CLASSIFYING THE ARGUMENTS AND METHODS TO INTEGRATE HISTORY IN MATHEMATICS EDUCATION: AN EXAMPLE

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ABSTRACT

The ICMI Study volume “History in Mathematics Education”, published in 2000, includes a comprehensive list of arguments for integrating history in Mathematics Education (ME) and methodological schemes of how this can be accomplished. To classify the above arguments, Jankvist distinguished between using “history-as-a-goal” and using “history-as-a-tool” in ME. Independently, Grattan-Guinness distinguished between “history” and “heritage” aiming –among other things - to help understanding better which history could be helpful and meaningful in ME. In a recent paper by the authors, these pairs of concepts are used to classify more finely and deeply these arguments and methodological schemes. The present paper aims to illustrate this 2X2 classification scheme by means of a specific example, namely logarithms and related concepts as taught to upper high school students.

1 Introduction

In the last decades, there has been a growing interest in integrating the history of mathematics (HM) in mathematics education (ME). Arguments for this integration have been put forward to refute possible objections and/or enhance the interest of the ME community, educational material has been produced, empirical research has been conducted and methodological schemes have been described & implemented.

For a long time, there were no coherent theoretical ideas and framework to place, see and compare all these activities. A serious attempt in this direction is the comprehensive ICMI Study volume (Fauvel & van Maanen 2000), but further important work followed:

Jankvist (2009a,b,c) reconsidered the general arguments for HM in ME (the *whys* in his terminology) and methodological schemes (the *hows*), introducing two interesting criteria:

- To classify the *whys* according to whether history appears as a ***goal***, or as a ***tool***, with emphasis on “*meta-perspective*” issues, or “*inner*” issues, respectively.
- To clarify the *hows* according to a 3-level distinction of the possible types of implementations: *illumination approaches*, *modules approaches* and *history-based approaches*.

Independently, Grattan-Guinness (2004a,b) introduced the distinction between “***history***” and “***heritage***” to interpret mathematical activities and their products. This is an important conceptual tool to revisit the issue of “which history is appropriate to ME?” (Barbin 1997), attempting to clarify existing conflicts and tensions between a mathematician’s and a historian’s approach to mathematical knowledge. Grattan-Guiness (2004b) gives several examples by contrasting the general characteristics of the two concepts.

History and *Heritage* should be seen as ***complementary*** ways to approach and understand mathematics as a human activity, in the sense that none of them, taken alone, can lead to a sufficiently wide and deep enough understanding of what (a specific piece of) mathematics is. Similarly, Jankvist’s distinction between the use of *history-as-a-tool* and *history-as-a-goal* in ME should be seen as complementary ways to classify the arguments and the methodological approaches to introduce a historical dimension in ME. The term “complementary” is used closely to that used by N. Bohr to describe the microphysical reality and subsequently raised to a general conceptual tool to understand reality (Bohr 1934, 1958).

Recently, these ideas have been described in more detail and an attempt has been made to classify the *whys* and *hows* more finely, by projecting them onto the 2X2 grid formed by the two

dipoles, namely (*history-as-a-tool*, *history-as-a-goal*) and (*history*, *heritage*) (Tzanakis & Thomaidis, to appear). The term "dipole" is used here to emphasize the interconnections between concepts, thus reflecting better their complementary character mentioned above. This paper, which **complements** the one above, aims to illustrate these ideas by means of an example. Therefore, a brief description of the conceptual dipoles above is given in the next section, together with a list of the ICMI *whys* & *hows* and Jankvist's distinction of possible implementations. Their corresponding classification in terms of these dipoles is given in section 3. Finally, these theoretical ideas are illustrated in section 4 by means of an example; logarithms and related concepts to be taught in high school.

