MATHEMATICAL CONNECTIONS AT SCHOOL

Understanding and facilitating connections in mathematics

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ABSTRACT

For nearly 20 years the National Council of Teachers of Mathematics (1989, 2000) has recommended that teachers enable pupils to recognise and to use connections among mathematical ideas (Presmeg 2006). This statement is consistent with developments in mathematical education globally. Combinations of epistemological and sociological approaches like those of Heintz (2002) to describe connections in mathematics as a science, provide useful tools for examining ways in which teachers support the making of connections, for instance between school mathematics and everyday life of pupils or between different fields of mathematics: geometry, algebra and stochastics.

In order to deepen the understanding of connections in the school environment the author starts with presenting some modern approaches to mathematics as a science. The author then goes on to draw conclusions for enabling connections in the mathematical classroom by constructing problem-nets as a special learning environment. On the basis of diverse examples of application at school the transfer to the practice of mathematical education is established.

1 Connections in mathematics as a science

Educational standards all over the world (for example NSC in South Africa, NTCM in USA, diverse curricula in Germany) recommend that teachers enable pupils to recognise and to make connections among mathematical ideas. Therefore the main question of this paper is: *How can teachers support the discovery of connections between different fields of mathematics: geometry, algebra and stochastics?* To understand the nature of mathematical connections it could be helpful to look at connections in mathematics as a science. Therefore, the author will first look at how the work of mathematicians from different fields is often necessary in order to produce new results. Then, the author will give an example from a learning environment which will illustrate how interdependence of epistemic and sociological aspects of mathematical connections can be taken into account when teaching mathematics.

In recent studies epistemological aspects of mathematics are seen in their interdependence with sociological aspects (Heintz 2000, Prediger 2002). According to that, interconnections in mathematics refer not only to mathematical objects and scientific topics but also to the cooperation among mathematicians. Thereby high theoretical coherence as epistemic component and wide spread of social consensus is recognised as characteristic qualities of mathematics.

Partly seriously and partly joking, mathematicians use the *Erdős number* to describe social structure of the mathematics. Paul Erdős has written over 1400 papers with over 500 co-authors. His productivity inspired the concept of the Erdős number. Jerry Grossman (see “The Erdős Number Project” 2009) and his colleges define Erdős number in the following way:

“In graph-theoretic terms, the mathematics research ***collaboration graph*** **C** *has all mathematicians as its vertices; the vertex* ***p*** *is Paul Erdős. There is an edge between vertices* ***u*** *and* ***v*** *if* ***u*** *and* ***v*** *have published at least one mathematics article together. We will*

usually adopt the most liberal interpretation here, and allow any number of other co-authors to be involved; for example, a six-author paper is responsible for 15 edges in this graph, one for each pair of authors. Other approaches would include using only two-author papers (we do consider this as well), or dealing with hypergraphs or multigraphs or multihypergraphs. The Erdős number of ***v****, then, is the distance (length, in edges, of the shortest path) in* C *from* ***v*** *to* ***p****. The set of all mathematicians with a finite Erdős number is called the Erdős component of* C*. It has been conjectured that the* ***Erdős component*** *contains almost all present-day publishing mathematicians (and has a not very large diameter), but perhaps not some famous names from the past, such as Gauss. Clearly, any two people with a finite Erdős number can be connected by a string of co-authorships, of length at most the sum of their Erdős numbers* (http://www.oakland.edu/enp/readme/ 10.12.2010).*"*

Based on information in the database of the American Mathematical Society's Mathematical Reviews, an automatic collaborations distance calculator was created to determine Erdős numbers. Referring to that, e.g. Andrew Wiles has Erdős number 3. Furthermore, the automatic distance calculator can help to find the distance between two different mathematicians in accordance to their co-authorships (see Hischer 2010). Even if Grossman sees Erdős numbers just as a silly game, which have nothing to do with a mathematician's status or the quality of his work, such games illustrate the importance of cooperation on a social level and show how intensely mathematicians are working with each other. However, it does not provide more detailed information about the content or topics of their communication. These are represented in the variety of mathematical publications listed for example in Zentralblatt MATH (Z-MATH). This is an online Database, which contains about 2.9 million entries drawn from about 3500 journals and 1100 serials starting from 1868 and even earlier to the present. To categorise items in the mathematical science literature, Z-MATH uses the Mathematics Subject Classification (MSC). The MSC is a hierarchical scheme with three levels of structure. At the top level there are 64 mathematical disciplines. Examples for such disciplines are Number theory (11-XX), Algebraic geometry (14-XX), Probability theory and stochastic processes (60-XX). As a tree structure MSC itself is not networked. The network qualities of Z-MATH come from the papers itself. Consequently, every paper is related to chosen disciplines and connects them by these means. Since every paper was written by one or more authors it connects people too. In this sense, we can see mathematical papers as representations of mathematical connections at the epistemic and social level at the same time. Furthermore, Z-MATH makes it possible to observe the development of mathematical inquiries from outside across time. An example for this idea is given below (see Figure 1). This entry from Z-MATH shows a paper that connects mathematicians and mathematical disciplines.

