CONCEPT MAPS AS VISUALISATION

Their role as an epistemological device for introducing and implementing History of Mathematics in the classroom

Leo ROGERS
University of Oxford, UK
Leo.Rogers@education.ox.ac.uk

ABSTRACT

A Concept Map is a graphical multi-layered and flexible metacognitive tool for organising and representing knowledge. Various forms of concept map have been developed for use in teaching (Novak and Canas 2008) The ability to construct such maps, particularly in a virtual environment, allows the juxtaposition of ideas and the creation of new connections. Visualisation of such possibilities has a powerful epistemic facility capitalising on the creative pedagogical potential of the teacher and the active engagement of the learner (Arcavi, 2003; Giaquinto, 2007). This paper describes the use of such maps in the mathematics classroom and offers a theoretical basis for practical intervention using Design Research (Swan 2006) allowing many points of entry into historical material that supports pupils' cognitive, affective and operative engagement with mathematical learning.

1 The UK Curriculum 2008

In 1999 the education system for England and Wales[1] was centralised by a government that produced a curriculum enshrined in traditional beliefs about 'levels' of knowledge, and a collection of disparate activities rarely connected in any sensible way producing little serious engagement with mathematical thinking. An Inspectors' report on secondary schools showed that too many pupils were taught formulas they did not understand, and could not apply:

> "The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently." (Ofsted 2008, p.5)

Since 2008 the government organisation QCDA[2] has worked with schools on a revised curriculum to publish guidance[3] complemented by video case studies where 'enrichment and enhancement' activities are used with a range of learners[4].

Recognising the Historical and Cultural Roots of Mathematics is now one of the Key Concepts in the new Programmes of Study for Secondary Schools (ages 11 – 18) (QCDA 2008). Consequently, it may now become easier to incorporate the teaching of this Key Concept in such a way as to enable the history to emerge from the discussion of *canonical situations* (be they images, texts, or conceptual problems) introduced by the teacher. This approach has the advantage of being able to link different areas of the curriculum by

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[1] Different systems apply in Scotland and the Republic of Ireland.
[2] The Qualifications and Curriculum Development Authority, the Government sponsored body set up to maintain and develop the national curriculum and associated assessments, tests and examinations. This organisation is now being disbanded by our coalition government.
[3] QCDA (2009) *Engaging mathematics for all learners.*
[4] The revised Programme of Study for Secondary Mathematics can be found at http://curriculum.qcda.gov.uk/key-stages-3-and-4/subjects/key-stage-3/mathematics/index.aspx

offering situations derived from historical sources, thereby enabling pupils to appreciate connections between parts of mathematics that have hitherto been concealed or ignored.

2. Heritage not History

In a series of papers, Radford (from 1997 to 2006) has demonstrated that mathematical knowledge is deeply embedded in a given culture, and that each new cultural phase re-interprets and changes the conceptions of earlier thinkers so that new ideas become possible. Furthermore, in our attempts to understand the history, we inevitably bring our modern socio-cultural conceptions of what the past was like with us. Marwick (2001) recognises that history, being about 'what happened in the past', and says that the best we can do is to make clear our basic assumptions and, acting on the evidence available, make as honest a story of it as we can. We confess our ignorance where we do not know, and make clear our speculations where evidence is sparse. History is a cumulative activity and accounts should always subjected to debate, qualification, and correction.

Concerning the use of history in education, Grattan-Guiness (2004) makes a distinction between the History and the *Heritage* of mathematics. History focuses on the detail, cultural context, negative influences, anomalies, and so on, in order to provide evidence, so far as we are able to tell, of what happened and how it happened. *Heritage,* on the other hand, address the question "How did we get here?" where previous ideas are seen in terms of contemporary explanations, and similarities with present ideas are sought. He says,

> "The distinction between the history and the heritage of [an idea] clearly involves its relation to its prehistory and its posthistory. The historian may well try to spot the historical *foresight* - or maybe lack of foresight - of his historical figures,…. By contrast, the inheritor may seek historical *perspective* and hindsight about the ways notions actually seemed to have developed."(2004:168)

and

> "…heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows foundations are dug down, and not necessarily into firm territory." (2004, p.171)

Referring to the apparent development of mathematics, the interpretation of Euclid's work as 'geometrical algebra' (as in Heath and van der Waerden for example) has been shown to be quite misguided5 as history, but as heritage is quite legitimate because it is the form in which Arab mathematicians of the 10th century justified the logic of their creation of algebra (Rashed 2007, p.18-24).

