MODELLING IN CLASSROOM

‘Classical Models’ (in Mathematics Education) and recent developments

Hans-Stefan SILLER
Universität Salzburg, Iffb-Fachdidaktik Mathematik & Informatik, Salzburg, Austria
hans-stefan.siller@sbg.ac.at

ABSTRACT

The influence of modelling in mathematics education is gaining in importance since applied mathematics has become popular in classrooms again. Today it is one of the most challenging topics for teachers and students in school. The extraordinary part of such modelling tasks in classroom is traversing the modelling cycle: therefore questions have to be formulated; data has to be found and connected to an appropriate model, so that conclusions could be drawn by certain interpretations.
In particular by taking ‘classical models’ into account in education, e.g. models in / of financial mathematics or applications in trigonometry, the idea of modelling in classroom can be shown as a traditional aspect in mathematics education with a long history.
Considering this idea mathematics can be shown as a useful and applicable instrument in daily life to students, like Freudenthal, Krygovska, Pollak or Steiner stated it in 1968 at the conference "How To Teach Mathematics so as to be Useful".

1 Interests for mathematical modelling

Arguments in the discussion of mathematics education for learning mathematical modelling are outlined and listed by Barbosa (2003). Modelling tasks should …

- motivate the learners / students,
- facilitate the learning of mathematical contexts,
- prepare the learners to apply mathematical knowledge in different areas,
- contribute to the development of general and mathematical competencies,
- guide the learners to understand the socio-cultural role of mathematics which is gaining importance in the community.

The arguments are comprehensible in the way mathematics educators are acting. But the specific advantages of modelling-tasks are not that obvious. The author does not offer any theoretical suggestions or empirical results to justify his arguments.

A possible motivation of the listed arguments can be found in Blum (2003). He argues that mathematical modelling is one of the reasons why people are learning mathematics and simultaneously describes the manner how people are learning mathematics. Hence mathematical modelling provides arguments for the importance of learning mathematics, it even supplies arguments for the whys and wherefores.

As a first conclusion it is possible to find two different perspectives combined in one idea: Mathematics is helpful to learn mathematical modelling on the one hand, and on the other hand mathematical modelling is helpful to learn mathematics (cf. Ottesen, 2001). Reasons for this perception are brought by mathematical history.

2 Modelling and its history

The importance of mathematical modelling founded through applied science, in particular applied mathematics. This discipline was strongly connected to its neighbouring disciplines like physics, astronomy or engineering sciences (cf. Blum, Galbraith, Henn & Niss, 2007). A symbiosis between real and applied mathematics can be detected since the 19th century. From there on trends preferring the real mathematics or the applied mathematics can be located (cf. Kilpatrick, 1992).

Such a trend can be identified in the 1960s to the 1970s, the ‘New Math Reform’. Representatives of this movement claimed that children should learn to think logically and abstract very early. As a consequence the theory of sets was introduced to primary schools. At the same time contrary views were brought to public, in particular of persons working in tertiary education. They claimed to consider more applications in mathematics. The reason for it can be found in the UK for example. Troubles got obvious, because alumni of mathematical studies were not able to take their knowledge for solving real problems despite excellent certificates.

The initial point for observing applications in mathematics (education) and mathematical modelling can be found in the conference of Freudenthal (1968) "WHY TO TEACH MATHEMATICS SO AS TO BE USEFUL". In the first edition of "Educational Studies in Mathematics" (1968) articels of participants presented at that conference can be found. The papers of Freudenthal (1968), Pollak (1968) and Klamkin (1968) should be mentioned eminently.

Pollak (1968) quotes – in opposition to the impacts of that time – the importance of applied mathematics: "To be against applications of mathematics in teaching is like being against motherhood and in favor of sin." Klamkin (1968) quotes the importance of mathematics for applications: "… being useful means not the teaching of applied mathematics but the teaching of mathematics such that it can be applied. Nevertheless, the two must be intertwined for each to be more meaningful."

The interdependence of mathematics and applications in German speaking countries is discussed not until the conference "Anwendungsorientierte Mathematik" (cf. Dörfler & Fischer, 1977) took place. Aspects of applied mathematics for education are included and discussed from a scientific perspective the first time. A next milestone for introducing mathematical modelling can be found in Pollak (1979). In his chapter XII, *The interaction of mathematics and other school subjects*, Pollak defines the field of applied mathematics that should be integrated to a modern mathematics classroom situation:

1. "Applied mathematics means classical applied mathematics.
2. Applied mathematics means all mathematics that has significant practical application.
3. Applied mathematics means beginning with a situation in some other field or in real life, making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation.
4. Applied mathematics means what people who apply mathematics in their livelihood actually do. This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times".