Zbl 0823.11030
Taylor, Richard; Wiles, Andrew
Ring-theoretic properties of certain Hecke algebras. (English)
Ann. Math. (2) 141, No. 3, 553-572 (1995). ISSN 0003-486X
Classification :
11G05 Elliptic curves over global fields
11F11 Modular forms, one variable
11D41 Higher degree Diophantine equations
13C40 Linkage, complete intersections and determinantal ideals
14M10 Complete intersections
14H52 Elliptic curves

Figure 1: Entry from Z-Math

The paper of 1995 contains last steps to the proof of Fermat's Last Theorem and marks an end of a long mathematical adventure, which started in the 17^{th} century with the conjecture of Fermat's Last Theorem. The conjecture states that no positive integers *a*, *b*, and *c* can satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than two. The 1670 edition of Fermat's *Diophantus' Arithmetica* includes his handwritten commentary, particularly his "Last Theorem".

This commentary was republished in 1932 and is listed in Z-MATH as well. Since older documents listed in Z-MATH are not classified according to MSC, it is not possible to say which fields of mathematics connect the originally formulated conjecture. Nevertheless the subsequent proof of this conjecture connects different fields of mathematics like the modular and the elliptic worlds. Furthermore, it connects mathematicians from different countries and from different historical periods. Some steps of this process are shown in the diagram below (see Figure 2). To create the diagram I used illustrations of elliptic curves and modular curves, mathematical formula and also historical facts from Kramer's short presentation of the proof in 1995.

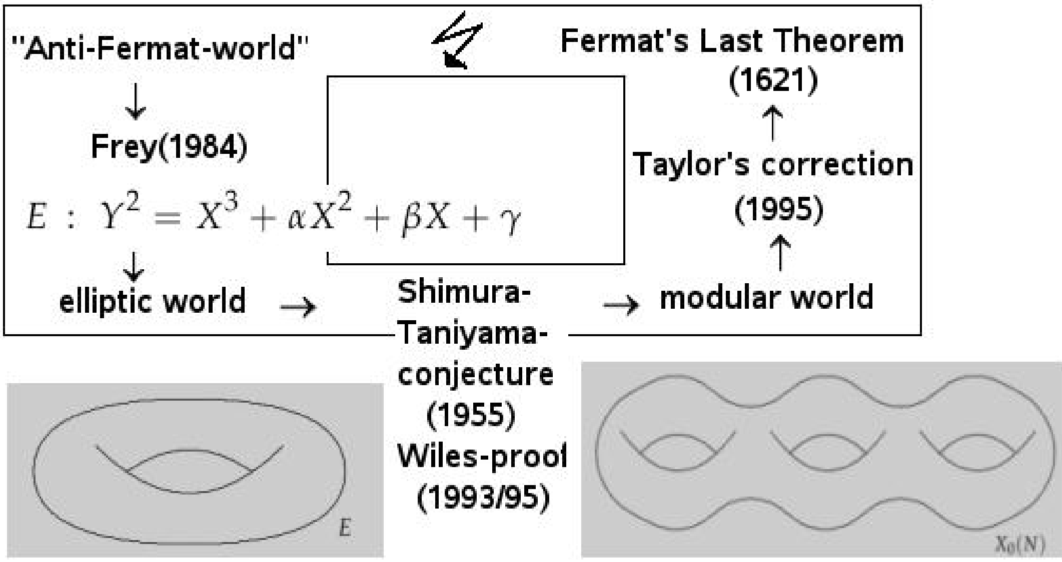

Figure 2: Fermat's Last Theorem

Consequently as a part of the proof, in 1984 Frey explored what would happen if Fermat's Last Theorem was false and there is at least one solution. Starting on the base of this hypothetical solution he found an elliptic curve and connected Fermat's Last Theorem with the Shimura-Taniyama-Conjecture (1955), which was not proven at that time. The Shimura-Taniyama-Conjecture was posed in Japan and connected the elliptic and the modular worlds. It was proved by Wiles in 1993 and the proof had to be corrected by Wiles and Taylor two years later. In this manner, different mathematicians shared posing and proving the main theorem and its parts.

In addition to the analyses and visualisation of social networks of mathematicians in Germany, Z-MATH was connected with Organizational Risk Analyser (ORA). ORA is a dynamic meta-network assessment and analysis tool with distance based, algorithmic, and statistical procedures for studying and visualising networks. To visualise social networks of mathematicians in Germany from 1990 until now, not only mathematicians but also mathematical topics according to the MSC as well were modelled with graph theory like vertices and publications as edges between these vertices (see Figure 3). This figure was created in cooperation with two students of informatics and Olaf Teschke from ZMATH, who helped me to link ORA with data from ZMATH. In order not to expand the document length unnecessarily, only the titles of selected MSC-disciplines are listed in here. Nevertheless, Figure 3 helps us to imagine mathematical interconnections not only as connections of scientific topics, but as connections within the scientific community of mathematicians too.