2 The two conceptual *dipoles* and the *whys* & *hows* of integrating history in mathematics education

2.1 The two conceptual dipoles

Jankvist introduced two broad ways in which HM could be helpful and relevant to ME: ***History-as-a-tool*** and ***History-as-a-goal***, which are intimately connected with issues within mathematics (*inner issues*) and issues that concern mathematics itself (*meta issues*):

> "***History-as-a-tool*** concerns the use of history as an assisting means, or an ***aid***, in the learning [or teaching] of mathematics…. in this sense, history may be an aid both …"[1] "as a motivational or affective tool, and … as a cognitive tool …"[2] "[It] concerns … inner issues, or ***in-issues***, of mathematics [that is] issues related to mathematical concepts, theories, disciplines, methods, etc.— the internal mathematics"[3].
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> "***History-as-a-goal*** does not serve the primary purpose of being an aid, but rather that of being an ***aim*** in itself … posing and suggesting answers to questions about the evolution and development of mathematics, … about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics and its history" (Jankvist 2009b §1.1). "[It] concerns … learning something about the meta-aspects or ***meta-issues*** of mathematics … [that is] issues involving looking at the entire discipline of mathematics from a meta perspective level" (Jankvist 2009c, pp239-240).

These two ways are mutually exclusive, in the sense that the emphasis put in each case is different and to a large extent incompatible with each other. But although "…history-as-a-goal 'in itself' does not refer to teaching history of mathematics ***per se***, but using history to surface meta-aspects of the discipline…in specific teaching situations, [it] may have the positive side effect of offering to students insight into mathematical in-issues of a specific history" (Jankvist 2009d, p.8). Conversely, using "history-as-a-tool" to teach and learn specific mathematics may stimulate reflections at a meta-perspective level, extrapolated from the particular subject considered; that is, an ***anchoring*** of *meta-issues* into the *in-issues* that constitute the study of the subject may result (Jankvist 2009b, §§5.3, 5.4, 6.1, 6.3). These are important interrelations, stressing the indispensability of both "history-as-a-tool" and "history-as-a-goal", which thus constitute what we call a coherent ***conceptual dipole***.

Independently, Grattan-Guinness distinguished between ***History*** and ***Heritage***:

> The ***History*** (***Hi***) of a mathematical subject N refers to "…the development of N during a particular period: its launch and early forms, its impact [in the immediately following years and decades], and applications in and/or outside mathematics. It addresses the question '*What happened in the past?*' by offering descriptions. Maybe some kinds of

[1] Jankvist 2009b §1.1.
[2] Jankvist 2009d, p8.
[3] Jankvist 2009c, p240.

explanation will also be attempted to answer the companion question '*Why did it happen?*''[4]. "[It] should also address the dual questions "*what did not happen in the past?*" and "*why not*?"; false starts, missed opportunities…, sleepers, and repeats are noted and maybe explained. The (near-)absence of later notions from *N* is registered, as well as their eventual arrival; *differences* between *N* and seemingly similar more modern notions are likely to be emphasized"[5].

The ***Heritage*** (***He***) of a mathematical subject *N* refers "…. to the impact of *N* upon later work, both at the time and afterward, especially the **forms** which it may take, or be **embodied**, in later contexts. Some modern form of *N* is usually the main focus, with attention paid to the course of its development. Here the mathematical relationships will be noted, but historical ones…will hold much less interest. [It] addresses the question "*how did we get here?*" and often the answer reads like "the royal road to me." The modern notions are inserted into *N* when appropriate, and thereby *N* is unveiled... *similarities* between *N* and its more modern notions are likely to be emphasized; the present is *photocopied* onto the past" (Grattan-Guiness, 2004a, p.165).

Grattan-Guinness argues that ME can profit equally well from **both** *Hi* and *He* and gives a detailed list of the differences between them (Grattan-Guinness 2004b, §1.3), showing their **incompatibility**:

"The distinction between history and heritage is often sensed by people who study some mathematics of the past, and feel that there are fundamentally different ways of doing so. Hence the disagreements can arise; one man's reading is another man's anachronism, and his reading is the first one's irrelevance. The discords often exhibit the differences between the approaches to history usually adopted by historians and those often taken by mathematicians." (Grattan-Guinness 2004b, p.8).