Through this characterizations of applied mathematics it is possible to describe this proceeding exchange from the "rest of the world" and "(applied) mathematics" the first time. A first modelling cycle is defined and can be described very simple by a little graphic:

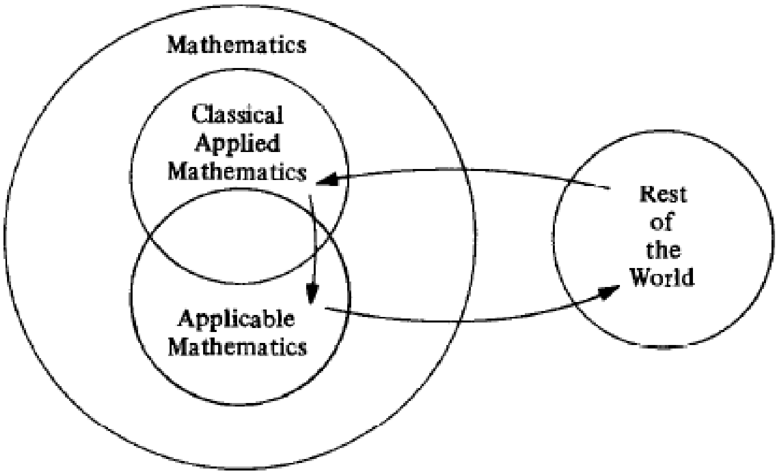

Figure 1. Modelling cycle of Pollak (1979)

Pollak (1979) illustrates the figure as follows: "In this picture the left-hand side shows mathematics as a whole, which contains two intersecting subsets we have called classical applied mathematics and applicable mathematics. Classical applied mathematics represents definition (1) and applicable mathematics, definition (2). Why doesn’t (2) contain all of (l)? The overlap between these is great, but it is not true that all of classical applied mathematics is currently applicable mathematics. There is much work in the theory of ordinary and partial differential equations, for example, which is of great theoretical interest but has no applications which are visible at the moment. Such work is included in definition (1) as classical applied mathematics, since this contains all work in differential equations; on the other hand, if it is not currently applicable, it does not belong in definition (2).

The rest of the world includes all other disciplines of human endeavour as well as everyday life. An effort beginning in the rest of the world, going into mathematics and coming back again to the outside discipline belongs in definition (3). Definition (4) involves, as will be seen, going around the loop many times."

3 Modelling and its visualization – modelling cycles

The discussion about modelling in mathematics education was established by the accomplishments of Pollak, in particular the introduction of real-world-problems in mathematics education was favored by mathematics educators. This basic idea (cf. Klika, 2003) was discussed intensively afterwards. A lot of suggestions for implementing mathematical modelling in education were created and can be found in literature (cf. ISTRON – www.istron-gruppe.de; cf. MUED – www.mued.de).

On the basis of the shown considerations different modelling cycles were created in addition. In 1985 two further modelling cycles were presented by Fischer and Malle (1985) as well as by Blum (1985). The last one is the one which is taken into account in its main features since today.

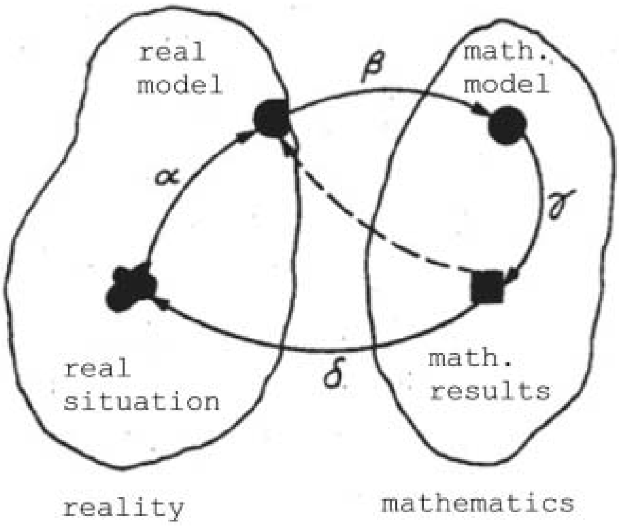

Figure 2. Modelling cycle of Blum (1985)

Fischer and Malle (1985) emphasize, that applying mathematics is a kind of process, which can be divided into several modules, which can also be passed through several times, so that the model itself can be advanced. The development of modules does not have to follow the specific schedule – sometimes some modules have to be observed, sometimes they are strongly connected to others, so that a differentiation is complicated, sometimes some module can be missed.

Following those instructions it is getting obvious that mathematical modelling is a complex activity for mathematics education. Students have to have a lot of different competencies in different (mathematical) areas (e.g. competencies in communication, mathematical basic competencies, competencies in reflection (mathematical) facts), especially if one wants to consider the complexity of this idea.