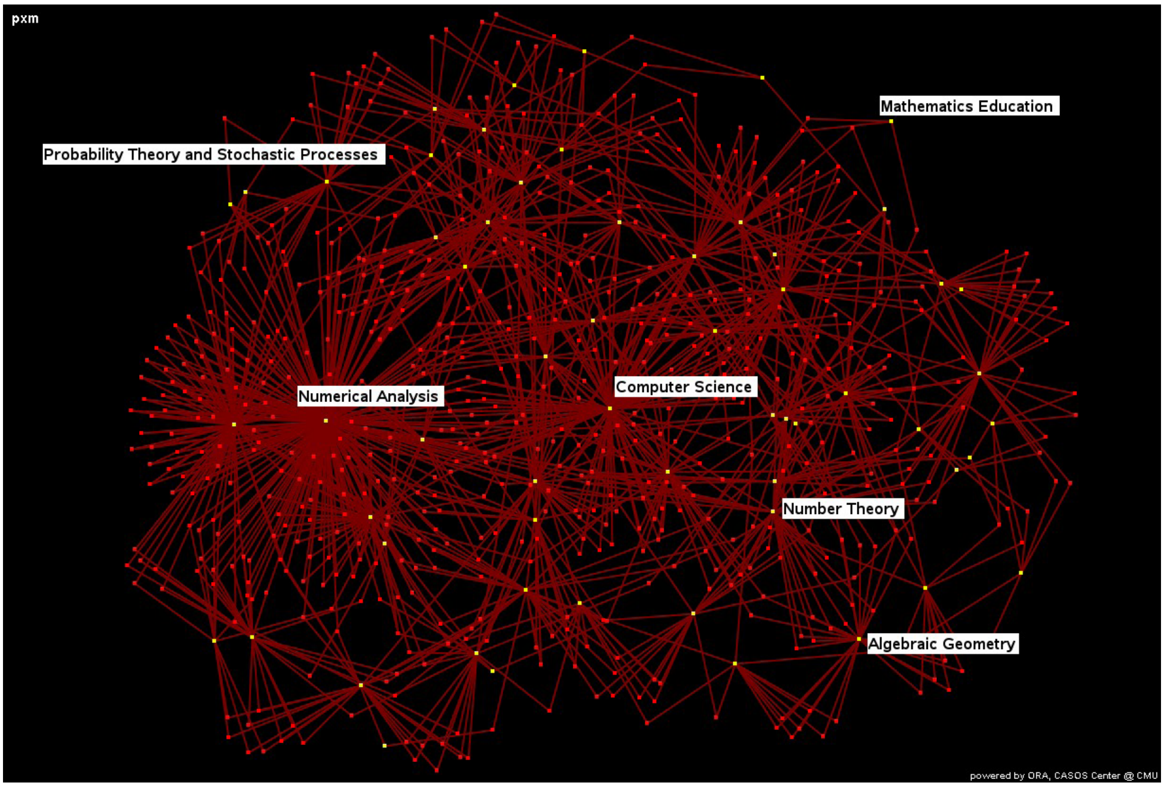

Figure 3: Mathematical connections in Germany (1990-2010)

Since every publication in ZMATH is linked to one or more mathematical discipline and one or more authors we modelled mathematicians (red) and disciplines (yellow) as vertices of the graph. The edges emerged through co-authorships and were modelled with help of ORA-algorithms. Since it is not the main subject of the paper I can not give here the explanation of the algorithms, which are used to visualise social networks. Very accessible introduction applied into theory of networks is the book of Horst Hischer. You will find it the list of references.

What can we learn from this about the nature of mathematical connections for teaching mathematics? Firstly, it shows those disciplines which have more connections to other fields like numerical analysis and computer science. If mathematicians cannot find classical solutions for the problems posed in their own discipline, they cooperate with their colleagues who are studying e.g. Numerical Methods or dealing with Computer Science. Secondly, even the titles of the disciplines like Algebraic Geometry reveal to school teachers, that many connections should be possible between algebra and geometry. Thirdly we see, that connecting the fairly recent discipline of probability theory with other disciplines can challenge not only pupils but experienced mathematicians as well. In addition to all of this, we see that segmenting mathematics as a scientific field in disciplines and topics is very important in order to classify mathematical objects on an epistemological level. This conclusion facilitates the segmenting of school mathematics in fields like geometry, algebra and stochastics as a premise for making and understanding connections. Hence, segmenting teaching material and the student's mathematical knowledge into categories as in the usual way in curricula, schoolbooks and teacher preparations should be maintained. In addition to segmenting school mathematics it is important to create situations where students can combine their knowledge from different segments of school mathematics. This is already being done when teachers deal with connections among geometrical and algebraic aspects for example, by using functions to describe geometrical concepts or calculating areas. Nevertheless, incorporating stochastics would be the most innovative step forward.

