AN IMPLEMENTATION OF TWO HISTORICAL TEACHING MODULES

Outcomes and perspectives

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ABSTRACT

Distinguishing between two different uses of history of mathematics in mathematics education (history as a tool to enhance the inner issues – in-issues – of mathematical topics, concepts, methods, etc.; and history as a sort of goal, in the sense that it is considered a goal to teach students something about the historical development of mathematics, its influence on society and culture as well as the other way round, and the fact that mathematics is a human endeavour and the development of mathematics therefore likewise – that is to say some of the more meta-perspective issues – meta-issues – of mathematics) the present presentation concerns an empirical investigation of the use of history as a goal. More precisely two longer historical teaching modules were designed and implemented in a Danish upper secondary mathematics class in order to answer the following research questions:

1. In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of 'history as a goal'?
2. In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?
3. In what way may teaching modules focusing on the use of 'history as a goal' give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?

In the presentation I shall mainly focus on the two first questions, displaying data from the implementation of the teaching modules to support possible answers to these. This involves also, for example, a discussion of certain design principles for using history in mathematics education. The third question shall only be touched upon briefly, outlining the main results.

The above mentioned research will be related to the body of empirical research already available in the field of using history in mathematics education, thereby also discussing the status of this field from an empirical point of view as well as the role and perspectives for future empirical research within the field.

I am honored to have been invited to give this opening plenary lecture at the 6th European Summer University in Vienna and I thank the organizers and the scientific program committee for this opportunity.

Three years ago I attended my first ESU, in Prague, and there I gave a short presentation which resulted in an even smaller abstract in the proceedings. Today I plan to 'expand' a little on that abstract. More precisely, I shall talk about the research and outcomes of my doctoral thesis (Jankvist, 2009d), which I handed in and defended last year, and related publications. At the end of the talk I shall present some of my own views in terms of perspectives for further research on history in mathematics education, in particular regarding the empirical dimension of this.

I shall structure my talk into four parts: first, an introduction; second, a discussion of students' meta-issue discussions and their anchoring; third, a brief discussion of students' beliefs about mathematics as a discipline; and finally, some perspectives for future research.

1 Introduction

When talking about history in mathematics education, we may generally distinguish between two different kinds of arguments for or purposes with using (or integrating) history in the teaching and learning of mathematics. I have previously referred to these as history as a tool and history as a goal (Jankvist, 2009a), and I shall do that here as well. When using history as a tool, history acts as an assisting means, or aid, for the teaching and learning of mathematics. We may distinguish between different roles that history may play when acting as a tool: it may be used as motivating or affective tool; it may be used as a cognitive tool; a kind of pedagogical tool; or as an evolutionary one, the latter referring to the arguments for using history that say that the learning trajectory of the students should more or less follow that of the historical development of some topic. Common for all of these, however, is the intention that students should come away having learned something about a selection of inner issues – in-issues – of mathematics, e.g. mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof, etc. When using history as a goal the in-issues are not necessarily of first priority, instead the meta-perspective issues – or meta-issues – are. The term 'goal' must not be misinterpreted in the way that it is a matter of teaching history per se. Rather it refers to the situation where it is considered a goal in itself to teach the students something about the historical meta-issues of mathematics: e.g. the historical development of mathematics; societal and cultural influences on this development and the other way around; mathematics' interplay with other areas of practice through history; that mathematics is a human endeavor that takes place in, and to some extent therefore also dependant of, time and space; etc.

In a similar manner as the 'whys' of using history may be split into two different categories, so may the 'hows', i.e. the approaches to using history, be split into three different categories. The first of these comprises the illumination approaches, where the ordinary teaching is somehow supplemented by historical information – in a way these are 'spices' added to the 'mathematical casserole' that is being served to the students. The second consists of the modules approaches which are instructional units devoted to history, often a specific case. These can be either small curriculum-tied modules or longer more free modules, as for example the student projects done at Roskilde University (Kjeldsen & Bjomhøj, 2009). Third we have the history-based approaches. These are approaches directly based on or inspired by the historical development of some mathematics, in a sense we can say that history sets the agenda for the presentation of some topic. Of course, each of these types of approaches may be scaled according to size and scope. For a longer and deeper discussion of history as a tool versus history as a goal, the three broad approaches to using history as well as a survey of the related literature, see Jankvist (2009a).

Now, integrating history in mathematics education is of course not a new idea and examples of older texts about this are laCour (1881), Poincaré (1899), Klein (1908), and Toeplitz (1927). But despite it being an old topic, only very few empirical studies have been made. When conducting a survey as part of my thesis, I was able to identify around eighty somewhat or clear-cut empirical studies. The vast majority of these concern a use of history as a tool. Only about ten percentages or so may be said to primarily address a use of history as a goal. Why this distribution one could ask, in particular when so much of the literature on history in mathematics education seems to value arguments of history as a goal (Jankvist, 2009a)? Well, maybe one reason is that mathematics education researchers concern themselves with the teaching and learning of mathematics and no some 'fuzzy' meta-issues of the subject and its history! Nevertheless, when we look into curricula, syllabi, and teaching plans we often find an emphasis on matters that address elements of history as a goal, e.g. in terms of students' view of mathematics etc.