Each culture had its own ways of defining the objects of their inquiry and recent historiography (Chemla 2004, Robson 2008, Plofker 2009) has shown that research from a historical-epistemological point of view can provide us with considerable information about the development of mathematical knowledge. This information is principally specific, but also has implications for wider theoretical developments. Radford and others have already suggested that:

> "The way in which an ancient idea was forged may help us to find old meanings that, through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of the elaboration of teaching

[5] Typically, this is done with Euclid II,4 and described as 'completing the square', see the examples in Katz (1998: Section 2.4.3) for a more nuanced interpretation. For a critique of Heath, see Netz (1999).

sequences …" (Radford 1997, p.32)

Nevertheless, even though locked into our own cultural contexts, with the sensitive use of modern historiography, we can work on the heritage we discover to develop materials that give new insights into learning in a modern curriculum.

3. Maps, Narratives and Orientations

A concept map is a graphical multi-layered metacognitive tool for organizing and representing knowledge. In view of a variety of experiences the original form has now been considerably modified by Novak & Canas (2008) and other users into a series of flexible ways of working, appropriate for a number of different disciplines. Concept Maps have considerable advantages over linear text. They can be used to support the collaborative development of knowledge, the sharing of vision and understanding, the transfer of expert knowledge, and the enhancement of metacognition. Burke & Papadimitriou (2002) have proposed that when faced with non-linear text, the *Map*, a user would need two more elements; a *Narrative*, that provides information on the general context and background, and an *Orientation* that describes the activities provided for pupils to 'find their bearings' in the map of the ideas presented.

This idea gives us the freedom to consider a map in a *virtual environment* where the arrangement of concepts, objects, events, propositions and actions may be partially ordered and even multi-layered, crucially breaking up the linear sequence and juxtaposing different ideas. No map is ever 'complete'; what may be chosen to be the principal concept(s) at one stage can be rearranged according to the needs of the learning process, and of the individuals involved. In contrast, most curriculum activities are presented to teachers as a *linear narrative* of topics, restricted to some imagined age-related 'levels of competence' of the pupils. Clearly, to be relevant and useful, Maps have to be developed collaboratively where, for example, a group of teachers, or a teacher and an 'expert' share knowledge and combine their vision. In this way we can present both pupils and teachers with Maps to be explored and interpreted. However, the *Map* needs some background *Narrative* and some suggestions for *Orientations* to help pupils address problems arising from the situation[6].

Maps clearly have both a metacognitive and an epistemological function. By organising ideas, concepts and events in a particular way, and examining the possible links between them in a visual display, maps can be used as plans for teaching and scaffolding for learning, leading us to new connections between ideas. Indeed, Barbin (1996, p.18) states that:

> "The history of mathematics shows that mathematical concepts are indeed constructed, modified and extended in order to solve problems. Problems come, as much as into the birth of concepts as into the different meanings attached to concepts as tools for the resolution of problems. This emphasis on the role of problems in the historical construction of knowledge can lead to a new way of conceiving history."

and, I would add, to a new way of finding problems suitable for the classroom.

The importance of visualisation in these activities is clear; from the representation of

[6] In the case of the relationship between teachers and pupils the use of Narrative and Orientation would vary according to the circumstances and these ideas were discussed in the Workshop accompanying this presentation.

objects, to manipulating them physically and learning to do so in the mind to bring out hidden properties. Furthermore, visualisation has linguistic and semiotic connections, and Barthes (1977) writing on the rhetoric of the image maintains that there are many different ways we read an image, (linguistically, iconically, coded and uncoded), and this idea is no less relevant to the structures and maps we design.