That is the reason why problems, which refer to different mathematical fields, should be given to the students. They should recognise and establish connections among mathematical ideas. Problems like those could be presented to the students for example at the end of the school year. It is with thanks to the cooperation of mathematicians that mathematical connections eventually emerge. Students, therefore, should be encouraged to cooperate by working on such problems. Segmenting school mathematics on one hand, and creating learning environments (where it is possible to weaken the borders between these segments) on the other are two interdependent factors of making connections at school.

2 Making connections with PYTHAGORAS-TREE

To review different “segments” of school-mathematics students could be divided into five or six small groups for solving problems together and sharing their solutions with the whole group. As the history of Fermat's Last Theorem teaches us, it is necessary to find mathematical problems or contexts, which would connect different segments of school mathematics (algebra, geometry and stochastics) on the one hand, and motivate students to cooperate, on the other hand. An example for a mathematical context, which could fulfil the

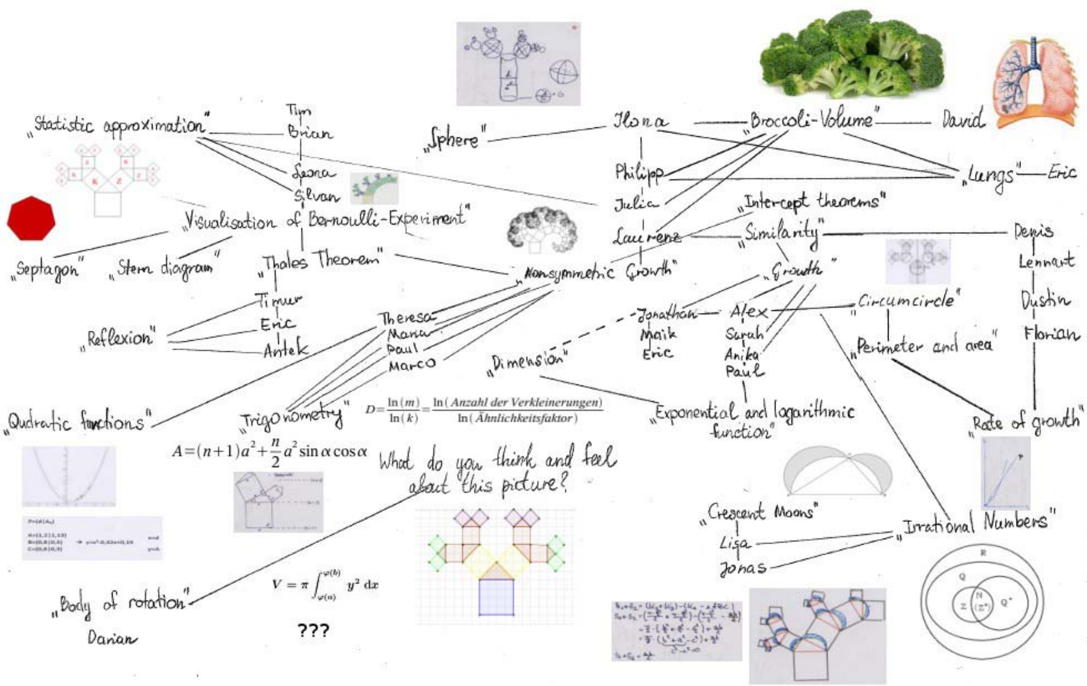

Figure 4: Pupils connecting mathematics with Pythagoras-Tree

above described pedagogical hopes, will be given in the next chapter. Figure 4 shows a

diagram which illustrates how students of 9th and 10th grade at a grammar school in Berlin worked on the given problems in the context of the Pythagoras-tree.
The students accomplished the work in two sessions lasting between two and four hours. Similar to the visualised networks of mathematicians, we can see "students-networks" around the Pythagoras-Tree. We can also see the names of students as "mathematicians", mathematical topics and titles of the problems illustrated with some pictures. To give an idea of how the context of Pythagoras-Tree could be used in the mathematical classroom in order to enable students to recognise connections by cooperating, a learning environment, which was tested at school, will be described in the following. The learning environment included a short text (see Figure 5) with general information on the Pythagoras-Tree and initial problems, which are related to major topics of school mathematics. The students were asked to work in jigsaw-puzzle.

> A Pythagoras tree is generated by adding to the top of a square (the 'trunk') a right-angled triangle sitting on its hypotenuse (branches). The 'twigs' are further squares added to the two sides adjacent to the hypotenuse. On the opposite sides, rectangular triangles are added again. These triangles are similar to the first one. And so it goes on. All growing branches end in squares (leaves). The picture on the work sheet stating "initial problem" shows a symmetrical Pythagoras tree with three levels (see Figure 6). Pick one of the following problems and solve it in small work groups. Do not forget to complete the table of competence for your chosen problem. Further squares can be added to each of the sides adjacent to the hypotenuse. These are the 'twigs'.