In particular Fischer and Malle are anticipating with the model of Blum and Leiss (2005, p. 19), which was created on the basis of Blum’s shown modelling cycle. This modelling cycle is the basement for all modelling activities and modifications on modelling cycles nowadays.

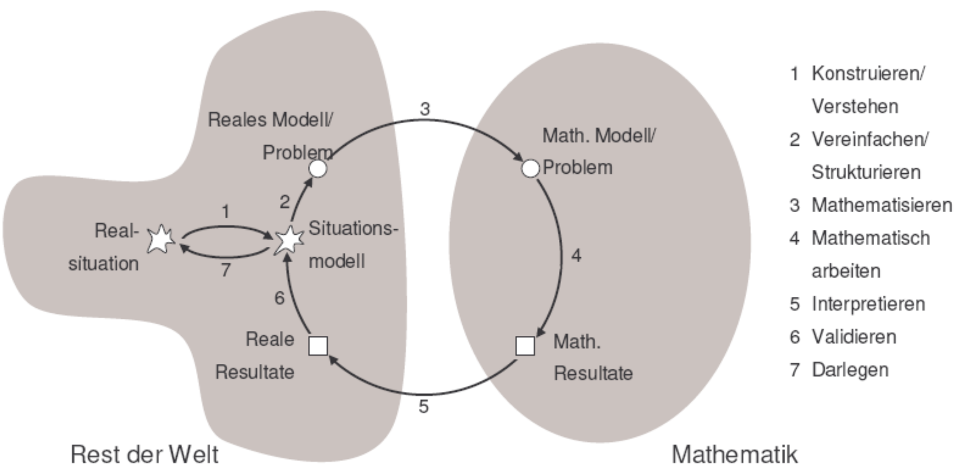

Figure 3. Modelling cycle of Blum & Leiss (2005)

In literature a lot of further modelling cycles can be found, e.g. those of Müller and Wittmann (1984), Schupp (1987) or Burhardt (1988). The last one seems to be worth mentioning because this modelling cycle is designed like a flow-chart-diagram. In particular it is possible to draw more attention to processes of system-dynamics with its help as well as the combination to other disciplines, e.g. computer-science (-education), can succeed.

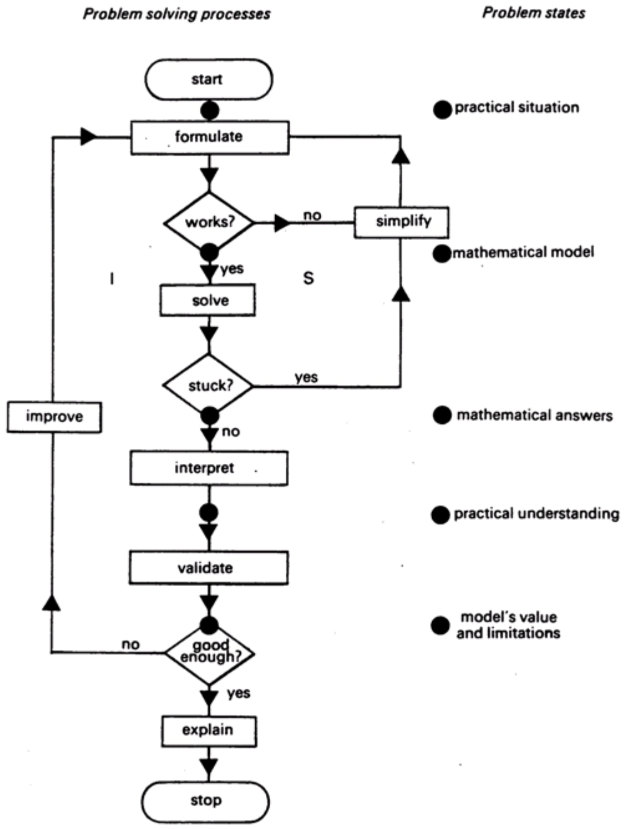

Figure 4. Modelling cycle of Burkhardt (1988)

In contrast to the mentioned modelling cycles Burkhardt's starting point is an arbitrarily one. That can also imply that the starting point cannot be found in a practical situation; it even can be situated in an inner-mathematical problem, e.g. modelling with the help of linear regression, in a list of measurement readings, with a certain amount of measuring points. It is not necessary that a real-life situation in the environment of the problem must be found. I do not concentrate on such a situation in the following part, although the chosen situations could be interpreted like this.