> "The image, in its connotation is constituted by an architecture of signs from a variable depth of lexicons …. The variability of readings … is no threat to the 'language' of the image if it be admitted that language is composed of idiolects, lexicons and sub-codes. The image is penetrated through and through by the system of meaning, in exactly the same way as man is articulated to the very depth of his being in distinct languages." (Barthes 1977, p.47)

Adapting a Map to explore links through the curriculum to historical contexts can act as part of a developable knowledge structure to be offered for integrating aspects of our mathematical heritage into a teaching programme, where the history then becomes integral to the exploration of the mathematics. A Map is there *to enable teachers to have the freedom to develop their own Narrative*, it can throw light on certain problems, suggest different approaches to teaching, and generate didactical questions. It is thus possible to offer ways in which teachers, starting from a particular point in the standard curriculum, could incorporate the teaching of 'Key Concepts' (QCA 2008) to link with some important developments in the history of mathematics.

4. Visualisation and Epistemology

Visualisation is an important function in mathematics teaching and learning, even more so now we have available a greater variety of media than hitherto. Arcavi (2003) regards visualisation as a vital aspect of our classroom communication of mathematical ideas and concept building. This ability is fundamental in our interpretation, creation and reflection upon a variety of images and diagrams that depict and communicate information, aid thinking, and help to advance our understandings. Arcavi's paper is important in describing the facilities of the human mind and their importance in the cognitive and affective aspects of learning and teaching. However, Giaquinto (2007) explores to what extent we can rely on our visual abilities to develop deeper cognitive awareness and justifications for epistemological advances. In particular, he explores the nature of our cognitive grasp of structures that may give us clear insights and develop new knowledge. While a large part of our knowledge comes from direct experience, and vicariously from someone else's experience, we can also derive knowledge from theory. Since theoretical structures can be depicted, he argues that *we can know structures by means of our visual capacities* and he forms the idea of 'visual templates' demonstrating the way we can appreciate them and manipulate them. A visual display can suggest connections that inspire theory. Once we have perceived the elements of a particular configuration, we have the ability to perceive configurations of other visual types as structured in the same way.

> "Thus, I suggest, we can have a kind of visual grasp of structure that does not depend on the particular configuration we first used as a template for the structure. … Once we have stored a visual category pattern for a structure, we have no need to remember any particular configuration as a means of fixing the structure in mind. …. There is no need to make an association or comparison. So this is more direct

than grasp of structure via a visual template." (Giaquinto 2007, p.221)

As a philosopher, Giaquinto is cautious of the word 'intuition' because he is interested in carefully exploring the cognitive aspects of our appreciation of structures, but in my view, ideas that come to us 'suddenly' (often after some months of incubation) could well be stimulated by 'metacognitive manipulation' of a (public or private) Concept Map.

5. A Practical Approach through Design Research

In response to the requirement of history in the new curriculum, the perceived lack of knowledge and the absence of appropriate resources among teachers, in 2009 a Working Group on History in the Mathematics Classroom was set up to "select, share, trial, evaluate and modify appropriate material in the light of teachers' experience so that together we may discover sensible ways of introducing the rich historical and cultural roots of mathematics to our pupils."7 The approach adopted by the Working Group has evolved from experiences of colleagues in presenting 'episodes' from the history of mathematics in workshop form, so that interesting and worthwhile problems can arise from the historical contexts8.

In order to address the question of teachers' lack of knowledge and experience, the principal focus lies in providing secondary teachers with professional development materials that start from the fundamental ideas that they are required to teach, and open up the possibilities of developing the concepts involved by finding *historical antecedents* to support the connections between and motivations for these ideas, and the possible links to be made between the mathematics and other subject areas. The project has two aspects; developing practical materials for teachers, and providing a context that offers qualitative research opportunities. This involves regular liaison with teachers[9] who wish to try out these ideas in their classrooms.