One example of such can be found in the mathematics program for Danish upper secondary school, where students now are to “demonstrate knowledge about the evolution of mathematics and its interaction with the historical, scientific, and cultural development” (UVM, 2007). This is to come about through some “teaching modules on the history of mathematics” (ibid.), which are included in the so-called supplemental curriculum that takes up 1/3 of the total teaching time.[1] The rhetoric in these new Danish regulations follows those of the Danish so-called KOM-report – ‘KOM’ is a Danish abbreviation for ‘competencies and mathematics learning’ (Niss & Jensen, 2002) – which also focus on history as a goal. More precisely the KOM-report lists eight first order mathematical competencies, which include for example mathematical thinking, reasoning, problem solving, etc., and three second order competencies, so-called types of overview and judgment.[2] These are (i) The actual application of mathematics in other practice and subject areas; (ii) the nature of mathematics as a subject (discipline); and (iii) the historical evolvement of mathematics, internally as well as in a societal context. It is of course the latter which is of our main interest, and about this the KOM-report says:

> The central forces in the historical evolution must be discussed including the influence from different areas of application. [...] The type of overview and judgment should not be confused with knowledge of ‘the history of mathematics’ viewed as an independent topic. The focus is on the actual fact that mathematics has evolved in culturally and socially determined environments, and on the driving forces and mechanisms which are responsible for this evolution. On the other hand, it is obvious that if overview and judgment regarding this evolution is to have *solidness* they must rest on concrete examples from the history of mathematics.” (Niss & Jensen, 2002, p. 168, 68; my translation and emphasis)

The KOM-report’s mentioning of ‘solidness’ (or solidity) suggests some kind of anchoring of the students’ meta-issue reflections in the related mathematical in-issues. By anchoring, I am referring to something that substantiates discussions and reflections about meta-issues on a basis of knowledge and understanding of the related in-issues, e.g. by revealing insights about the meta-issues that could not have been accessed or uncovered without knowing about the in-issues, or by providing in-issue evidence for meta-issue claims or viewpoints (Jankvist, forthcoming(a)).

In the wake of the new Danish regulations for the upper secondary mathematics program, new textbook systems came out from different publishes. An analysis of three of these systems first and foremost revealed that the treatment of history in the new generation of textbooks often is quite anecdotical, that it is most often detached from the related mathematics (in-issues), and that when it appears it often does so in a manner that seems ‘pasted on’, e.g. in special colored boxes etc. (Jankvist, 2008a) – putting it on the edge: on the right hand page there will be the regular presentation of core curriculum mathematics, and on the left hand page there will be a colored box telling some anecdote about a old mathematician, perhaps mentioning some mathematical results due to him or her, but these

[1]The supplemental curriculum is tested on a local oral exam, whereas the 2/3 core curriculum are tested on a national written exam.

[2]No English translation of the KOM-report is available yet, but a presentation and discussion of the eight mathematical competencies and the three types of overview and judgment may be found in Jankvist & Kjeldsen (preprint).

will have nothing to do with the curriculum mathematics on the opposite page.[3] This observation together with that of the distribution of empirical studies and the KOM-report's mentioning of 'solidness' led me to ask the following three research questions in my doctoral thesis:

1. In what sense, to what extent, and on what conditions is it possible to have upper secondary students engage in meta-issue discussions and reflections of mathematics and its history in terms of 'history as a goal'?
2. In what sense and on what levels may an anchoring of the meta-issue discussions and reflections in the taught and learned subject matter (in-issues) be reached and 'ensured' through a 'modules approach'?
3. In what way may 'history as a goal' modules give rise to changes in students' beliefs about (the discipline of) mathematics, or the development of new beliefs?

2 Students' meta-issue discussions and their anchoring

The way of trying to provide a basis for answering these questions was by designing two historical teaching modules to be implemented in a Danish upper secondary level mathematics class. The first of these was on the early history of error-correcting codes and was based on the works of Shannon (1948), Hamming (1950), and Golay (1949). The second module was on the history of public-key cryptography (Diffie & Hellman, 1976) and RSA (Rivest, Shamir & Adleman, 1977). The historical case of the first module may be considered a story of modern applied mathematics, whereas the case of the second may be considered a story of a modern application of old mathematics proper (number theory). And an important element in the selection of these cases is that they both concern applications which are present in students' everyday life (Jankvist, 2009b). Another important element concerns one of the design features of the modules, namely what I shall refer to as general topics and issues in the history and historiography of mathematics. Some examples are e.g. inner and outer driving forces in the development of mathematics, multiple discoveries, the discussion of pure versus applied mathematics (Jankvist, 2009a). Yet an example, one from the historiography of mathematics, is the so-called epistemic objects and epistemic techniques (Rheinberger 1997; Epple, 2000, Kjeldsen, 2009). For example, when Hamming was developing his error-correcting codes at the Bell Laboratories around 1946-47 then these codes were the epistemic objects under investigation, but in order to develop them he relied on a lot of already well-established epistemic techniques such as the notion of metric due to Frechet and elements of linear algebra which we may ascribe to Grassman. These general topics and issues may be – and in this case were – used to identify exemplary 'local' cases in the history illustrating more general 'global' features in the historical development of mathematics. Other design features included a use of translated excerpts from original sources and a strong focus on both meta-issues and in-issues in the material that the students were to work with, the latter came about for example with mathematical tasks on relevant in-issues of the historical case as well as a use of so-called essay assignments. I shall exemplify the idea of essay assignments later, but for now it shall suffice to say that these dealt explicitly with the meta-issues of the historical case in question.