4 A showcase for modelling with historical material

If we have a closer look at an egg, we will see that its shape is very harmonic and impressive. Considering a hen's egg it is obvious that the shape of all those eggs is the same. Because of the fascinating shape of eggs I tried to think about a method to describe the shape of such an egg with mathematical methods. Searching the literature I found some material from Münger (1894), Schmidt (1907), Wieleitner (1908), Loria (1911), Timmerding (1928) and Hortsch (1990). The book of Hortsch is a very interesting summary about the most important results of 'egg-curves'. He also finds a new way for describing egg-curves by experimenting with known parts of 'egg-curves'. The modality how the authors are getting 'egg-curves' is very fascinating. But none of them has thought about a way to create a curve by using elementary mathematical methods. The way how the authors describe such curves are not suitable for mathematics education in schools. So I thought about a way to find such curves with the help of well known concepts in education. My first starting point is a quotation of Hortsch (1990): "The located ovals were the (astonishing) results of analytical-geometrical problems inside of circles." The second point of origin are the definitions of 'egg-curves' found by Schmidt (1907) and presented in Hortsch (1990).

Definition 1 (cf. Schmidt, 1907):

Schmidt quotes: "An ‚egg-curve' can be found as the geometrical position of the base point of all perpendiculars to secants, cut from the intersection-points of the abscissa with the bisectrix, which divide (obtuse) angles between the secants and parallel lines in the intersection-points of secants with the circle circumference in halves. The calculated formula is $r = 2 \cdot a \cdot \cos^2\varphi$ or $(x^2+y^2)^3 = 4 \cdot a^2 \cdot x^4$".

In education the role of modelling is gaining in importance. Different systems, like computer-algebra-systems (CAS), dynamical-geometry-software (DGS) or spreadsheets, are used in education. With the help of technology it is possible to design a picture of the given definition immediately. In the first part I use a DGS because with its help it is possible to draw a dynamical picture of the given definition. The DGS I am using is GeoGebra. It is free of charge and very suitable in education because of its handling.

First of all the one point P of such an egg as it is given in the definition has to be constructed.

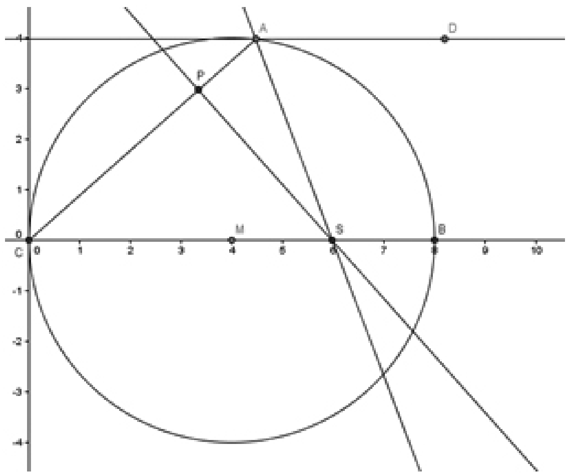

Figure 5. Construction by obtaining the definition of Schmidt (1907)

According to the instruction a circle (center and radius arbitrarily) is constructed first, then a secant from C to A (points arbitrarily). After that a parallel line to the x-axis through the point A (= intersection point secant-circle) is drawn and the bisecting line CAD is determined, which is cut with the x-axis. So the point S is achieved. Now the perpendicular to the secant through S can be drawn. The intersection point of the secant and the perpendicular is called P and is a point of the 'egg-curve'. By activating the "Trace on" function and using the dynamical aspect of the construction the point A moves towards the circle and the 'egg-curve' is drawn as Schmidt (1907) has described it. This can be seen in the following figure:

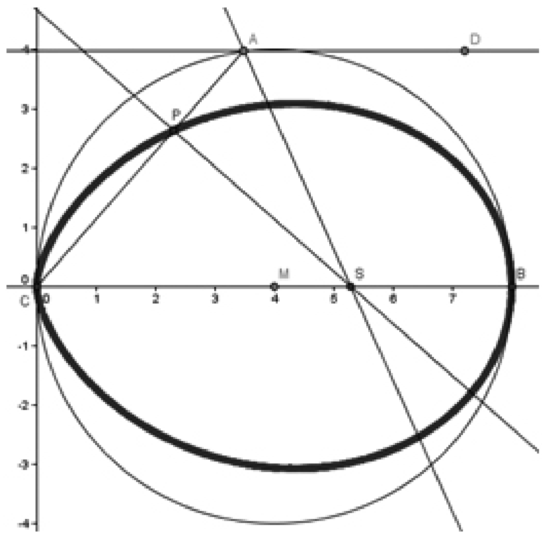

Figure 6. An ‘egg-curve’

Now a way to calculate the formulas $r = 2\cdot a\cdot \cos^2\varphi$ or $(x^2+y^2)^3 = 4\cdot a^2\cdot x^4$ as mentioned above has to be found.
Let us start with the following figure:

