' responsiveness and availability to work with such sequence. We found difficulties in associating an irrational number with a measure of one square side and also difficulties in mathematical problem proposing. We confirmed the need to complement the referred sequence with extra reading material, specific recommendations, as well as proposing tasks to support teachers in their work with the sequence in their own classrooms. This study is part of an ongoing work and aims to bring contributions to the debate about the effectiveness of the History of Mathematics in Mathematics Education.

1 Introduction

In related studies on the History of Mathematics and Mathematics Education there is a variety of approaches. According to what was observed by Siu (2007), most of these studies discuss the role and the importance of the History of Mathematics in teaching and learning process. However, few studies discuss that the History of Mathematics contributes, in fact, for mathematics teaching. This instigated us to investigate the History of Mathematics effectiveness in Mathematics teaching. In this study the term effectiveness refers to our intention to clarify if the history of mathematics is a good ally (or not) to mathematics teachers in the performance of his professional actions.

Some researchers (Kjeldsen, 2010; Kjeldsen and Blomhøj, 2009) have published articles that show ways to integrate the History of Mathematics in Mathematics Education in undergraduate courses. These articles present different integration aspects and show how it leads to an improvement in Mathematics learning. Our study, however, focuses on Mathematics teaching.

To innovate education, many teachers are looking for a special support. This is the case of introducing History of Mathematics in teaching process, as shown in the article written by Siu (2007). Based on obtained data from 360 mathematics teachers in 41 schools, it shows that teachers consider important to use the History of Mathematics in classroom. However, these teachers say they do not use this feature.

According to the data obtained by Siu (2007), from 608 issued responses by in-service math teachers and future teachers, there are at least 15 reasons for this. Among these reasons, some are stronger (and other less), justified by the respondents. In this article we focus on only two of them:

1. The lack of material resources, mentioned by 64.47% of respondents.
2. The lack of teacher's preparation, mentioned by 82.89% of respondents.

Our many years experience, working with in-service and future math teachers show that obtained data by Siu (2007) correspond to our country reality. We believe that the arguments used to justify non-using the History of Mathematics is presented as a problem that requires investigation.

These data reinforce the importance of continuing our study whose aim is to bring contributions to the debate about the effectiveness of the History of Mathematics in Mathematics Education. Our study is divided into five parts:

1. To investigate teacher's responsiveness and availability to deal with a *method based on history*;
2. To provide historical documentation for in-service teachers use in the classroom;
3. To predispose future teachers to use historical materials in the classroom;
4. To observe the use of historical materials in the classroom;
5. To verify the History of Mathematics efficacy in Mathematics Education.

The term *method based on history* is here used in the perspective pointed by Jankvist (2009). The *method* refers to a teaching approach inspired on the History of Mathematics.

The present article discusses the responsiveness and availability of the participants regarding to the use of the referred *method* which approach the History of Mathematics in an indirect way. Here we present some results that led us to a first reflection on the effectiveness of the History of Mathematics in Mathematics Education.

We have taken as reference a prepared material by Radford and Guérette (2000), whose title is "*Second Degree Equations in classroom: the Babylonian approach*". The authors develop and utilize a teaching sequence based on Babylon mathematics, whose purpose is to lead students to reinvent the formula that solves a general quadratic equation.

Analyzing the referred teaching sequence we asked ourselves: the in service teachers and future teachers are prepared to use this sequence in their classrooms or we need to modify it somehow? To answer this question, we came up to short courses specially designed for these teachers. In these courses we challenged participants to find the formula for quadratic equation solution by using the sequence. Our initial goal was to assess the responsiveness of the participants and their willingness to work with such sequence in the classroom.

2 Concerns about using the History of Mathematics

We considered relevant to refer to the theoretical approach that involves the discussion of *whys* and *hows* to use the History of Mathematics in teaching and learning processes. We also considered relevant to refer to the arguments that support this type of the History of Mathematics use in Mathematics Education.

A study developed by Jankvist (2007) highlights the importance of conducting empirical research involving the History of Mathematics, so that positive experiences reached by a teacher, a classroom or a school can be transferred and shared. Considering this, it is possible to be clear about the arguments that support such History of Mathematics use.

In another study Jankvist (2009) proposes a way to organize and structure the discussion of *why* and *how* to use the History of Mathematics in mathematics teaching and learning. To organize and structure this discussion he develops a deep and wide research, through which identifies two categories: the arguments for the use of the History of Mathematics (the *whys*) and the different ways to make use of history (the *hows*).