Regarding the experimental setup, written textbooks was prepared for both modules

[3]One exception is in the book system by the publishing house Gyldendal, where they have invited professional historians of mathematics, Tinne Hoff Kjeldsen and Jesper Lützen, to write a chapter each on a historical topic somehow related to the curriculum. But unfortunately such elements in the new book systems are a rarity.

(Jankvist, 2008b, 2008c). The first module was implemented in the class' second year of upper secondary level (students age 17-18) and the second module in their third and final year. The modules were taught by the class' regular mathematics teacher and each module had a duration of approximately fifteen double lessons (one double lesson being 90 minutes). An essential element of the implementation (and design) was that the students worked in groups and handed in their essays in groups as well. The generated data from the implementation (to be used in answering the three research questions) consisted of: (i) questionnaires and interviews with both students and teacher; (ii) students' written mathematical tasks and essay assignments; and (iii) videos of the teaching and in particular of one focus group of students.[4]

In my talk today I shall mainly focus on the second historical teaching module, because it provided the basis for an answering of research question 2 about anchoring. However, a few comments about the first teaching module are in order. In terms of research question 1, the first module provided an existence proof that it is possible to have students discuss and reflect upon meta-issues of mathematics and its historical development. But in order for this to happen a setting must be provided. The use of essay assignments and general topics and issues seems promising elements for such a setting (Jankvist, 2010). Also, if general topics and issues are introduced probably they may assist in anchoring the students' meta-issue discussions in the in-issues (Jankvist, 2009d) – and thereby help preventing the integration of history becoming anecdotical and anachronical.[5] But in terms of research question 2, what the first module did do was that it showed the existence of anchoring being present – the second module made this existence proof more constructive.

In order for us to be able to follow the students' discussions of the meta-issues regarding the history of public-key cryptography and RSA and to provide an answering of research question 2, a brief introduction to this historical case is required.[6] Our story begins at Stanford around 1975 where Whitfield Diffie, Martin Hellman, and Ralph Merkle as a team are driven by a desire to solve the so-called key-distribution problem. This is a problem of private-key cryptography that deals with the distribution of the private encryption/decryption keys. One way of phrasing it is to say that in order for two parties to share a secret (the message) they must already share a secret (the key). During post World War II the world became much more international which resulted in increasing demands for secure communication, first with banking and later with the beginning of electronic mail, the Internet, etc. Diffie was one of the first to truly realize this and as a result he came up with the idea for public-key cryptography: a cryptosystem where the encryption-keys were made publically accessible, but where the person to receive the encrypted message held a personal private decryption-key. The system was based on an idea of a mathematical one-way function; that is a function for which calculating f is a straightforward operation, but for which it for all practical purposes is impossible to calculate f – 1. Of course, the phrase 'for all practical purposes' is not really something to be used in a mathematical definition, but what it means is that it may take a computer a second to calculate f whereas it may take it an eon to calculate f – 1 – and in, say, a million years who is going to care about the secret message sent anyway. So this is not the same as saying that f – 1 does not

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[4]For an elaborated discussion of how the focus group students were selected, see Jankvist (2009d).

[5]The latter is also referred to as 'Whig' history. For a discussion of this in relation to history in mathematics education, see Fried, 2001.

[6]A longer version may be found in Jankvist (2009d). And of course a much more elaborated account may be found in the book by Singh (1999) which is dedicated to the subject of cryptography.

exist, because it does; only that it is difficult to find. In fact, even coming up with a one-way function in the first place for which f and f – 1 satisfied the system requirements proved so difficult that Diffie and Hellman eventually gave up and published their idea for public-key cryptography in 1976 without having a concrete implementation of it. Fortunately three other researchers got hooked on the idea after reading Diffie and Hellman's paper. These were Ronald Rivest, Adi Shamir, and Leonard Adleman from MIT. Rivest and Shamir, being mainly computer scientists, would come up with ideas for one-way functions, and Adleman, begin a mathematician, would then put them to the test. After some 42 attempts Rivest finally came up with something that would work, after he had deciding looking into number theory. The idea, as many of you will know, is that it is a straightforward operation to take to very large prime numbers, say 2-300 digits each, and multiply these together to get a larger integer. However, being given this large integer and then finding its prime factorization is for all practical purposes impossible. Hence, the one-way function. The three researchers patented their solution in 1977, which got named RSA after their initials, and they eventually became millionaires, but that is a different story. What is interesting from our point of view is the mathematics that Rivest, Shamir, and Adleman used in their solution. One thing is finding the one-way function, but around this a cryptosystem must be built up, one that generates the public encryption and private decryption keys. This system builds on old, even ancient, number theory. I will not give a detailed description of the system here, since many of you will already have seen it before, and if not you may find a description in Jankvist (2009d), but I shall mention the three historical theorems. These are: the Chinese remainder theorem, which is ascribed to Sun Zi around the year 400;[7] Fermat's little theorem from 1640;[8] and Euler's generalization of this, known as Euler's theorem, from 1735-36.[9]