But, their **indispensability** in understanding the development of mathematics is clearly emphasized:

"The claim put forward here is that *both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not*." (Grattan-Guinness 2004b, p.8)

Hence, the two concepts are **complementary** in the sense of section 1, constituting the poles of another *conceptual dipole*.

As far as the introduction of a historical dimension in ME is concerned, the distinction between *history* and *heritage* is close to similar distinctions between pairs of methodological approaches; *explicit* & *implicit* use of history, *direct* & *indirect* genetic approach, *forward* & *backward heuristics* (Fauvel & van Maanen 2000, ch.7, pp.209-210). Hence, this distinction is potentially of great relevance to ME (Rogers 2009), serving -among other things- to contribute towards answering the recurrent question "Why and which history is appropriate to be used for educational purposes?" (Barbin 1997).

2.2 A concise list of the *whys* & *hows*

In this section the *whys* & *hows* are listed according to the ICMI Study volume (Fauvel & van Maanen 2000, §§7.2,7.3) and Jankvist (2009c, §6), noting that the *whys* correspond to didactical *tasks* to be attempted and the *hows* correspond to *methodological approaches* to be followed.

2.2.1 The "ICMI Study *whys*"

The areas in which the HM is beneficial for the teaching and learning of mathematics are

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[4]Grattan-Guiness, 2004b, p.7.

[5]Grattan-Guiness, 2004a, p.164.

listed below.

A. ***The learning of Mathematics***

1. *Historical development vs. polished mathematics*: To uncover/unveil concepts, methods, theories etc.

2. *History as a re-source*: To motivate, to raise the interest, to engage the learner by linking present knowledge and learning process to knowledge and problems in the past.

3. *History as a bridge between mathematics and other disciplines/domains*: From where and how did a great part of mathematics emerged? To bring-in new aspects, subjects and methods.

4. *The more general educational value of history*: To develop personal growth and skills, not necessarily connected to mathematics.

B. ***The nature of mathematics and mathematical activity***

1. *Content*: To get insights into concepts, conjectures & proofs, by looking from a different viewpoint; to appreciate "failure" as part of mathematics in the making; to make visible the evolutionary nature of meta-concepts.

2. *Form*: To compare old and modern; to motivate learning by stressing clarity, conciseness and logical completeness.

C. ***The didactical background of teachers***

1. *Identifying motivations*: To see the rationale for introducing new knowledge and progress.

2. *Awareness of difficulties & obstacles*: To become aware of possible didactical difficulties and analogies between the classroom & the historical evolution.

3. *Getting involved and/or becoming aware of the creative process of "doing mathematics"*: To tackle problems in historical context; to enrich mathematical literacy; to appreciate the nature of mathematics.

4. *Enriching the didactical repertoire*: To increase the ability to explain, approach, understand specific pieces of mathematics and on mathematics.

5. *Deciphering and understanding idiosyncratic and/or non-conventional approaches to mathematics*: To learn how to work on known mathematics in a different (old) context; hence to increase sensitivity and tolerance towards non-conventional, or "wrong" mathematics.

D. ***The affective predisposition towards mathematics***

1. *Understanding mathematics as a human endeavour*: To show and/or understand evolutionary steps.

2. *Persisting with ideas, attempting lines of inquiry, posing questions*: To look in detail at similar examples in the past.

3. *Not getting discouraged by failures, mistakes, uncertainties, misunderstandings*: To look in detail at similar examples in the past.

E. ***The appreciation of mathematics as a cultural endeavour***

1. *Appreciating that mathematics evolves under the influence of factors intrinsic to it*: To identify and appreciate the role of internal factors.

2. *Appreciating that mathematics evolves under the influence of factors extrinsic to it*: To identify and appreciate the role of external factors.

3. *Appreciating that mathematics form part of local cultures*: To understand a specific piece of mathematics through approaches belonging to different cultures.