The first category (the *whys*) is subdivided into two forms: history as a *tool*, which involves mathematics internal learning issues and history as a *goal*, which involves history use as a purpose contained within it.

The second category (the *hows*) is subdivided into three different ways to make use of history: the *illumination* approaches, based on motivational and affective arguments; the *modules* approaches, which builds on cognitive arguments; and the *history-based* approaches to makes reference to arguments of their own evolutionary History of Mathematics.

Besides that, Jankvist (2009) states that interrelationships knowledge between the *whys* and *hows* may become easier to analyze education material that makes use of the History of Mathematics, to see if it meets certain requirements and goals.

Longing for a theoretical and general exposure Jankvist (2009) explores a central point in his analysis: he separates strictly, the categorizations of the *whys* and *hows*. This is due to the fact that by making a literature review this author finds an image "opaque" in *whys* and *hows* of discussion. Given that he seeks such categorization.

It is important to bear in mind that the categorizations of the *whys* and *hows* are not so absolute. They serve the following purposes: create conditions for a real analysis of the History of Mathematics in Mathematics Education.

Taking as reference this theoretical approach, we observed that the knowledge of the interrelationships among the *whys* and *hows* can be used in decision making about the content, presentation and organization form of the History of Mathematics uses in Mathematics Education, as well as the materials to be used by both teachers and students.

Ahead of this, we decided to start our own investigation with reference to this theoretical approach and teaching sequence, inspired by the History of Mathematics prepared by Radford and Guérette (2000). The choice was a function of its form and organization of the History of Mathematics and materials used by these authors.

Although this sequence was originally used by its authors, in basic education students, it did not present as an obstacle to our purposes. We worked with in-service teachers and future mathematics teachers, seeking to clarify issues related to their responsiveness and availability to work with a teaching methodology based on the History of Mathematics.

Below we present the result of the teaching sequence that came out to meet the requirements and purposes of our study.

3 The teaching sequence

Regarding the integration of history in mathematics teaching, there are studies indicating that the sequences inspired by the History of Mathematics can be used to teach some math concepts. This is the basic idea behind the article "*Second Degree Equations in classroom: the Babylonian approach*" Radford and Guérette (2000). These authors developed a teaching sequence aimed to lead students to reinvent the formula that solves a general quadratic equation.

The sequence takes as a reference a problems solving method that uses geometrical figures (represented on paper) that can be cut, moved and pasted. We call this *cut-and-paste geometry* method. Jens Høyrup (1990a, 1990b) was the first to suggest that Babylonians scribes devised this method to solve geometric problems and called it *naive geometry*.

It is not our intention to discuss here the method itself, since it is adequately addressed in Radford and Guérette paper (2000). We are interested in presenting a summary of the teaching sequence. Here, the sequence will be presented in five parts, as it appears in the original article.

Part 1. Naive geometry introduction

The goal of the first part of the sequence is to present *naive geometry* to students. To achieve this goal, students will work in cooperative groups to complete three tasks, which are discussed below by steps:

Step 1.1

The teacher asks the students to solve the problem below using any chosen method they want.

What are the dimensions of a rectangle whose semi-perimeter is 20 and whose area is 96 square units? (Problem 1A).

Students are encouraged to solve the problem using any method. After completing the task the teacher returns to geometric context and uses geometric figures represented on paper, placed on the blackboard. Doing this, he shows students the technique of the *naive geometry*.

Step 1.2

Once presented the technique, the teacher explores similar problems. For example, the following problem could be studied:

What are the dimensions of a rectangle whose semi-perimeter is 12 and whose area is 30 square units? (Problem 1B).

Step 1.3

To help students achieve a better understanding, the teacher asks them to bring a description (written for next day class) of the steps that must be followed in solving these problems. The description should be clear enough to be understood by any student from any other class, of the same year.

Part 2. Written text discussion and new problems formulation

Step 2.1

This part begins with a discussion of the requested descriptions. Working in cooperative groups, students should compare the descriptions given by them and reach an agreement on the points that could cause conflict or that could allow a better understanding. When all group members reach an agreement, the teacher chooses one student from each group to present their findings to the other groups.

Step 2.2

Next, students are encouraged to develop problems considering the following restriction: the both sides of the rectangle should be expressed by integers. Then, as a second exercise, the rectangle sides not necessarily need to be expressed by integers. Thus students are challenged to find answers containing rational numbers.