The essay assignments, as mentioned previously, consisted of a main essay assignment where the students were to provide two different accounts of the historical case; one focusing on 'who and when' and another on 'why and how'. In a way, this was of course meant as a comment to the new textbook systems' approaches, since these usually include history from a 'who and when' point of view and rarely touches upon the 'why and how'. In order to prepare the students for their answering of the main essay they were to do three supportive essay assignments first and use these in their answers of the main essay. The first of these concerned inner and outer driving forces in the history of public-key cryptography and RSA; the second concerned the fact that public-key cryptography and RSA are actually multiple discoveries, both were discovered more or less simultaneously within the British Government Communication Headquarters (see Singh, 1999); and finally, the third was concerned with the discussion of pure and applied mathematics, taking as its departure point the students' reading of 2/3 of the English edition of G.H. Hardy's A Mathematician's Apology from 1940. For now, I shall only consider the third supportive essay assignment (for elaborations of both the main essay and the supportive essay, see Jankvist, 2009d). In this essay the students were, among other things, asked to do the following: Discuss Hardy's view on pure and applied mathematics and relate this to the case of RSA. For the focus group this resulted in the following discussion.[10]

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[7]In a modern formulation the Chinese remainder theorem may be formulated as: let $m_1, m_2, \ldots, m_n$ be positive integers relatively prime by pairs. The system $x \equiv a_1 \pmod{m_1}$; …; $x \equiv a_n \pmod{m_n}$ then has a unique solution x modulo $m = m_1 m_2 \cdots m_n$, meaning there is a solution x for $0 \leq x < m$ and all other solutions are congruent to this solution modulo m.

[8] Fermat's little theorem may be formulated as: if p is a prime, then for every n: $n^p \equiv n \bmod p$.

[9] Euler's theorem may be formulated as: $\gcd(n,m)=1$, then $n^{\varphi(m)} \equiv 1 \bmod m$, where $\varphi(m)$ is Euler's φ-function.

[10]The students' names have been changed, but the sexes of the students remain the same.

Harry: It says [in the teaching material] that one of the problems in number theory is to decide if a number is a prime or not. And in this book [the *Apology*], he [Hardy] says that finding primes is pure mathematics, because when you are a mathematician then you already have the frame for the area you are working within, you know what a prime number is, in contrast to a physicist or a chemist who works with some applied mathematics, they have to work with things relatively, I would say: Here is a table [points to the table], but to you... for some other person this might not be a table, or for some other thing in a different universe, this might not be a table. But a prime number will always be a prime number.

Andrew: So the pure mathematics cannot be discussed, you might as well say.

Lola: That depends on how you conceive that the history of mathematics has been developed, because if someone else had been sitting and thinking about a number and had found some other connections, well then a prime might suddenly not have the same meaning.

Harry: But now they have this frame...

Andrew: You could say that, what we are kind of working with... it is our frame, it is the numbers we have, it is our frame, and then within these there is a lot of pure mathematics, for example prime numbers.

Lola: Yes, primes can never be different, if you look at how we look at mathematics.

Andrew: No, if you stick within our frame.

Lola: But if we imagine that the numbers had some different values or whatever you'd say, then...

Andrew: Yes, but then you are changing to a new frame, and then there is a connection within this frame.

Lola: Well... but then you could also say...

Harry: In that way it might be kind of relative, I can see that. But what he says is that you can't discuss it. He says that in this world we live in here there can be two different physicists who tell you what this is [points to the table] while two different mathematicians [equally] can tell you what a prime number is.

Lola: Yes, and that is what Hardy he says, right? Do you want me to write that?

Andrew: But then you could also say that the pure mathematics is as objective as anything can ever become, right, because it isn't colored by anything.

Harry: But we have to see it in connection to RSA.

Before entering into the analysis of the above discussion, a bit of explanation regarding the students' references to Hardy's Apology will have to be given. One of the passages that the students' refer to and base their discussion upon is the following:

> A chair or a star is not in the least like what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surrounds it; but '2' or '317' has nothing to do with sensation, and its properties stand out the more clearly the more closely we scrutinize it. […] Pure mathematics […] seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is*, because mathematical reality is built that way. (Hardy, 1992, p. 130)

Of course, one of the more famous quotes from Hardy, which is often referred to when discussing number theory in relation to its application in cryptography is the following (ibid.):

> Real mathematics has no effects on war. No one has yet discovered any warlike purpose

> to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.