The methodology of Design Research (Swan 2006) being used in this project involves a systematic series of interventions to transform a classroom situation. It is a collaborative and iterative approach to research and development in which theoretical arguments and reviews of existing studies are brought together in the design of new teaching approaches. Proposals for intervention are evaluated in classrooms using standard methods (Robson 1993)[10] where outcomes lead to further refinement of the theories and approaches. Revised plans are further tested and the emerging results reveal ways in which teaching methods may be designed to become more effective. This methodology involves discussions on approaches, different for different classrooms, including draft lesson plans with notes, copies of historical materials, and links to appropriate resources that offer flexibility and allow for modifications. In this way, a 'lesson' can be seen as a series of 'micro tasks' each offering serious epistemological challenges both mathematically and pedagogically. This historico-epistemological research demonstrates the potential for providing data on cognitive, affective and operative aspects of

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[7] This is an official Working Group of the British Society for Research in Learning Mathematics (BSRLM). Members are practicing teachers, teacher trainers, research students and trainee teachers. See http://www.bsrlm.org.uk/

[8] Some of these problems were used in the Workshop accompanying this presentation.

[9] The teachers involved are volunteers who usually work in schools local to members of the Working Group who access schools through their job as teacher –trainers.

[10] For example, forms of action research such as interviews, questionnaires, case studies, participant observation, etc.

learning thereby offering a range of affordances (Greeno 1998)[11].

6 Negotiating Meanings in the Mathematics Classroom

Living in the world is a constant process of negotiation of meaning (Wenger, 1999). Meaning does not lie in the individual nor in the world but in the dynamic interaction between the two. Participation in meaning is an active process of taking part in activities as a member of a community. It therefore has both social and personal dimensions and thus shapes the individual and the community in which the individual participates (Rogers, 2002).

These principles were applied to the mathematics education community by Adler (2000). However, Douady (1986, 1991) had already provided an application of the principle of negotiation of meaning to mathematical learning as a ‘dialectique’, between pupil and teacher. Douady discussed the interface between teaching and learning in mathematics, and examined the role of the teacher in providing opportunities for learning. She considered the teacher's role principally to set up conditions for learners to make connections between various aspects of mathematics, and then to mediate the learning of new concepts by drawing on learners' ideas and negotiating these in discussions in the classroom.

In the formation of concepts, there is a *dialectic* between a mathematical concept as a tool (*outil*) for use in solving a problem, and as an object (*objet*) when it has the status of an independent mathematical entity. This idea of moving between two states is central and is included as part of the recognised process of building mathematical knowledge. Douady sees learners using objects as tools, and also working with tools to build objects. Her model is cyclic, where concepts, used as tools, give rise to new tools which in turn become objects, and the role of the teacher is central in providing learners with a task that requires them to make use of their existing knowledge. In their writings to teachers, Watson & Mason (2005) show how cognitive conflict, negotiation and concept building, together with encouraging justification and the extension of ideas, can be given clear practical guidance for operating in the classroom.

Many episodes from the history of mathematics show how the protagonists are offering an idea for discussion and different positions are taken up, according to the understandings of the correspondents. Recently, a dialogic approach by Barbin (2008) illustrates the negotiations to be found in many historical materials.

7 Working with Teachers and Pupils

The implementation of the Design Research approach first requires participants who are going to be sympathetic to the general aims of the project. A number of colleagues in the Working Group have access to schools through their work as teacher-trainers, so much cooperation is already established.

Here I outline the stages of evolution of the planning and general procedures in the context of one particular class of 13/14 year olds in an English state secondary school, who have an experienced teacher who worked with a researcher to introduce some historical elements into her mathematics teaching. However, this teacher was constrained by the demands of the curriculum, as well as having to maintain elements of school policy,

[11] Greeno sees *affordances* as “qualities of systems that can support interactions and therefore present possible interactions for an individual to participate in” (1998, p.9)

and run her teaching programme parallel with another class. The pupils were near the end of their school year, and about to pass on to the next class.[12]

Initial discussions took place between the teacher and researcher about the curriculum and some relevant historical contexts that finally focussed on the general idea of ‘completing the square’ (a subject that is part of the pupils’ normal programme of study). The teacher’s awareness of the pupils’ background knowledge and skills enabled particular pedagogical strategies to be considered.

For this group of pupils, their background knowledge and skills were fairly well known (though not consistently nor entirely secure), and so some revision was arranged to cover the use of items such as brackets and multiplication of binomials. For many pupils moving too quickly into generalisation with symbols can lead to confusion, and the expansion of expressions like (a + b)(c + d) can be very awkward and confusing. The ‘Grid Method’ (Fig. 1) is a useful device commonly taught in English Primary schools.