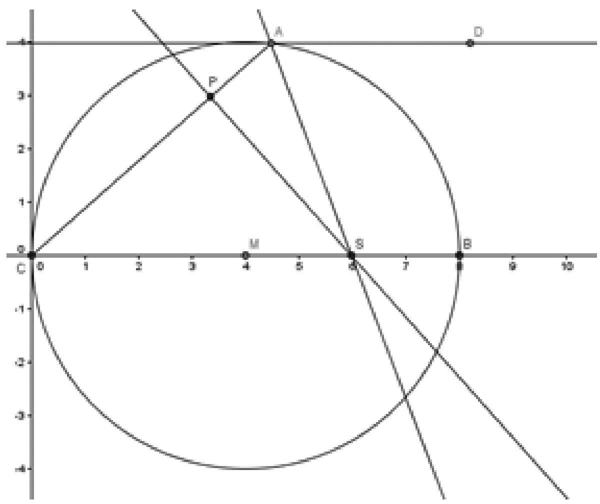

Figure 7. Initial situation for calculating the equations of the egg-curve

Because of the construction the triangle CPS is right-angled. Furthermore it can be recognized that the distance CP and PS is the same and that the triangle CAB is also rectangular, because it is situated in a semicircle. This can be seen in the following figure, where I have also drawn the real ‘egg-curve’ as dashed line.

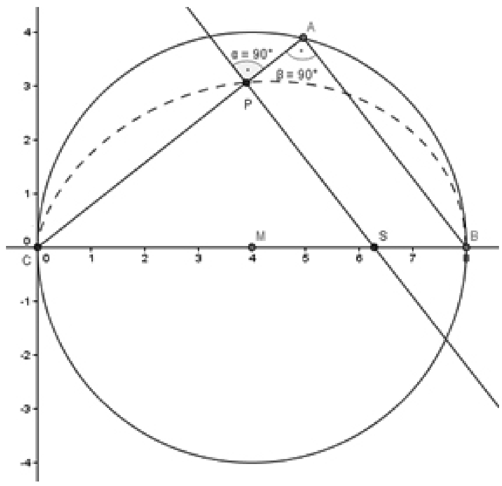

Figure 8. Observing the triangles

Because of the position of the points C (0, 0), A (x, y), B (2·r, 0) and the construction instruction the coordinates of point S and P can be calculated. Therefore only a little bit of vector analysis is necessary. The calculation can be done in the CAS Mathematica. First of all the points and the direction vector of the bisecting line w need to be defined:

```
{c = {0, 0}, a = {x, y}};
a - c
{x, y}
1/Norm[a - c] (a - c)
```

$$\left\{\frac{x}{\sqrt{\mathrm{Abs}[x]^2+\mathrm{Abs}[y]^2}},\ \frac{y}{\sqrt{\mathrm{Abs}[x]^2+\mathrm{Abs}[y]^2}}\right\}$$

$$w = -1\Big/\sqrt{(x)\wedge 2 + y\wedge 2}\ \{x,\ y\} + \{1,\ 0\}$$

$$\left\{1-\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}}\right\}$$

$$w = \left\{1-\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}}\right\};$$

Now the equation of the normal form of the bisection line can be calculated. It is cut with the x-axis and defines the intersection-point S.

```
vn = {y, √(x)^2 + y^2 - x};
vn.{u, v} == vn.a
```

$$u\,y + v\left(-x + \sqrt{x^2 + y^2}\right) == x\,y + y\left(-x + \sqrt{x^2 + y^2}\right)$$

```
vn.{u, v} == vn.a /. v → 0
```

$$u\,y == x\,y + y\left(-x + \sqrt{x^2 + y^2}\right)$$

```
First[%] / y == Last[%] / y
```

$$u == \frac{x\,y + y\left(-x + \sqrt{x^2 + y^2}\right)}{y}$$

```
Simplify[%]
```

$$\sqrt{x^2 + y^2} == u$$

```
s = {√x^2 + y^2 , 0};
```

Now the intersection point P of the secant and the perpendicular through S is calculated.

```
Solve[{x u + y v == (x) (√(x)^2 + y^2), -y u + (x) v == 0}, {u, v}]
```

$$\left\{\left\{u \to \frac{x^2}{\sqrt{x^2 + y^2}},\ v \to \frac{x\,y}{\sqrt{x^2 + y^2}}\right\}\right\}$$

$$p = \left\{\frac{x^2}{\sqrt{x^2 + y^2}},\ \frac{x\,y}{\sqrt{x^2 + y^2}}\right\};$$