In my analysis of the students' discussions in relation to an answering of research question 2, I adopted and adapted Anna Sfard's discursive approach to learning (Sfard, 2008). Roughly speaking, Sfard begins with the assumption that understanding/learning is a form for change. What changes she asks, and based on a survey of literature on philosophy and learning psychology she concludes that it is one's thinking that changes. Thinking, she argues, is a type of (inner) communication. She then defines her term commognition, as a contraction of communication and cognition. Discourse is defined as "the different types of communication, and thus of commognition, that draw some individuals together while excluding some others" (Sfard, 2008, p. 91). Mathematics is a discourse – and from our point of view, we may of course equally argue that so is history, philosophy, and history of mathematics. This line of thought by Sfard leads to the conclusion that understanding is change in discourse. By drawing the parallel that reflection is also change in discourse, we may then be able to identify what I shall refer to as potential anchoring points in the students' discussions, the term 'point' referring to a point in time in the discussion. The students' discussions may follow two different discourses: meta-issue discourses, which are those that address history, philosophy, sociology, psychology, etc.; and in-issue discourses, which are the mathematical discourses. The potential anchoring points are those where the students' discussion change from a meta-issue discourse to an in-issue discourse – i.e. places where the meta-issue discussions and reflections may be anchored in the related mathematical in-issues.

So, what discourses may we observe in the discussion of the focus group students as displayed above? Well, there is a historical discourse present in which the students are trying to link Hardy's statements to RSA. There is also a mathematical discourse with their talk related to number theory and in particular prime numbers. And there are elements of a somewhat philosophical discourse with their talk of 'frames'. The potential anchoring point(s) of this discussion are the place(s) where they change from either the historical or the philosophical discourse to the mathematical one and begin reasoning by means of their knowledge of prime numbers. It is important to stress again that the way of identifying anchoring as illustrated above only reveals potential anchoring points. Once the potential points have been identified they will have to be either verified or rejected by means of methodological triangulation with other data sources. In the case above, we could for example look into the students' work on mathematical tasks, test or interview questions, etc. related to prime numbers (or related concepts such as relatively prime, Euler's φ-function, etc.) to get an idea of Harry's, Andrew's, and Lola's understanding of the in-issues of the mathematical discourse. Thus, identifying anchoring points, and hence anchoring, is a two step process.

Carrying out such investigations for both teaching modules (see Jankvist, 2009d, also for the analysis of the excerpt above) revealed the presence of at least four different 'levels' regarding anchoring: (1) the non-anchored; (2) anchored comments, which are comments that can be verified as being anchored by triangulation, but which give rise to no further discussion; (3) anchored arguments, which is where an in-issue discourse is used to underpin a meta-issue point (of view) or argument; and finally (4) anchored discussions, where anchored comments or anchored arguments are taken up by other group members and eventually come to make up the basis for a meta-issue discussion. The latter (4) is exemplified by the excerpt above of the focus group's discussion of pure and applied mathematics with reference to Hardy, number theory, and in particular prime numbers, the in-issue that comes to make up a point of reference for them in their treatment of the meta-issues.

3 Students' beliefs about 'mathematics as a discipline'

This brings me to part 3 of my talk, which is concerned with students' beliefs about mathematics as an academic discipline. Beliefs are generally considered to be a kind of lenses through which one looks when interpreting the world; psychologically held understandings; and beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual (Philipp, 2007). Lester, Jr. (2002, pp. 352-353) asked the question: "Do students know what they believe?" Of course, one of Lester's reasons for asking this question in the first place may be to provide the answer himself, which he does stating that: "I do not think most students really think about what they believe about mathematics and as a result are not very aware of their beliefs" (ibid.). Furinghetti and Pehkonen (2002) argue that one should take into consideration beliefs students hold consciously as well as unconsciously.[11]

The question, of course, when trying to study students' beliefs and any changes that may occur in these becomes how to take things like the above into account. What I will do is that I will tell you the approach I took in doing so, and then it will be up to you to decide if you find this to be appropriate. The methods I used to try and access students' beliefs consisted of questionnaires, follow-up interviews, and videos. The class of 23 students did four rounds of questionnaires: one before the first module; one in between the modules; one immediately after the second module; and one sometime after this. After each round of questionnaires, 12 students, selected in such a way that their questionnaire answers represented the answers of the class as was best possible, was interviewed about their questionnaire answers, essay assignments, etc. Besides being a method for accessing the students' beliefs, these rounds of questionnaires and follow-up interviews to some degree also resulted in the students' awareness of their (consciously held) beliefs being developed some. The focus group was video filmed whenever they worked together on tasks, either mathematical tasks or essay assignments, and these films came to function as a way of trying to uncover some of these particular students' more unconsciously held beliefs. All the four rounds of questionnaires and interviews, and the implementation of the modules, took place over a period of one year.

Allow me to give you some examples of the questions, and type of questions, that the students were asked in their questionnaires, thus also implicitly illustrating what elements I consider to be part of 'mathematics as a discipline' (for a more clear definition see Jankvist (2009d), where you may also find the questionnaires in full length). The questions the students were asked may be divided, for matters of clarity, into to three groups – three groups not completely unrelated to the KOM-report's three types of overview and judgment. The first of these I shall refer to as the historically oriented questions, and as examples of such I provide the following two:

- When, how, and why do you think the mathematics in your textbook came into being?
- What do you think a researcher in mathematics does?