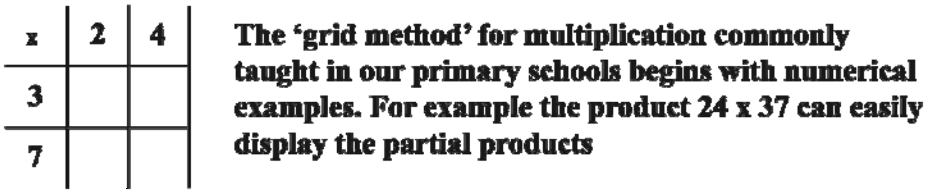

Fig. 1. Grid Method for Multiplication

Pupils can also have practice in manipulation of templates representing squares and rectangles (Rogers 2009), and visualisation comes into play here where metacognitive possibilities are exposed:

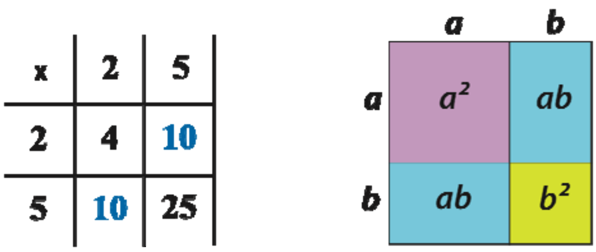

Fig. 2. Grid and Euclid II, 4.

From the grid multiplication display it was decided to progress to identical binomial products such as (2 + 5)(2 + 5) that produce an arithmetical square [4 + (2x10) + 25] = 49, and contemplating a geometrical square with two squares and two identical rectangles symmetrically placed. During discussion with the pupils, the geometrical square can be labelled with algebraic expressions showing products (Fig. 2).

The initial curriculum discussions involve making concept link sketches and noting possible situations for the introduction of references to historical contexts. The use of the historical Narrative in conjunction with the teachers’ knowledge of the curriculum in writing the Orientation is essential. Choosing what to do and deciding on prerequisites, to a large part determines the style of the Orientation. Preliminary discussions took place in

- choosing a general topic area from the current teaching programme

[12] Pupils in the UK system progress through the school according to their age, so Year 9 are 14-15 years (Key Stage 3) and Year 10 are 15-16 years old (Key stage 4).

- discovering what historical links with the central concepts might be possible (The Map)
- considering the contextual and pedagogical approach including pupils' assumed knowledge background and skills pre-requisites
- understanding and establishing something of the historical background (The Narrative)
- planning a sequence of lessons (The Orientation)
- providing for feedback and modifications

The general area of the curriculum chosen was section '3.1 Number and Algebra' from QCDA (2009). In particular:

"rules of arithmetic applied to calculations and manipulations with rational numbers (KS3)"

"linear, quadratic and other expressions and equations (KS4)"

As well as considering the pupils' knowledge, provision needs to be made for the Key Concepts and Processes (QCDA 2009), and it is now left entirely to the teacher to decide how to interpret these statements in terms of classroom tasks and activities.