Part 3. A different use for *naive geometry*

Step 3.1

In Part 3 is presented to the students a problem that requires another *naive geometry* approach. The problem is the following:

> *The rectangle length is 10 units. Its width is unknown. We place a square on one side of the rectangle, as shown below. Together, the two figures have an area of 39 square units. What is the given rectangle width? (Problem 2).*

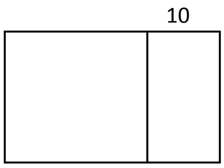

FIGURE 1

The teacher asks students to solve the problem 2 with similar ideas to those used to solve problem 1. If the students fail, the teacher can introduce a new solving problems method. Here the procedure used was "*complete the square*" which is detailed presented in Radford and Guérette (2000).

Briefly the procedure is as follows: using large cardboard figures, placed on the blackboard, the teacher cuts vertically the given rectangle (length 10, see Figure 1) by dividing it into two equal parts, then grabs one of the pieces and places it on the square base, represented in Figure 1. Students should realize that the geometric resulting shape is almost a square. Then the teacher points out that this geometric form can be completed in order to become a new square.

Step 3.2

Then, similar problems are presented to the students to be solved in group.

Step 3.3

As in Part 1, students are asked to prepare a written description containing the steps that must be followed in solving this kind of problem.

Part 4. Discussing procedures and proposing new problems

Step 4.1

As in Part 2, we start a discussion about the written descriptions prepared by the students that contains all necessary steps to solve problems explored in Part 3.

Step 4.2

Then the teacher asks them to propose some problems (analogous to problem 2) involving a specific condition to the rectangle sides:

(i) the rectangle sides must be expressed in integers numbers,
(ii) the rectangle sides must be expressed in rational numbers,
(iii) the rectangle sides must be expressed in irrational numbers.

Part 5. Finding the formula

In this part students think about problems similar to those discussed in Parts 3 and 4. However, it is not assigned a specific number for the rectangle base nor to the area of the two given figures. The aim is to encourage students to discover the formula that solves quadratic equations.

Step 5.1

The teacher explains to the students that the interest now is to find a formula that provides an answer to the problems explored in Parts 3 and 4. The teacher can suggest that they take as reference the written procedure in part 4 and that they use letters instead of words. To facilitate the formulas comparison developed by students, in a next phase, the teacher may suggest the use of the letter "***b***" to the rectangle's base and "***c***" to square and rectangle's area together (see Figure 2):

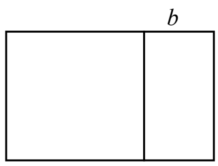

FIGURE 2

The obtained expressions are discussed in cooperative groups. The formula obtained is:

$$x = \sqrt{c + \left(\frac{b}{2}\right)^2} - \frac{b}{2}$$

Step 5.2

In this step the teacher can use procedures to implement geometrical problems to the algebraic language: if the side of the square is 'x', then its area is equal to x^2 and the rectangle area is bx, so the sum of both areas is equal 'c', then $x^2 + bx = c$. Now, in order to relate the equations with the formula, the teacher presents some real equations (such as $x^2 + 8x = 9$, $x^2 + 15x = 75$) and ask students to solve them using the obtained formula.

Step 5.3

In this step we consider the equation $ax^2 + bx = c$ and asks the students to find a formula to solve this equation. They should note that if this equation is divided by a (let's assume $a \neq 0$) we obtain the first term of the previous equation. You must then replace the 'b' for 'b/a' and '**c**' for 'c/a', in the previous formula, coming to the new formula:

$$x = \sqrt{\frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}.$$

Step 5.4

The last step is to consider the general equation $ax^2 + bx + c = 0$ and find the formula that solves it. The formal relationship with the previous equation $ax^2 + bx = c$ is clear: we can rewrite this equation as $ax^2 + bx - c = 0$. Thus, to obtain the equation $ax^2 + bx + c = 0$ we must substitute '***c***' for '**-** ***c***' and do the same in the formula. By doing this we get:

$$x = \sqrt{\frac{-c}{a} + \left(\frac{b}{2a}\right)^2} - 2a$$

Naturally, this formula is equivalent as to the well-known formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

For all numerical solutions we must consider the negative square root of $b^2 - 4ac$. This leads to the general formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4 Working with math teachers

In search of evidence of responsiveness and availability from teachers in relation to the uses of a teaching sequence, based on the History of Mathematics, we offered short courses. The courses content took as reference the sequence developed by Radford and Guérette (2000). Besides that, we offered participants some basic historical information.