The second group is the sociologically oriented questions, illustrated by:

- Do you think mathematics has a greater or lesser impact in society today than 100 years ago?
- Where is mathematics applied in society and (your) everyday life?

[11] A long discussion of the beliefs literature in relation to the present study may be found in Jankvist (2009d) and Jankvist (forthcoming(b)).

And finally the third group, the philosophically oriented questions, of which I mention:

- Do you believe mathematics, or parts of it, can become obsolete?
- Do you believe mathematics is something you discover or invent?

Of course this last question stems from an old, long, and still ongoing discussion, but it actually turned out to be one of the more interesting questions to ask students at this level, because it puzzled them and the more they thought about it, the more it confused them. But it also made them reflect about, sometimes change, their view of things, and it made them try to justify and/or exemplify their beliefs, also by referring to elements of the two historical cases of the teaching modules. And it would not only be so that they settled on one or the other – discovery or invention – some students would by themselves argue that one would precede the other, which is even along the lines of what Hersh (1997) argues. So indeed it seems to me that such open-ended questions may play an important role in the clarification and/or development of students' beliefs.

Another, to some degree also open-ended, question that we will have to ask is whether the beliefs as I could detect them, with the methods described above, are then actually beliefs. One of the practicalities you are faced with in a study like this is that you have to start somewhere: I took the students' answers to the first round of questionnaires and follow-up interviews as their virginal beliefs. Yes, perhaps one year may not be enough time to see actual changes in these beliefs – because in the literature beliefs are considered to be entities somewhat persistent. Thus, it may be more correct to speak about changes in views – that is if we take 'views' to be something less persistent than beliefs. Such views, however, may have the potential to develop into beliefs over time. We may also speak about students' images of mathematics. In my study I have taken students' images of mathematics to consist of their beliefs as well as their views.

For a longer, more deep and elaborated discussion and display of the students' beliefs as I found them, you are referred to Jankvist (2009d, forthcoming(b)). Here I just summarize the main findings in order to provide you with an answer to research question 3. So, on a basis of explicitness – if students are not explicit about their beliefs/views, we cannot say anything about them – I was able to identify three dimensions of change in the students' beliefs/views: (1) a growth in consistency in students' beliefs/views regarding related questions and issues were noticed; (2) an increase in students' justification of the beliefs/views they were expressing; and (3) students provided a larger amount of exemplifications in support of their beliefs/views, e.g. by referring to the two historical cases of the modules.

4 Perspectives: Further research

When being given the chance to give a plenary lecture like this, I think it is only fair that I provide you with my personal opinion on possible further (empirical) research in this area. Although the study, I have reported on above is concerned with history as a goal, I shall begin by making some observations regarding studies on the use of history as a tool.

When history does play the role as a tool, it is most often in the sense of stimulating students' learning of specific mathematical concepts, etc. or in terms of trying to motivate the students through meta-issue matters to try and learn the mathematical in-issues dictated by curriculum. That is to say, most empirical studies on history as a tool in mathematics education seem to focus on history as either a cognitive tool or history as a motivational or affective tool. But perhaps the development of students' mathematical competencies is a more 'natural setting' for the use of history as a tool. Kjeldsen (2010) has discussed how history may be used (as a tool) in this way through a study of original sources relating to curriculum topics, and furthermore how to do so in a non-Whig manner (cf. Fried, 2001). The KOM-report's eight mathematical competencies in relation to the use of history as a tool are also discussed at length in Jankvist & Kjeldsen (preprint).

When history (as a goal) is part of a curriculum, as in the Danish case, the problem becomes to avoid its treatment being anecdotical, on the one hand, and to ensure some kind of anchoring in the in-issues, on the other hand. The study presented in this paper shows that students can indeed deal with meta-issues related to the history of mathematics in a reflective manner and that they, on different levels, are able to anchor their treatment of these in the related mathematical in-issues. Furthermore, the study also shows that historical modules like the two presented here did have as an outcome that some students ended up with more balanced, multifaceted, reflected and profound images of mathematics. Now, it is not always important whether students believe one or the other, for instance whether they believe in invention or discovery of mathematics. What is important is that they reflect upon their beliefs, try to accommodate in case of conflicts, and hold their beliefs as evidentially as possible. The students were invited to do this in the modules. But in order for this to happen a scene needs to be set: the students must be provided with (meta-issue) aids (e.g. the 'general topics and issues') and some evidence (the two historical cases) before these things can take place. Because, as I have phrased it in my thesis,

> ... we cannot expect students' (core) beliefs to change in any way are they not confronted with some 'evidence'. Not until students have access to evidence – or counter-evidence – are they likely to criticize rationally, reason about, and reflect upon their beliefs, and possibly accommodate and change them. (Jankvist, 2009d, p. 257)

Regarding further research related to the study presented here, the results for research question 3 could perhaps serve as a model for what we will consider to constitute students' reflected images of mathematics as a discipline. This is illustrated in figure 1. By considering the three dimensions of consistency, justification, and exemplification as components in a model for students' reflected images (beliefs and views), we may to a larger degree be able to assess the development of such images – a development which is implicitly called for in many curriculum descriptions when talking about enlarging students' appreciation, awareness, etc. of mathematics. Possibly such a model could also serve as input to operationalize the KOM-report's three types of overview and judgment – thus, not only the one of them regarding history.