2.2.2 The "ICMI Study *hows*"

The *hows* for integrating HM in ME according to the ICMI Study volume are:

A. *Learning* **history** *by providing* **direct historical information**

Isolated factual information; historical snippets; separate historical sections; whole books and courses on history etc.

B. *Learning* ***mathematical topics*** *by following a teaching* ***approach inspired by history***
Teaching modules inspired by history; worksheets based on original sources; historical-genetic approach; modernised reconstructions of a piece of mathematics etc.

C. *Developing*
1. *Awareness of the intrinsic nature of mathematical activity* (***intrinsic awareness***) and
(i) The role of general conceptual frameworks
(ii) The evolutionary nature of all aspects of mathematics
(iii) The importance of the mathematical activity itself (doubts, paradoxes, contradictions, heuristics, intuitions, dead ends etc);
2. *Awareness of the extrinsic nature of mathematical activity* (***extrinsic awareness***)
(i) Relations to philosophy, arts and social sciences
(ii) The influence of the social and cultural contexts
(iii) Mathematics as part of (local) culture and product of different civilizations and traditions
(iv) Influence on ME through ME history.

2.2.3 "Jankvist's *hows*"

A similar account of Jankvist's possible types of implementations is:

A. ***Illumination*** *approaches*: Teaching and learning of mathematics, in the classroom or the textbooks used, is supplemented by historical information of varying size and emphasis.

B. ***Modules*** *approaches*: Instructional units devoted to history, and often based on the detailed study of specific cases. History appears more or less directly.

C. ***History-based*** *approaches*: Directly inspired by, or based on the HM. Not dealing with studying the HM directly, but rather indirectly; the historical development not necessarily discussed in the open, but often sets the agenda for the order and way in which mathematical topics is presented.

Both types of *hows* correspond to possible implementations of the HM into ME, of a different character, however; the ICMI Study *hows* focus on different emphases, whereas, Jankvist's focus strictly on the adopted methodologies.

3 The conceptual dipoles as a means to classify the *whys* & *hows*

Taking into account the description of the conceptual dipoles in §2.1, a 2X2 table results, composed by the elements of each dipole. Then, according to the description in §§2.2, 2.3, each of the *whys* and *hows* can be placed in at least one cell, depending on how sharply and clearly it has been described (Fauvel & van Maanen 2000, §§7.2, 7.3; Jankvist 2009c, §6). Thus, the two dipoles act as a "magnifying lens", either requesting a more complete description of each *why* and *how*, or/and providing a clearer orientation of the way each *why* and *how* could be implemented. Items appearing more than once in Table 1 are shaded and those placed with reserve appear with an interrogation mark[6], suggesting that the *whys* are not **irreducible** with respect to the two dipoles, but consist of **simpler** elements, as explained in more detail in Tzanakis & Thomaidis (to appear). Hence, they should be further analysed, so that they fall into only one cell of Table 1. But this remains to be shown and further work is needed (A.4 is not in the table; it should be analysed further).

[6]This convention is applied to all subsequent tables.

Table 1: The classification of the ICMI ***whys***

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	C.2, C.3(?) E.1, E.2, E.3	A.3; B.1, B.2 D.1
***History as a tool* (emphasis on *inner-issues*)**[7]	A.3; C.1, C.3, C4, C.5 D.2, D.3	A.1, A.2 B.1, B.2(?) C.2, C.3 E.3

This classification of the *whys* is **finer** to the extent that the dipoles have been determined as sharply as possible, which presupposes the detailed study of the *whys* and each *conceptual dipole* in the context of specific examples. In addition, this and the following tables can be considered in relation to the **target population** to whom they are addressed, specifying which entries are better suited to whom: mathematics teachers, curriculum designers, producers of didactical material, mathematics teachers' trainers and advisors.