Figure 5: General Information on the Pythagoras-Tree

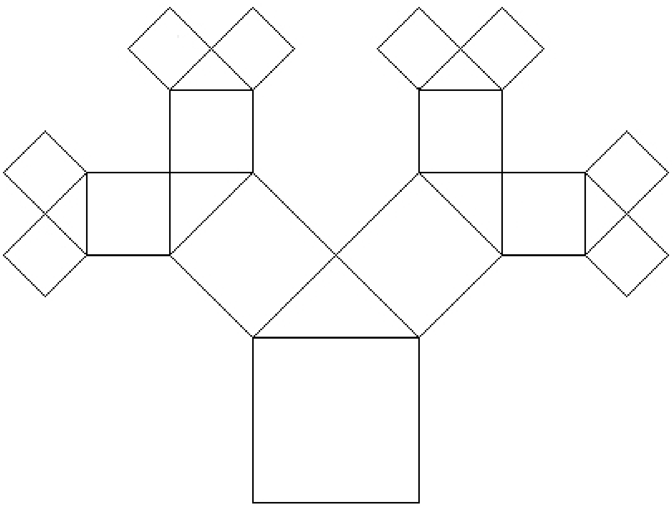

Figure 6: 3-Level-Pythagoras-Tree

Students were introduced to all the initial problems. They were then asked to choose one problem, to work on it in small groups (with a maximum number of five students) and to prepare a presentation of the solution for the whole class. First seven problems are presented without solution. To the last three problems are given students' own solutions. These solutions should illustrate my intentions in the area of making mathematical connections - in the same way as professional mathematicians. I hope therefore to offer new ideas to teachers about what they can expect from their students or how they can adapt the environment to their own classroom.

1. Figures and Lengths

Explore whether rational numbers are sufficient for the description of the possible side length. How do these lengths relate? Give reasons for your guess.

2. Dependencies

Look at the picture. Which functions describe the dependence of the area/perimeter on its trunk width? Use all possible ways of presenting a function.

3. Similarity

Which similar shapes do you recognise in the picture? Why are those figures similar?
Give the parameters of similarity. It is said that Pythagoras trees have a similar appearance to broccoli. What do you think?

4. Estimate

Ask your classmates and teacher to estimate the area of the drawn figure. Find the mode, median, mean and the span of your sample. Present your results in a boxplot. Work out the area by using the width 'a' of the trunk (use your ruler to measure it). Compare the estimate with the calculated value. What do you see?

5. Continuation

The symmetrical Pythagoras tree can be expanded with no limitations. Which function would you use to describe the relationship between the area of the leaf and the number of levels? Find a function to describe this equation and draw a graph. Find the inverse function. Starting at the trunk, how many levels of a symmetrical Pythagoras tree do you have to add, in order to end up with a leaf size (area) of 1/128 of the area of the trunk? Find the number of levels of a Pythagoras tree which possesses more than one million leaves.

6. General formula for calculating the area

Find the formula for calculating the area of non-symmetrical Pythagoras trees. Satisfy yourself that this formula is correct. Note: 'a' is the width of the trunk, 'n' the number of levels and 'α' stands for the base angle.
Additionally: Find the correct formulae for the Pythagoras trees with base angles of
30°, 45° and 60°. Is there anything particular here? How would you explain this?

7. General formula for calculating the circumference

How could you calculate the perimeter of non-symmetrical Pythagoras trees, when the width of the trunk, the number of levels, and one base angle of the biggest possible triangle are given? Find a general formula.

Solutions of selected problems

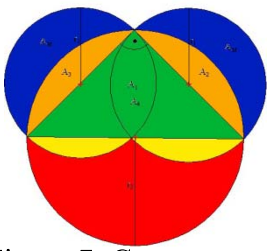

Figure 7: Crescent moons

9. Crescent moons

Small crescent moons appear (blue coloured) due to overlapping of the circles of the two sides adjacent to the hypotenuse and the hypotenuse of a right-angled triangle. Draw the corresponding crescent moons for all the levels of the shown picture. In which way can you calculate very quickly the total area of all the drawn crescent moons?

What did students make from this problem? To solve this problem Jonas and Lisa worked together. Their solution presented contains algebraic and geometrical elements (see Figure 8). Jonas shows his strong point in geometry. Lisa is doing better in algebra.