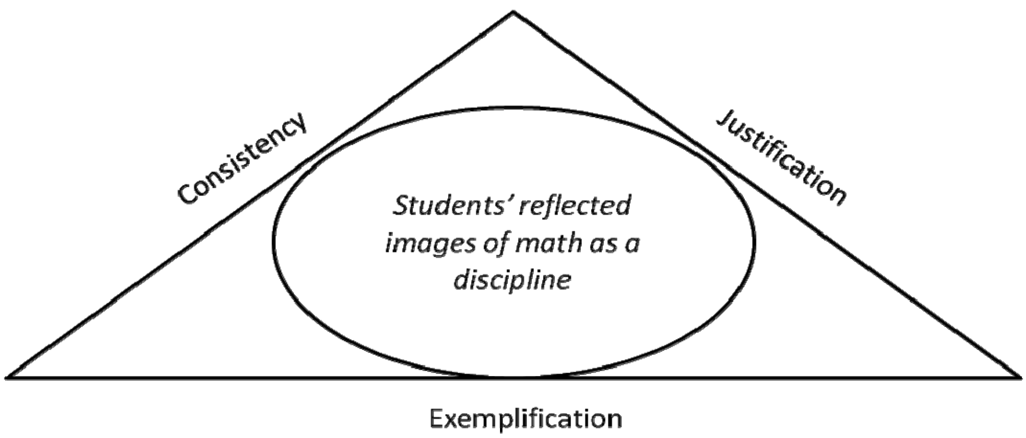

Figure 1. A model of students' reflected images of 'mathematics as a discipline' as made up by the three dimensions of consistency, justification, and exemplification.

I shall round my talk off with some statements regarding research on history in mathematics education. First of all, we need much more empirical research within the History and Pedagogy of Mathematics (HPM) community. Surely, I am not the first and only to state this (see also the discussion in Jankvist, 2009c), but I think that it needs to be repeated, in particular if we want to spread the message of history in mathematics education. And I hope that we do! Secondly, if one scans the relatively few empirical studies that do exist, one will observe that only rarely do these relate to general mathematics education research (MER). This is a least a pity and a most a major flaw on our behalf, I think, because MER has a lot to offer HPM, in particular in terms of theoretical constructs, conceptual frameworks, methodology, etc. that we can use and benefit from when conducting our studies. And HPM would have something to offer MER in return(!) – at any rate, more than just the usual statements regarding epistemological obstacles and historical parallelism, which is what one most frequently finds whenever history is mentioned in the MER literature. One example, although only speculative, could perhaps turn out to be the model for students' reflected beliefs (figure 1). If this model, which is based on findings from using history in mathematics education, i.e. HPM related research, can be used to assess students' development of the KOM-report's two other types of overview and judgment (the actual application of mathematics and the nature of mathematics), then that would be an example of HPM contributing to MER. But in order for something like this to happen and for mathematics education researchers to realize what HPM has to offer MER, we need to link the two tighter together – beginning with our own research![12]

REFERENCES

- Diffie, W. and M. E. Hellman, 1976, 'New Directions in Cryptography'. *IEEE Transactions on Information Theory* **IT-22**, 644–654.
- Epple, M., 2000, 'Genisis, Ideen, Institutionen, mathmatische Werkstätten: Formen der Mathematikgeschichte – Ein metahistorischer Essay'. *Mathematische Semesterberichte* **47**, 131–163.
- Fried, M. N.: 2001, 'Can Mathematics Education and History of Mahtematics Coexist?'. *Science & Education* **10**, 391–408.
- Furinghetti, F. and E. Pehkonen, 2002, 'Rethinking Characterizations of Beliefs'. In: G. C. Leder, E. Pehkonen, and G. Törner (eds.): *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht: Kluwer Academic Publishers, pp. 39–57. Chapter 3.
- Golay, M. J. E., 1949, 'Notes on Digital Coding'. *Proceedings of the IRE* **37**, 657.
- Hamming, R. W., 1950, 'Error Detecting and Error Correcting Codes'. *Bell System Technical Journal* **29**, 147–160.
- Hardy, G. H., 1992, *A Mathematician's Apology*. Cambridge: Cambridge University Press, canto edition. Foreword by C. P. Snow.
- Hersh, R., 1997, *What is Mathematics Really?* Oxford: Oxford University Press.
- Jankvist, U. T., 2008a, 'Den matematikhistoriske dimension i undervisning – gymnasialt set'. *MONA* **4**(1), 24–45. English translation of title: The Dimension of the History of Mathematics in Teaching – The Case of Upper Secondary Level.
- Jankvist, U. T., 2008b, *Kodningsteoriens tidlige historie – et undervisningsforløb til gymnasiet*, No. 459 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: The Early History of Error Correcting Codes – a Teaching Module for Upper Secondary School.
- Jankvist, U. T., 2008c, *RSA og den heri anvendte matematiks historie – et undervisningsforløb til gymnasiet*, No. 460 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: RSA and the History of the Applied Mathematics in the Algorithm – a Teaching Module for Upper Secondary School.
- Jankvist, U. T., 2009a, 'A categorization of the 'whys' and 'hows' of using history in mathematics education'. *Educational Studies in Mathematics* **71**(3), 235–261.