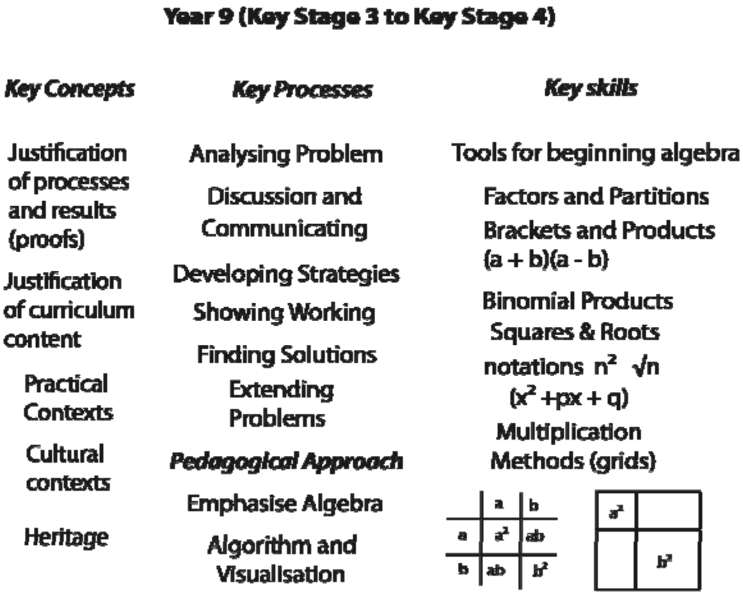

Fig. 3. Transcription of Syllabus Content Map Discussion

The transcribed Sketch Map above (Fig. 3) represents the results of an early discussion with the teacher on the three Key areas of the curriculum; Concepts, Processes and Skills in the transition from KS3 to KS4. The heading "Year 9" refers to the class of 13/14 year olds, and the 'transition' in the curriculum (in terms of algebraic skills) involves a progression from solving linear equations to developing techniques for solving quadratic equations. The ideas expressed here intend to build on the pupils' current knowledge and use their abilities of visualization and representation to work on tasks that will assist that progression. Below, in the ideas for developing the lessons (Fig.4), 'Number and Algebra' can be seen in the centre of the picture, while on the left appear the activities potentially linking to some historical contexts.

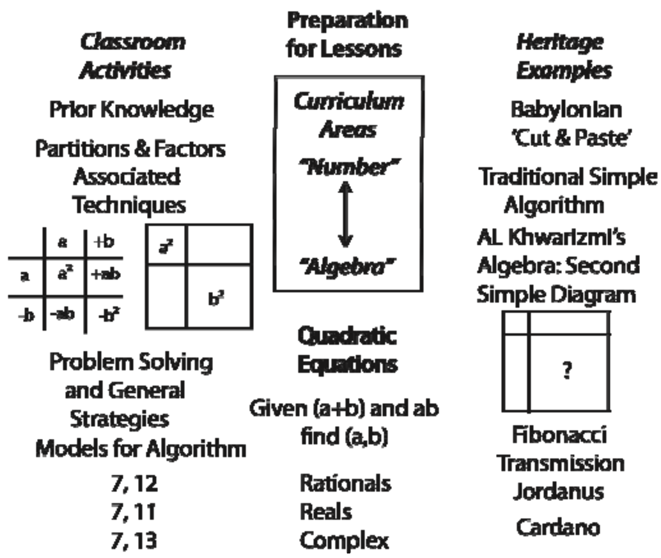

Fig.4. Transcription of Lesson Preparation Map

It is commonly the case that school algebra has been presented as generalised arithmetic, thereby missing the most important aspect of algebra as powerful structural tool for identifying and operating with relationships. In this case having studied linear equations, a possible progression to quadratics shows how two approaches, one from geometrical structure and the other from an arithmetic algorithm can combine to produce general ways of solving the problem. *the general strategy here is to develop the idea that algorithms derived from arithmetical practices can be transformed into geometrical and wider structural concepts.*

A draft lesson sequence was then created that consisted of a series of tasks each allowing opportunities for discussion and contemplation of different aspects of these ideas. Two of these tasks are shown in Appendix A (Jordanus de Nemore's Solution[13]) and Appendix B (Thinking of Two Numbers).

8 Pedagogical Freedom and Curriculum Progression

Typically, when good teachers work mathematically with pupils, they elicit their pupils' knowledge through interactions that prompt pupils to express their own ideas and lead them towards improved and more efficient methods and clearer definitions. The use of Concept Maps however initially sketchy and tentative, can enable us to organise material in a way that allows originality, flexibility, and potential for developing new pedagogical avenues. Viewing mathematics lessons as series of questions, prompts and 'micro tasks' rather than a collection of procedures to be practiced, opens new possibilities and allows connections to be made between curriculum topics as well as with mathematical history and the wider culture.

Asking questions like "What if?" and "Then what?" and encouraging pupils to visualise, seek patterns, compare and classify, explore variations in structures, test

[13] The diagram for completing the square often used in History of Mathematics texts is found in (Rashed 2009: 110) but for this case Al-Khwarizmi drew another diagram a page later showing a gnomon with a space to be filled (Rashed 2009: 112). This procedure is shown in (Hughes 1981: 16)

conjectures, identify properties and relations and make meaning from involvement in a particular task, requires focussing attention on the details of contexts and developing understandings and structural relations that are generic and transferable.