All important parts for finding the ‘egg-curve‘ are calculated. Let us have a closer look at figure 8. It is easy to recognize that there are two similar triangles – triangle CPS and triangle CAB. The distance CP shall be called r and the radius of the circle shall be called a. The distance CB has now the length 2·a. The other two distances which are needed CA and CB have the length $\sqrt{x^2 + y^2}$. The similarity of the triangles is applied:

$$\frac{CA}{r} = \frac{2 \cdot a}{CS} \Leftrightarrow \frac{\sqrt{x^2 + y^2}}{r} = \frac{2 \cdot a}{\sqrt{x^2 + y^2}}$$

Transforming this equation gives:

$$x^2 + y^2 = 2 \cdot a \cdot r$$

By using the characteristic of the right-angled triangle CAB and calling the angle ACB φ the cosine of this angle is obtained:

$$\cos\varphi = \frac{\sqrt{x^2+y^2}}{2\cdot a} \Leftrightarrow$$

$$4\cdot a^2\cdot \cos^2\varphi = x^2+y^2$$

Inserting this connection in the equation gives:

$$4\cdot a^2\cdot \cos^2\varphi = 2\cdot a\cdot r$$

Shortening this equation shows:

$$r = 2\cdot a\cdot \cos^2\varphi$$

By substituting r and cosφ it is possible to get the implicit Cartesian form, mentioned in the definition:

$$\sqrt{x^2+y^2} = 2\cdot a\cdot \cos^2\varphi$$

$$x^2+y^2 = 4\cdot a^2\cdot \frac{x^4}{r^4}$$

$$x^2+y^2 = 4\cdot a^2\cdot \frac{x^4}{(x^2+y^2)^2}$$

respectively

$$(x^2+y^2)^3 = 4\cdot a^2\cdot x^4$$

As it can be seen the ‘egg-curve’ has been modelled by elementary mathematical methods. Through using technology teachers and students get the chance to explore such calculations by using the pivotal of modelling. Through such calculations the necessity of polar-coordinates can get obvious.

Definition 2 (cf. Münger, 1894):

Another construction instruction is formulated by Münger (1894). He quotes:

> "Given is a circle with radius a and a point C on the circumference. CP_1 is an arbitrarily position vector, P_1Q_1 the perpendicular to the x-axis, Q_1P the perpendicular to the vector. While rotating the position vector around C point P is describing an egg-curve. The equation of this curve is $r = a\cdot\cos^2\varphi$ in Cartesian form $(x^2+y^2)^3 = a^2\cdot x^4$."

As it is given in the construction instruction a circle (radius arbitrarily) and a point C on the circumference of the circle is constructed. Then an arbitrarily point P_1 on the circumference of the circle can be construed. The perpendicular to the x-axis is construed through P_1 which is cut with the x-axis and delivers Q_1. After that the perpendicular to the secant CP_1 through Q_1 is construed. All these facts can be seen in the following picture:

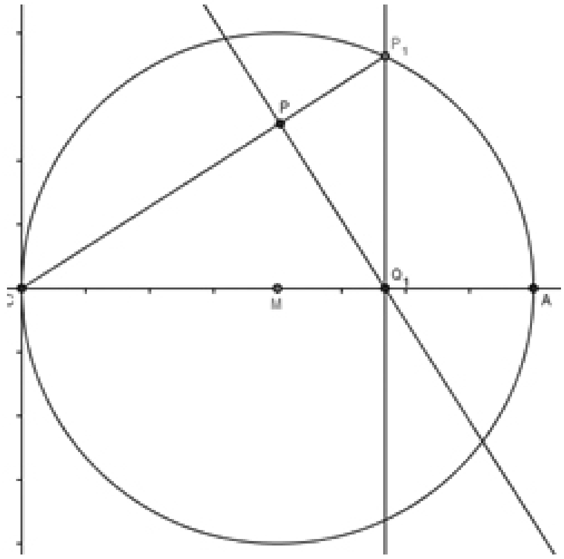

Figure 9. Construction by obtaining the definition of Münger (1897)

If the point P_1 is moved toward the circle, P will move along the ‘egg-curve’. It will be easier to see if the "Trace on" option is activated.

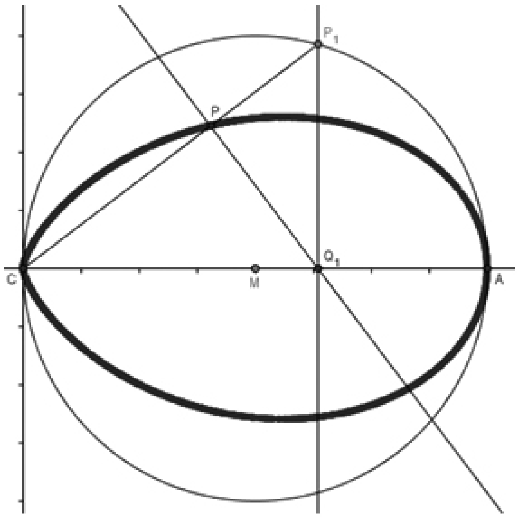

Figure 10. Another ‘egg-curve’

The formula given by Münger (1894) can be found in a similar way as the other formula was found. The most important fact which has to be seen here is that in this picture two rectangular triangles CPQ_1 and CP_1A exist. Those triangles are similar.