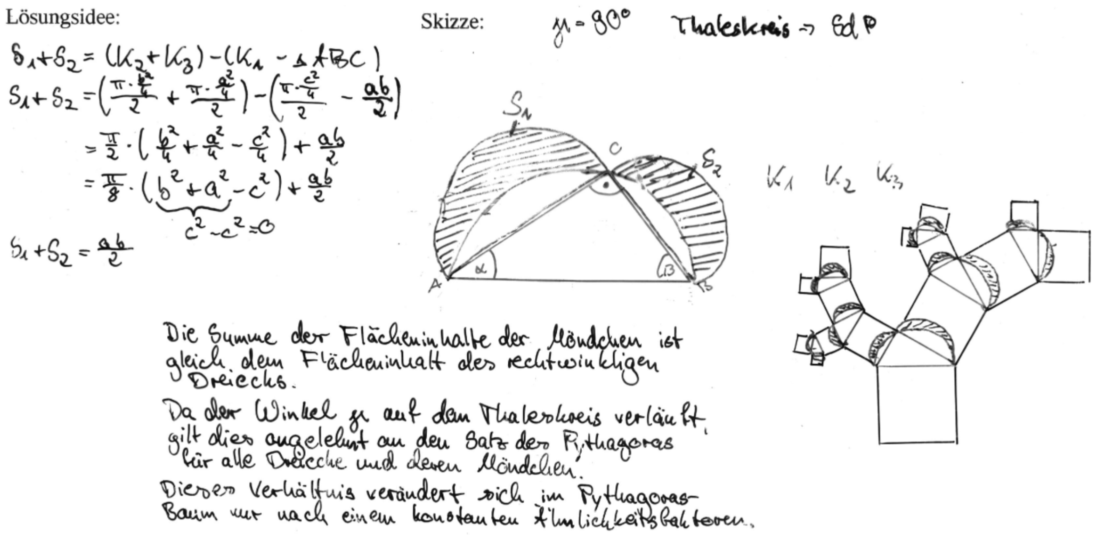

Figure 8: Pupils connect algebra and geometry

In order to solve the problem the students have drawn a picture first. Jonas then translated different areas of the figure into algebraic terms, but he could not simplify them. Lisa transformed the formulas with algebraic tools. She worked on the formal level and lost connection to the geometrical meaning of the formula. At the end of the algebraic transformations, the students asked the teacher to help them. The teacher reminded them of the Pythagorian Theorem, which helped them to translate variables a, b and c into the lengths of the sides of the rectangular triangle and they simplified the expression using the equation from the Pythagorian Theorem. As the result they wrote down the equation in the fifth line. Afterwards they translated the results into words, where they said that the area of the crescent moons is equal to the area of the triangle. They extended this observation to the following steps of the Pythagoras-Tree. Furthermore the students noticed similarity of the parts of Pythagoras-Tree and drawn the appropriate picture. This example shows how students complemented each other to solve the problem combining their geometrical and algebraic knowledge.

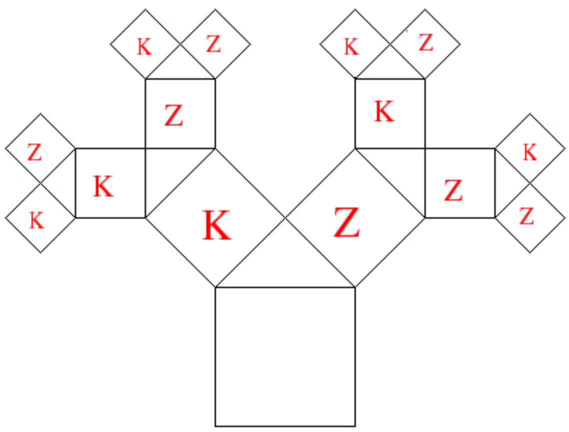

Figure 9: Bernoulli-Experiment

10. Multiplication rules

Possible outcomes of a Bernoulli-Experiment can be visualised with the Pythagoras tree, very similar to the probability tree diagram. The Pythagoras tree has to be read from the bottom to the top in order to follow the time line. As at the start of a Pythagoras tree, there is a square available for the visualisation of an experiment at a higher level.

On each level, the area of each of the newly generated squares will be half that of the original square. The total of the square area stays the same on each level. The drawing shows a threefold toss of a coin presented in a Pythagoras tree. What are the benefits and disadvantages of this way of visualisation in comparison to using a tree diagram?

What did students make from this problem?

After they had discussed the problem above, Eric, Antek and Timur were engaged in solving the following question: We throw a dice four times in a row and want to know how often the number six comes up. However, we are unable to show this experiment in a symmetrical Pythagoras tree. What would a Pythagoras tree look like according to this experiment? Using Thales' Theorem they have replaced the isosceles right-angled triangle with a non-isosceles right-angled triangle. This has resulted in the generation of two squares in level two with a ratio of 1:5. Is this model transferable to other Bernoulli-Experiments? Find further examples.

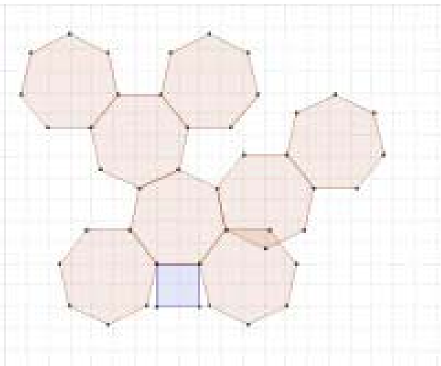

Figure 10: Heptagon

Results of Eric, Antek and Timur were presented to Leona, Silvan, Tim and Brian. To find a representation for the experiment with the dice they tried to find an analogy to the triangle in the coin-experiment. They suggested a construction from heptagon to visualise the dice-experiment. By producing heptagons with computer tools like GeoGebra (http://www.geogebra.org/cms/ 10.12.210) they found very quickly, that it was not an appropriate model. The conjecture students produced was proven false.