Table 2: The classification of the ICMI ***hows*** (cf. §2.2.2)

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	Direct historical information: A Intrinsic awareness: C.1(ii) Extrinsic awareness: C.2(ii)	Direct historical information: A Extrinsic awareness: C.2(i) (iii) (iv)
***History as a tool* (emphasis on *inner-issues*)**[7]	Intrinsic awareness: C.1(i) (iii) Learning mathematical topics (explicit use of history): B	Learning mathematical topics (implicit use of history): B

Labels (i)-(iv) in this table refer to the sub-items in §2.2.2.C and provide an example of the "irreducibility" idea mentioned above; the development of mathematical awareness has been described clearly in the ICMI Study volume, allowing for a clearer classification of its various aspects[8]. The same holds for learning mathematical topics by following an approach explicitly, or implicitly inspired by history (cf. §2.1, last paragraph), but not so for learning history by providing direct historical information.

Table 3: The classification of Jankvist's ***hows*** (cf. §2.2.3)

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**[7]	*Modules* approaches: B	*Illumination* approaches: B *History-based* approaches(?): C
***History as a tool* (emphasis on *inner-issues*)**[7]	*Illumination* approaches: A *Modules* approaches: B	*History-based* approaches: C

[7]Relating *History-as-a-goal* and *History-as-a-tool* with *inner-issues* and *meta-issues*, respectively is done keeping in mind the possible cross-interrelations mentioned in §2.1!

[8]E.g. items 3.2.3(b) definitely concern *meta-issues*; (ii) requires to consider issues in their historical context of a particular period; (i) (iii) (iv) touch upon issues that connect the present to the past and are likely to be based on a *heritage*-like approach, though this is clearer for (ii) & (iv) than for (i). On the other hand, 3.2.3(a)(i), (iii) mainly concern specific historical examples whose mathematical content should be explored in a way that awareness of *meta-issues* may be developed (or be *anchored* there, in the sense of §2.1).

This table suggests that Jankvist's *hows* constitute a promising identification of broad categories of approaches to be analysed into more sharply described ones (this is already apparent in Jankvist 2009c, §6).

4 An example

To illustrate the general classification of section 3, we consider a specific example, namely, the network of interrelated notions

Power of numbers–exponent–logarithm & (logarithmic) base–exponential function–logarithmic function

In teaching and learning these concepts, or any other mathematical subject, questions are often raised, whose answer presuppose both teacher's knowledge of the historical development and (meta)knowledge of how to deal with this historical knowledge in the classroom. The range and depth of this knowledge is closely related not only to the subject itself and the questions raised, but also to the learners' age, the level of instruction, the didactical aims and how the subject fits into the curriculum. These factors determine to a large extent the relation between HM and ME in the specific subject to be taught and learnt.

We approach such possible questions as specific didactical *tasks* to be accomplished didactically, thus reflecting the *whys*[9] and corresponding answers as an outline of possible *approaches* to do that, thus reflecting the *hows* in section 3[8]. In this way we will illustrate the fitting of the *whys & hows* into the 2X2 classification scheme of section 4. However, we would like to emphasize that the formulation of the questions below and the outline of their answer constitute only one **choice** among various possibilities and that different questions and/or different answers could be provided, depending on the factors mentioned in the previous paragraph. We simply aim to indicate how the suggested classification is applied in a particular case, **once** we have specified the questions raised and their possible answers.

For the reader's convenience, we indicate *questions* by • and *answers* by ► and we refer directly to their relation with the numbering of *whys & hows* in section 2, respectively. We note that the questions and answers as they appear below have been formulated based on the literature of the last 25 years or so, on a historically motivated teaching of logarithms (e.g. Katz 1986, 1995; Thomaidis 1987; Toumasis 1993; Fauvel 1995; van Maanen 1997; Clark 2006; Stein 2006; Barbin et al. (2006); Panagiotou 2011).

The concept of logarithm, which is usually, introduced for the first time to high school students (16-17 years old) in the context of teaching of the exponential and logarithmic functions, raises several questions, already from its definition.