[12]One such initiative has been taken by the CERME (Congress of the European Society for Research in Mathematics Education) working group on history in mathematics education, where the following theme has been added to the call for papers for CERME-7 to take place in Rzeszów, Poland, February 2011: "Relationships between (frameworks and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education." See: http://www.cerme7.univ.rzeszow.pl/

- Jankvist, U. T., 2009b, ‘History of Modern Applied Mathematics in Mathematics Education’. *For the Learning of Mathematics* **29**(1), 8–13.
- Jankvist, U. T., 2009c, ‘On Empirical Research in the Field of Using History in Mathematics Education’. *ReLIME* **12**(1), 67–101.
- Jankvist, U. T., 2009d, ‘*Using History as a ‘Goal’ in Mathematics Education*’. Ph.D. thesis, IMFUFA, Roskilde University. No. 464 in Tekster fra IMFUFA, Roskilde. http://milne.ruc.dk/ImfufaTekster/pdf/464.pdf
- Jankvist, U. T., 2010, ‘An empirical study of using history as a ‘goal”. *Educational Studies in Mathematics* **74**(1), 53–74.
- Jankvist, U. T., forthcoming(a), ‘Anchoring students’ meta-perspective discussions of history in mathematics’.
- Jankvist, U. T., forthcoming(b), ‘Changing students’ images of mathematics as a discipline’.
- Jankvist, U. T. & T. H. Kjeldsen, (preprint), ’New avenues for history in mathematics education: Mathematical competencies and anchoring’. *Science & Education*.
- Kjeldsen, T. H., 2009, ‘Egg-Forms and Measure-Bodies: Different Mathematical Practices in the Early History of the Modern Theory of Convexity’. *Science in Context* **22**(1), 85–113.
- Kjeldsen, T. H., 2010, ‘Using history as a means for the learning of mathematics without losing sight of history: The case of differential equations’. In: *Proceedings from the CERME6 Working Group 15*. CERME, pp. 2742-2751.
- Kjeldsen, T. H., M. Blomhøj, 2009, ‘Integrating history and philosophy in mathematics education at university level through problem-oriented project work’. *ZDM Mathematics Education* **41**, 87–103.
- Klein, F., 1908, *Elementarmathematik vom höheren Standpunkte aus*. Leipzig.
- la Cour, P., 1881, *Historisk mathematik*. Kolding: Konrad Jorgensens Bogtrykkeri.
- Lester, Jr., F. K., 2002, ‘Implications of Research on Students’ Beliefs for Classroom Practice’. In: G. C. Leder, E. Pehkonen, and G. Törner (eds.): *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht: Kluwer Academic Publishers, pp. 345–353. Chapter 20.
- Niss, M., T. H. Jensen (eds.), 2002, *Kompetencer og matematiklæring – Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Undervisningsministeriet. Uddannelsesstyrelsens temahæfteserie nr. 18. English translation of title: Competencies and Learning of Mathematics – Ideas and Inspiration for the Development of Mathematics Education in Denmark.
- Poincaré, H., 1899, ‘La Logique et L’Intuition’. *Dans la Science Mathématique et dans L’Enseignement* **1**, 157–162.
- Rheinberger, H.-J., 1997, *Toward a History of Epistemic Things: Synthesizing Proteins in the Test Tube*. Stanford: Stanford University Press.
- Rivest, R. L., A. Shamir, L. Adleman, 1978, ‘A Method for Obtaining Digital Signatures and Public-Key Cryptosystems’. *Communications of the ACM* **21**(2), 120–126.
- Sfard, A., 2008, *Thinking as Communicating*. New York: Cambridge University Press.
- Shannon, C. E., 1948, ‘A Mathematical Theory of Communication I, II’. In: D. Slepian (ed.): *Key Papers in The Development of Information Theory*. New York: IEEE Press, pp. 5–18; 19–29.
- Singh, S., 1999, *The Code Book – The Science of Secrecy from Ancient Egypt to Quantum Cryptography*. New York: Anchor Books.
- Toeplitz, O., 1927, ‘Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen’. *Jahresbericht der deutschen Mathematiker-Vereinigung* XXXVI, 88–100.
- UVM (Undervisningsministeriet), 2007, ‘Vejledning: Matematik A, Matematik B, Matematik C’. Bilag 35, 36, 37. English translation of title: Ministerial order of 2007 – Mathematics levels A, B, and C.