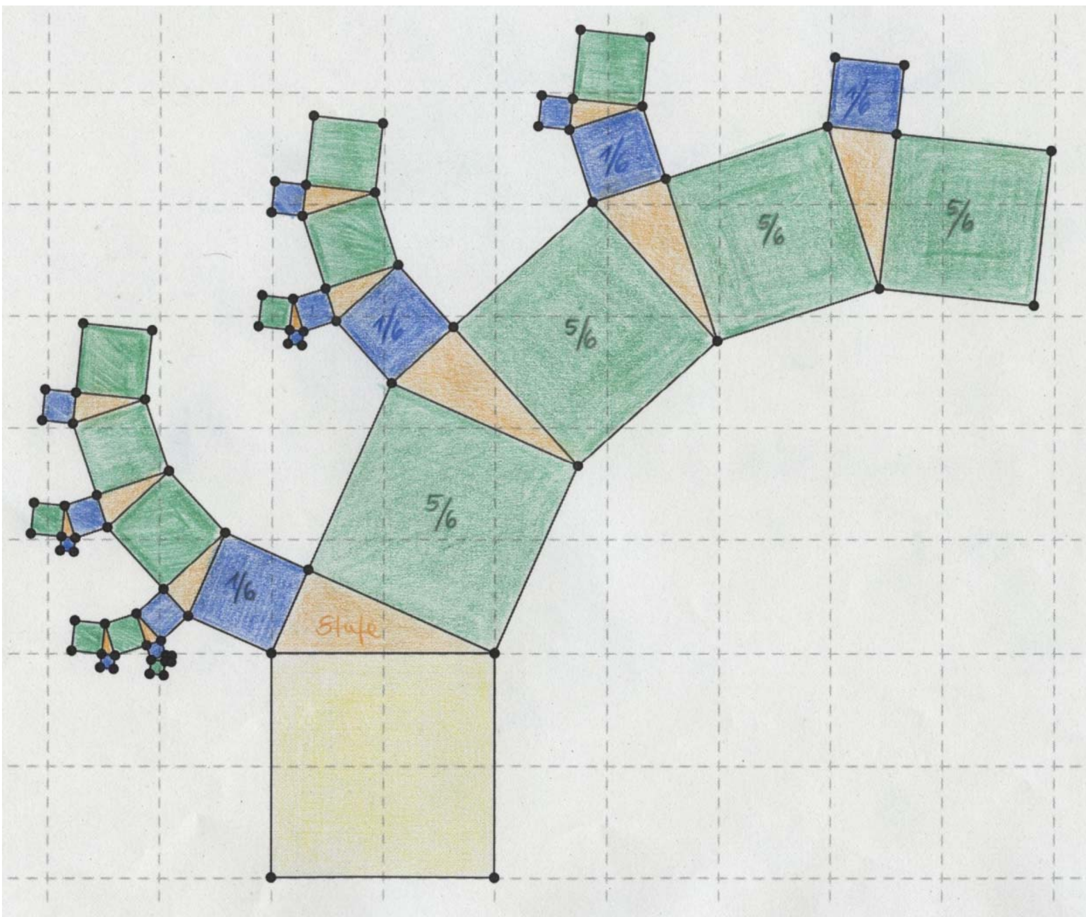

Figure 11: Visualisation of dice-experiment

Leona, Silvan, Tim and Brian constructed according to suggestions from Eric, Antek and Timur, a non-symmetrical Pythagoras-Tree to visualise the dice-experiment (see Figure 11). As an advantage of this model in comparison to the well known tree-diagrams in stochastics Leo said: “You can see that the probability to throw the number six, four times in a row, is very small. In this picture it is almost a point, but it is not a point. As a result the Pythagoras-Tree is not very useful, if you have many more steps”. Students then made their generalisation and concluded that the Pythagoras-Tree is very suitable to visualise the Bernoulli-experiments, since it has two branches and the Bernoulli-Experiment has two possibilities: “success” and “failure”. However, this generalisation is valid only for experiments with few steps. We can see that the students connected important ideas of geometry and stochastics. They used equality of the areas of figures to visualise equality of probabilities in every step of the Bernoulli-experiment. Students shared their ideas, produced and tested their own conjectures and studied the statements of other students. Starting with the given problem, students somehow cooperated like professional mathematicians to connect different “worlds” of school mathematics.

Broccoli

People say that Pythagoras trees look similar to broccoli. Draw a broccoli using known geometrical shapes. Find a possible function, which describes the volume of a broccoli in terms of the diameter of its trunk.