• **q_1**: Why the exponent of a number's power is called the *logarithm* of that power relative to that number as a base? (A1, B2)

Any attempt to answer this question cannot avoid direct, or indirect reference to the HM:

►**a_1**: When the concept of logarithm was invented in the early 17th century, the modern exponential notation of powers did not exist yet, hence the logarithm was **not** defined as the exponent of a power relative to a base. (2.2.2.A, 2.2.3.A)

This answer immediately leads to new questions:

• **$q_{2\text{-}1}$**: Why was the concept of logarithm introduced at all? (C1)
• **$q_{2\text{-}2}$**: How was the logarithm defined originally? (A2)
• **$q_{2\text{-}3}$**: Why was the **term** "logarithm" used/introduced? (A2)

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[9]Cf. the first paragraph of §2.2.

To answer such questions, sufficiently deep historical knowledge is required, hence the choice and use of appropriate historical sources is raised (a book on the HM, a treatise on the history of logarithms, relevant original sources etc) and whether the answers to be used will be accompanied by historical references or not, e.g.

►**a_2**: The answer could be limited to a modern, strictly mathematical framework, if the correspondence between an arithmetical and geometrical progression is used, from which it can be easily explained the usefulness of the logarithm as a tool to simplify numerical calculation and the etymology of the word "logarithm"[10]. History is used implicitly; no traces are left in the classroom. (2.2.2.B-implicit use of HM; 2.2.3.C)

This answer raises new questions:

• **q_3**: How can this model be useful in the classroom? (C4)

►**a_3**: The answer to this question too can also be restricted in a modern, strictly mathematical context, explaining technically how to construct a 'dense" geometrical progression and a corresponding "dense" arithmetical one. (2.2.2.B-implicit use of HM, 2.2.3.C)

Though it may be explained that at that era numerical calculations with many-digit numbers was time-consuming, tedious job – hence there was vivid interest to exploit any idea on their simplification, like the correspondence of arithmetic and geometric progressions -, it is readily appreciated that the same holds for the construction of two practically useful "dense" progressions. Hence, new questions arise naturally:

• **$q_{4\text{-}1}$**: What was the motivation to get involved in the construction of two practically useful "dense" progressions? (C1)

• **$q_{4\text{-}2}$**: Is it really true that this problem is of such great mathematical interest that it is worthwhile to get involved in it, ignoring the practical difficulties inherent to its solution? (C3)

►**a_4**: The problem of simplifying tedious numerical calculations was posed in the context of dealing with economic relations (e.g. using tables of interests) and astronomical measurements (use of trigonometric tables). Therefore, answering these questions leads outside mathematics, and refers directly to the relation of Mathematics to other disciplines in that historical period, which evidently, cannot be considered solely in mathematical terms. (2.2.2.A; 2.2.3.B)

This leads to new questions.

• **q_5**: How did people cope with elaborated calculations in various disciplines, including Mathematics? (A3, E1, E2)

►**a_5**: The answer will definitely refer to the reciprocal relation between Mathematics and other disciplines which are based/using Mathematics, elaborating on the use of groups of calculators paid by financial institutions or observatories, the development of new methods to construct numerical tables of higher accuracy, the invention of tricky methods to simplify calculation (like the trick of "prosthaphairesis"[11]) etc. (2.2.2.C.2(ii), 2.2.3.B)

• **q_6**: Who was the first to construct logarithmic tables, and how did he achieve to that? (C4)

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[10]*Logarithm*: From the Greek *logos* (ratio) and *arithmos* (number); every term of the arithmetic progression shows the number (multitude) of the ratios of successive terms of the geometrical progression up to the corresponding term of this progression. E.g., 6 in the arithmetic progression 0,1,2,3,4,5,6 (which equals $log_2 64$) indicates that to get the term 64 of the geometric progression 1,2,4,8,16,32,64 starting from its first term, six ratios are inserted, namely, 2:1, 4:2, 8:4, 16:8, 32:16, 64:32.

[11]From the Greek *Prosthesis* = addition and *aphaeresis* = subtraction; a trick based on trigonometric relations to transform the product of two trigonometric numbers into sums of such numbers (Smith 1959, pp.455-472; Barbin et al 2006, ch.II; Thomaidis 1987, §3).