The first course was designed for in-service teachers and future teachers, all together. After that, courses were offered for in-service teachers and future teachers, separately, in order to observe any differences that might exist between the two groups. Each course lasted for a different period of time due to limitations in schedules of the participants. The courses were developed in periods of 4, 5 or 6 hours.

In all the courses we've attended in the classroom. During the course we observe the participants performance in attempt to overcome the challenges they were proposed to. Also took note on the teaching sequence steps that brought difficulties to the participants.

At the end of the course we asked them to answer several questions in order to consider how the participants assess the possibility of using the sequence in the classroom. The questions were:

a) Is it feasible to use this result at school?
b) Do you see yourself applying this sequence at school?
c) At what moment do you think the students would face difficulties?
d) What sequence part(s) would be more difficult for students?
e) It would require some adjustment? Which ones?
f) The History of Mathematics should or should not, appear explicitly when working with the teaching sequence?

The table below shows the offered courses. It makes focus on the date, duration and number of participants in each course:

Course	Date / Duration	Participants
1	October 2009 4 classes of 90 minuts duration	14 in-service mathematics teachers 23 future mathematics teacher
2	February 2010 2 classes of 3 hours duration	20 in-service mathematics teachers
3	April 2010 3 classes of 100 minuts duration	15 future mathematics teacher
4	April 2010 3 classes of 100 minuts duration	14 future mathematics teacher
5	April 2010 2 classes of 2 hours duration	4 in-service mathematics teacher in middle school, at the same time, post-graduation students
6	May 2010 2 classes of 2 hours and 30 minuts duration	14 in-service mathematics teachers

TABLE 1

5 Some results discussion

The collected data indicate the difficulties that the participants had to perform the proposed tasks and are related to how they judge the teaching sequence itself. We will highlight, in the next subsection, the main difficulties faced by teachers.

5.1 Difficulties in understanding the sequence during the course

Firstly we observed that the difficulties faced by in-service and future mathematics teachers are similar. The problems differ from one course to another only in intensity; therefore they are of the same nature. Below we discuss the most notable difficulties.

On the first part of the sequence the difficulties appeared in step 1.2. In seeking problem 1B solution, using the *naive geometry*, you must have a square whose side measure is equal to six and cut from this figure, another square whose side is equal to $\sqrt{6}$. In all courses the participants had some difficulties in obtaining the solution. In some cases, the researchers had to intervene by providing appropriate questions in order to instigate the solution searching. However, there were cases where this difficulty seems to have been more problematic than previously thought. One of the researchers, after provide the number 3 and 4 courses, wrote in his journal field:

> *"Most participants sought an integer to the square side. When they realized that they could start from a square with side equal to 6 and from this square they could remove a small square of area equal 6, they found out that the side of the square should be equal to*$\sqrt{6}$ *. However, some of them did not accept this as a truth value. Even after verifying that this result was consistent with problem data, some students did not accept this value to one square side, because they thought the square side should be linked only to an integer. At the beginning of activity at any of the groups, only one component tried to solve the problem using* $\sqrt{6}$*, but when*

encountered resistance from other group members, he stopped working with that number.

Another moment that special attention was demanded from the researchers was linked to step 2.2. At this time we asked the participants to prepare two problems. First: the rectangle sides should be expressed by integers. Second: the rectangle sides should *not* necessarily be expressed by integers.

They were instructed to write the problems on a paper sheet and hand over to the investigators. Upon receipt the problems the researchers mixed the sheets and distributed to the participants, so each group received the developed problem by another group. Then each group had to correct the statement (if necessary) and solve the proposed problem.

In accomplishing this task, it became clear that to propose a mathematical problem of this nature is not a simple task. We observed that many of the produced problems could not be solved without a proper correction in the statement. Furthermore, we observed that only some of the participants proposed problems whose answer would be an irrational number.

Another difficulty could be observed in the development of steps 5.2 and 5.3. In these steps the participants had to translate a problem of geometric field to algebraic field and solve it. Although it was possible to make use of geometric figures, we found that most participants had difficulty to perform that transposition. In all courses, the participants could only solve the proposed problems with the researchers support. It was observed that join the side of a geometric figure to a generic value (a symbol) is not immediate. Furthermore, the detachment of the geometric representation of a concept and its algebraic representation is not something that happens naturally, which requires special care by the teacher.