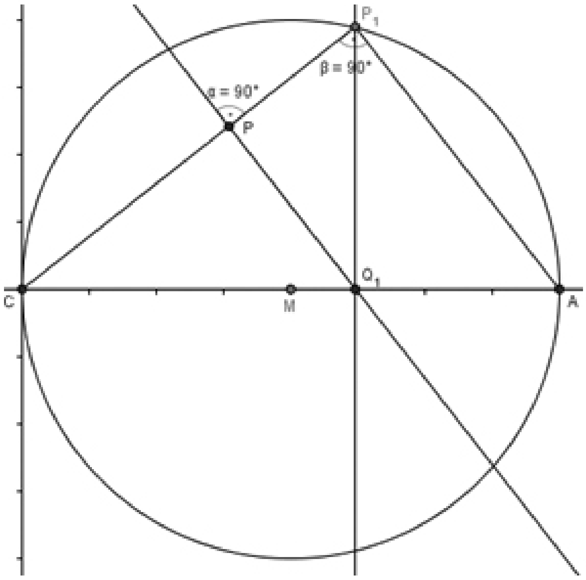

Figure 11. Observing the triangles

The coordinates of the points can be found mentally – without any calculation:
C (0, 0), P_1 (x, y), A (2·a, 0), Q_1 (x, 0)
If the distance CP is called r, then the coordinates of P will not be used. Otherwise they can be calculated analytically. For the sake of completeness I write down the coordinates of P:

$$P\left(\frac{x\cdot y^2}{x^2+y^2}, \frac{y^3}{x^2+y^2}\right)$$

By using the similarity of the both triangles the following equation will be obvious:

$$\frac{\sqrt{x^2+y^2}}{r} = \frac{2\cdot a}{x}$$

Through elementary transformation, because of the mathematical fact $\cos\varphi = \frac{\sqrt{x^2+y^2}}{2a}$ in the triangle CP_1A and substitution of the term $\sqrt{x^2+y^2}$ by 2·a·cosφ the following equation is calculated:

$$2\cdot a\cdot x\cdot \cos\varphi = 2\cdot a\cdot r$$

The result is:

$$r = x\cdot \cos\varphi$$

Because of the fact (assumption in the calculation) that the x is part of our circle – it is the x-coordinate of P_1 – x can be substituted by x = a·cosφ, with a as the radius of the starting circle. So the formula of Münger is found with polar-coordinates:

$$r = a\cdot \cos^2\varphi$$

If the implicit cartesian form should be stated, another substitution has to be done. The result is:

$$(x^2+y^2)^3 = a^2\cdot x^4$$

Epilogue

As it can be seen mathematical modelling is – more or less – an important part in mathematics and mathematics education (cf. Pollak, 2003, 2007). The importance of application-oriented mathematical classroom situation is unquestioned. In combination with mathematical modelling any use of mathematical concepts, methods for describing real-life-situations with the help of mathematics and solving of them is intended. One goal of mathematics education should be a demand for the development of modelling-skills of students. These skills can be met if one is able to recognize the modelling-capacities of mathematical contents and features and if one is able to apply heuristical strategies for supporting the mathematization of real contexts and the application of mathematical contents and features. But such skills can only be developed successfully if students are working active, independently on their own. The usage of historical material respectivelly historical ideas of mathematicians or historical techniques in mathematics can support this.

By bringing 'classical models', e.g. Bateman-function(s) (cf. Siller, 2010), Pell's equation or the Easter formula, into classrooms two important aspects of curricula are met: the role of history in mathematics education and mathematics is increased in students' minds as well as the importance of mathematical modelling documented by the history of mathematics itself. Questions students have to deal with should always be answered and discussed by considering (scientific) findings of a connated reference discipline and a strong focus on the practical implementation by observing mathematics education in reality. So students are able to recognize that the history of mathematics education is important

- for mathematics itself, because it would be hard to answer the question "What ist mathematics?" without knowing anything about its history.
- for the learning of mathematics itself, because students are able to recognize the usefullness and acceptability of this subject by recognizing mathematics as a historical phenomenon.