►**a_6** This is a complex question the answer to which includes references to the not rare fact of "independently made similar or identical discoveries/inventions" in mathematics; the arithmetical background of Bürgi's "*red numbers*" and the kinematical-trigonometric background of Napier's "*logarithms*". A historically complete answer should stress the essential differences between these two approaches; that is, the fact that they essentially concern different conceptions of "logarithmic" notion, though of course they have as a common starting point the correspondence between arithmetic and geometric progression and serve the same purpose. In addition, it is important from a didactical point of view to study biographical elements of the scientists involved in the invention of logarithms. (2.2.2.A; 2.2.3B)

Though the logarithmic tables solved the crucial – at that time - problem faced by those involved in complex arithmetical calculations and their use was adopted with enthusiasm, nowadays their use has banished and what remains is the concept of logarithmic function relative to a given base. Given that at the time of the invention of logarithms both the idea of a base and the function concept were nonexistent, the following questions naturally arise:

- **$q_{7\text{-}1}$**: What was the reason and/or questions that led to the connection of the concept of logarithm with those of exponent and base? (B1)
- **$q_{7\text{-}2}$**: What was the reason and/or questions that led to the connection of the concept of logarithm with the concept of a "logarithmic function"? (B1, C1)

►**a_7: ** Answers to these questions could be given in a strictly mathematical context with no, or only limited reference to the historical development. However, given that this historical development is closer to contemporary Mathematics, many of the original texts are suitable for reading and discussion in the classroom. In this way, the integration of the HM could be more direct, efficient and demanding, contributing to the development of a classroom discourse, resembling a "community of researchers" which explores mathematical questions and problems and deepens its understanding of Mathematics by studying historical texts. (2.2.2.B-explicit use of HM; 2.2.3.B)

The above questions and answers can be rearranged and seen in the context of the classification scheme of section 4 in the following table.

Table 4: The classification of questions and answers on logarithms

	History	***Heritage***
***History as a goal* (emphasis on *meta-issues*)**	a_4 (2.2.2.A, 2.2.3.B) q_5 (E1, E2) a_5 (2.2.2.C, 2.2.3.B) a_6 (2.2.2.A, 2.2.3.B)	a_1 (2.2.2.A, 2.2.3.A) $q_{7\text{-}1}$ (B1) $q_{7\text{-}2}$ (B1)
***History as a tool* (emphasis on *inner-issues*)**	$q_{2\text{-}1}$ (C1) q_3 (C4) $q_{4\text{-}1}$ (C1) $q_{4\text{-}2}$ (C3) q_5 (A3) a_5(2.2.2.C, 2.2.3.B) q_6 (C4) a_6(2.2.2.A, 2.2.3.B) $q_{7\text{-}2}$ (C1) a_7(2.2.2.B, 2.2.3.B)	q_1 (A1, B2) $q_{2\text{-}2}$ (A2) a_2 (2.2.2.B, 2.2.3.C) $q_{2\text{-}3}$ (A2) a_3(2.2.2.B, 2.2.3.C) q_6 (A2)

Remarks: (a) If the questions and answers were formulated more sharply and in more detail, we expect that they would not appear in more than one cell each.

(b) The arrows link questions, which refer to meta-issues with answers connected to inner issues; maybe this illustrates the *anchoring* process mentioned in section 3. This is an idea on which to elaborate more.

5. Concluding remarks

This paper is theoretical and much work remains to be done by analysing specific examples to check the validity of the basic ideas, their usefulness in actual implementation and their efficiency to better understand different aspects of which and how HM could be integrated in ME and for what purpose. A preliminary illustration of the classification schemes by means of a specific example was given in section 4, which should be considered only as a first step towards a better understanding and sharpening of this classification. We believe this is a promising line of inquiry that will sharpen the arguments for and approaches of integrating HM into ME and will better reveal possible interrelations of the conceptual dipoles introduced here.

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