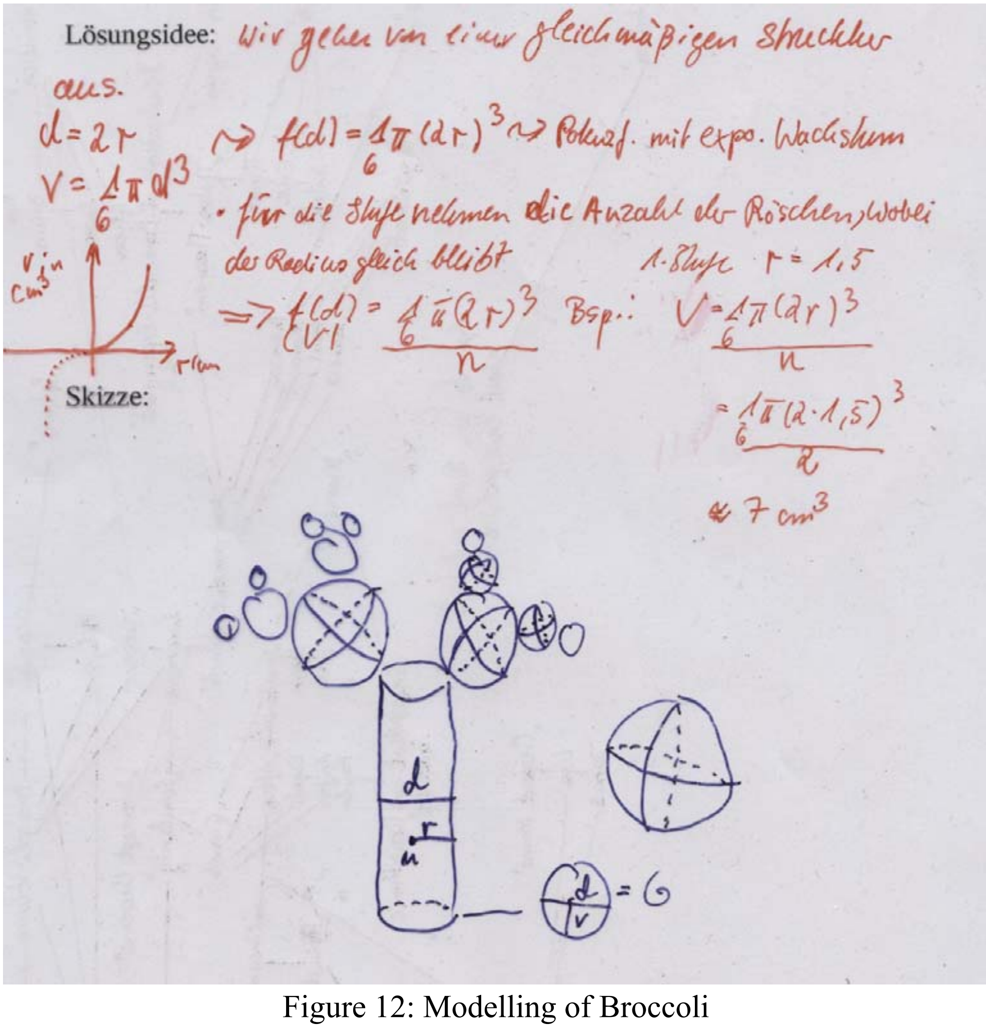

Figure 12: Modelling of Broccoli

What did students make from this problem?

Eric suggested drawing a human lung by using the Pythagoras tree. He has based his idea on the similarity between broccoli and a lung. Ilona said, that Eric's suggestion is more suitable e.g. for the tumour, because lungs only have finite space to grow. She suggested using other fractals like the Serpienski-Triangle to model lunges. Since the teacher made short remark about fractals as self-similar structures and gave an example from Serpinski-Triangle helping out in the small groups.

Ilona, Laurenz, Philip and Julia were modelling broccoli with geometrical solids (spheres). They then translated them into functions and used various representations of Functions (graph, table, equation). The students noticed that the sum of the volumes is equal at every step of the broccoli. But if you look at the solution carefully, you will notice a mistake. The students divided the volume of the spheres by n (amount of the steps) instead of multiplying it by n. Another mistake concerns the notes, where students say that power functions have exponential growth. However, for their presentation of the results to the whole class they received some critical comments. To describe concepts from biology it was necessary to combine ideas from geometry, like those from the area of spheres and from algebra and functions like the power-function. Likewise professional mathematicians' results were presented to the "scientific community" and mistakes were found and discussed. A similar case took place in the story of Fermat's Last Theorem, where a mistake appeared even after the presentation of the proof and had to be corrected afterwards. Mistakes of mathematicians can help teachers to allow students to make mistakes by making connections and correcting them.

3 Conclusions

The students' results commented above showed that they applied their knowledge of different mathematical segments to study the Pythagoras-Tree. They varied different mathematical representations, such as the number of steps of the Tree, length of the side, angles of the tree. They co-operated by proving, producing or falsifying conjectures. The students did not only solve mathematical problems referring to different areas of school mathematics and connecting them, but posed their own questions. Therefore, the example of a "Pythagoras-Tree" as a learning environment gives an idea of how teachers can put their students in situations similar to those of mathematicians'.

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