Summing up the idea of modelling under respect to historical developments in mathematics (education) allows to find interesting problems and a methodological plurality so that one is gaining a deep and serious insight into the discussed topic(s) and problem(s) instead of using standard methods.

REFERENCES

– Barbosa, J. C., 2003. "What is mathematical modelling?", in S. J. Lamon, W. A. Parder &. K. Houston (Eds.*), Mathematical modelling: a way of life,* Chichester: Ellis Horwood, 227-234.

– Blum, W., 1985, „Anwendungsorientierter Mathematikunterricht in der didaktischen Diskussion" in *Mathematische Semesterberichte. Zur Pflege des Zusammenhangs zwischen Schule und Universität,* Band XXXII, (195-232). Vandenhoeck & Ruprecht: Göttingen.

– Blum, W.,(2003, *ICMI Study 14: Applications and Modelling in Mathematics Education. In Educational Studies in Mathematics*, no. 51, New York: Springer, 149-171.

– Blum, W., Galbraith, P., Niss, M., Henn, H.-W., 2007, *Modelling and applications in mathematics education*. New ICMI Studies Series, no. 10. New York: Springer.

– Blum, W., Leiss, D., 2005, Modellieren im Unterricht mit der „Tanken", *Aufgabe. In: mathematik lehren*, Heft 128, 18-21.

– Burkhardt, H., 1988, *The Real World and Mathematics*, Blackie and Son Limited: Glasgow.

– Dörfler, W., Fischer, R., 1977, *Anwendungsorientierte Mathematik in der Sekundarstufe* II. Heyn: Klagenfurt.

– Fischer, R., Malle, G., 1985, *Mensch und Mathematik*. Wien: Bibliographisches Institut.

– Freudenthal, H., 1968, „Why to teach mathematics so as to be useful" *Educational Studies in*

Mathematics, **1**, pp. 3-8.

- Hortsch, W., 1990, *Alte und neue Eiformeln in der Geschichte der Mathematik*, München: Selbstverlag.
- Kilpatrick, J., 1992, "A History of Research in Mathematics Education.“, in Grouws, D.A. (ed.), Handbook *of Research on Mathematics Teaching and Learning*, New York: Macmillan, 3-38.
- Klamkin, M.S., 1968, „On the teaching of mathematics so as to be useful”., Educational Studies in Mathematics, **1**, 126-160.
- Klika, M., 2003, „Zentrale Ideen - echte Hilfen“, Mathematik lehren, **119**, (4-7.
- Loria, G. (1911). Ebene Kurven I und II, Berlin: Teubner.
- Müller, G., Wittmann, E. Ch., 1984, *Der Mathematikunterricht in der Primarstufe*, Braunschweig Wiesbaden: Vieweg.
- Münger, F., 1894, *Die eiförmigen Kurven*, Dissertation, Universität Bern.
- Ottesen, J. T., 2001, "Do not ask what mathematics can do for modelling", in D. Holton (Eds.), *The teaching and learning of mathematics at university level: an ICMI study*, Dordrecht: Kluwer, 335-346.
- Pollak, H.O., 1968, "On Some of the Problems of Teaching Applications of Mathematics.“, Educational Studies in Mathematics, **1**,24-30.
- Pollak, H.O., 1979, "The interaction between mathematics and other school subjects", in The international commission on mathematics instruction, IV, Paris: ICMI, 232-248.
- Pollak, H.,O., 2003, "A history of the teaching of modelling", in G. Stanic & J. Kilpatrick (Eds.), *A History of School Mathematics*, Reston: NCTM, 647-671.
- Pollak, H.O., 2007, "Mathematical modelling: A conversation with Henry Pollak", in W. Blum, P. L. Galbraith, H. Henn & M. Niss (Eds*.), Modelling and Applications in Mathematics Education: The 14th ICMI Study*, (Berlin: Springer, 109-120).
- Schmidt, C.H.L, 1907, „Über einige Kurven höherer Ordnung, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht“, 38. Jg..
- Schupp, H., 1987, "Applied mathematics instruction in the lower secondary level: between traditional and new approaches.“, in Blum, Werner et al. (Eds.):, *Applications and modelling in learning and teaching mathematics*, Chichester: Horwood.
- Siller, H.-St., 2010, "Consuming Alcohol - a topic for teaching mathematics?", in A. Araujo, A. Fernandez, A. Azevedo, J. F. Rodriguez (Eds.), *EIMI 2010 Conference: Educational Interfaces between Mathematics and Industry*.,Lisbon: Centro Internacional de Matemática, 481-491.
- Timmerding, H.E., 1928, *Zeichnerische Geometrie*, Leipzig: Akad. Verlagsgesellschaft.
- Wieleitner, H., 1908, *Spezielle ebene Kurven*, Sammlung Schubert, Leipzig.