5.2 Teaching sequence assessment

In general, participants considered important to work with the teaching sequence. They emphasize that problems solution through geometry cut-and-past is important because it would allow students a better understanding of certain mathematical concepts. However, in-service teachers added that, for them to use this new approach in schools, it would be necessary to make some changes and overcome some difficulties, such as:

a) *To work with cut-and-paste geometry n schools would be better to work with, scissors, paper or cardboard instead of animated slides as used in these courses;*
b) *before applying the sequence at school, they need to restore, in more detail, some basic concepts, to gain a deeper understanding of the most important points;*
c) *they must obtain school permission to use a new approach;*
d) *they must find a way to overcome students resistance, who consider such activities as not appropriate for a math class;*
e) *they have to find extra time to prepare lessons to use the new method;*
f) *they must find an extra reading material to help preparing the lessons;*
g) *they must find a way to overcome the student algebraic weaknesses.*

In relation to "c item" above, was observed that this is a relevant issue for private schools teachers. There is evidence that these teachers have little autonomy to innovate mathematics teaching. However, it seems that teachers who work in public schools have more autonomy to do so.

Another issue that we considered important is the participant's opinion on the History of Mathematics presentation in mathematics classes. Most participants felt that the Babylonian mathematics history should be fully displayed. However, we tried to make clear that teaching sequence was inspired by the history of Babylonian mathematics, but the uses of the sequence does not, necessarily, includes a full resumption of the Babylonian mathematics history.

With intent to observe the effectiveness of the History of Mathematics in Mathematics Education, we changed the order of the provided content for the courses. In course 1, we started with a discussion of Babylonian mathematics. After that, we explored the teaching sequence. In courses 2-6 we started with the sequence implementation and only introduced the discussion of Babylonian mathematics at the end.

At the end of each course we required an appraisal of the teachers on the following question: "The History of Mathematics should or should not appear explicitly in teaching sequence work?”. In the debate on this issue was clear that the use of the teaching sequence does not necessarily presuppose the introduction of historical issues. However, they insisted that the introduction of historical themes would be relevant to capture and maintain students' interest in math classes.

6 Implications in this investigation

Firstly we would like to emphasize that the future mathematics teachers, as well as in-service teachers, are our main partners in this research. So to continue our research project, we must take into account the views, wishes and recommendations by them expressed.

So our next step will be "*rewriting*" the teaching sequence, integrating the recommendations resulting from our previous work. We do not intend to modify the original sequence. However, those points, that brought difficulties to the participants, we intend to supplement with extra reading material, propose specific recommendations or tasks with the aim of supporting teachers in working with the sequence in their own classrooms.

For example, we intend to add to the original sequence, a text about the history of Babylonia mathematics, to be written especially for teachers. Our experience indicates that many teachers have not had the opportunity to attend any courses on the History of Mathematics during graduation. So if they are willing to explore the History of Mathematics related to this particular topic in their classrooms, we must provide them with appropriate reading materials.

Once concluded these adjustments we will come back to our work within-service mathematics teachers. We will invite them to take the course again. After that, we will ask them to apply the same sequence in their classrooms and evaluate the obtained data with us. Thus, we expect to have more elements to examine the question of effective use of the History of Mathematics in a classroom.

Moreover, considering theoretical reference, we note that in this study, participants conceive using history as an *illumination* approach, which relies on motivational and emotional arguments. This conception is linked to the *hows* category of using history. Thus, in reciprocity with what was proposed by Jankvist (2009) we believe that a discussion on history use can be beneficial for creating a new platform through which becomes possible the discussion of the History of Mathematics potential in Mathematics Education.

To these authors, the interest in the History of Mathematics use on teaching and learning should be based on a systematic and organized theoretical framework that takes the following questions into consideration:

- ✓ *Why can / should history be used for mathematics teaching and learning?*
- ✓ *How can and should history be used for mathematics teaching and learning?*
- ✓ *In what ways the arguments for history use of and the used approaches (different ways) are inter-related?*

With that in mind, on our study continuity, we intend to make a contribution in order to elucidate the following question: *is there any empirical evidence that teachers teach best when they make use of the History of Mathematics in the classroom*? We believe that this contribution may bring more elements to the debate about the effectiveness of the History of Mathematics in Mathematics Education.

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