The Design Research approach does not have to be undertaken on a grand scale, these methods are available to anyone who by evaluating their own pedagogy engages in action research, and by taking small steps can achieve pedagogical freedom while at the same time can satisfy the demands of the school curriculum.

What is equally important is the knowledge that there is a community of teachers and researchers working towards similar aims, and that combined with a sensitive and purposeful pedagogical approach, the history of mathematics can provide teachers and pupils at all levels with contexts that can open new visions of the nature of mathematical activity.

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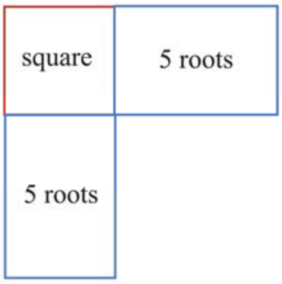
Appendix A.
Jordanus de Nemore’s solution for Al-Khwarizmi’s problem:

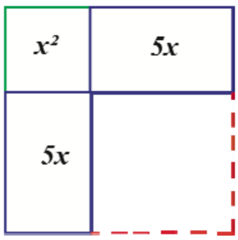
“The Mal plus ten roots are equal to thirty-nine dirhams.”

1. Read this through carefully, and explain Jordanus’ method to your neighbour.



Solution method according to Jordanus de Nemore (1225 - 1260).

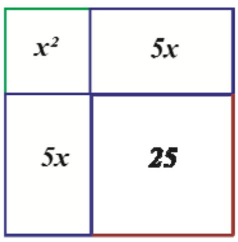
Since the square and the ten roots are equal to 39, and use the two equal halves of the roots to make a gnomon.
(Euclid II, 4).



Our Algebraic Representation
$[x^2 + 10\,x\,] = 39$
The roots are shown as:
$x^2 + [5x + 5x] = 39$

What is the size of the missing square?

Completing the square:



$[x^2 + 5x + 5x] + 25 = 39 + 25$

$x^2 + 10x + 25 = 64$
$(x + 5)(x + 5) = 64$
$(x + 5)^2 = 64$

$x + 5 = 8 \qquad x = 3$

2. Try these problems using the method above:

$x^2 + 6x = 45,\quad x^2 + 14x = 15,\quad 4x^2 + 7x = 15,\quad x^2 + 2/3x = 35/60.$

Make up some more equations of your own and solve them using Jordanus’ method.
What problems arise and what questions could you ask to improve this situation?

(The ‘Mal’ is a word still used today for the unknown and is sometimes called ‘the treasure’. A dirham is a unit of money.)

Appendix B: Thinking of Two Numbers

This is a class activity, managed by the teacher.

"I am thinking of two numbers; their sum is seven and their product is twelve, what are the numbers?"

The first part can be done orally, calling out the number pairs, writing them on the board, or using a visual display:

Sum	7	9	15	14	17	20
Product	12	20	56	33	60	64

The first pairs of numbers are usually easy, if pupils know their tables.
Even so, pupils need time to think and it is important not to lose anyone at this stage as the numbers get bigger, and they can be asked to write down the solution instead of calling out.
Pupils need to experiment like this for a few more numbers.

Sum	26	32	36 …	31	40	53	50 ...
Product	168	252	323 …	234	384	696	609 …

It is up to the teacher to decide at what stage whether to allow calculators.

The point of this game is to help pupils to develop a simple arithmetic strategy for finding the two numbers.

With the early examples, finding the Partitions of the Sum number gives a clue to the Factors of the Product number
7 = (3 + 4) and 3 x 4 = 12 etc.
26 = (13 + 13) but 13x13 = 169 however (12 + 14) gives 12x14 = 168
Partitions of 32: (16,16), (15,17), (14,18) and 14x18 = 252
Partitions of 31: (15 ½ , 15 ½), (15,16), (14,17), (13,18) and 13x18 = 234

So we have a general strategy

Take half the Sum number, then add and subtract in stages to find pairs for testing the Product.

If we think of the sum as a semi-perimeter of a rectangle, and the product as an area, we have a version of a 'geometrical algebra'.