THE USE OF PERIODICITY THROUGH HISTORY

Elements for a social epistemology of mathematical knowledge

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ABSTRACT

This is a historical review related to the use of periodicity in order to form a significant basis that will broaden the current educational schemes for the periodic property. We intend to exhibit the *periodic aspect* as part of an epistemology of practices so the meaning for periodicity can be redefined and its teaching in school could be differentiated from the sole need for acquiring, recognizing or handling the periodic property.

1 Introduction: educational problems in regard to periodicity

Periodicity is a concept present in the development of scientific thought. Starting with pattern observation, human beings are capable of abstracting this property in order to generate scientific knowledge. Examples of its use while developing scientific knowledge are numerous: Pannekoek (1961) identifies the systematic observation of celestial bodies' periodic behaviour as the origin of astronomy as a scientific activity; Whitehead (1983) points out periodicity as the property which favours an analysis of the analogies between different physical phenomena. It turns out to be a property of different kinds of objects that begins in the everyday individual experiences (the year seasons, night and day), it enters into school's mathematics from the very beginning (periodic decimal numbers) and goes through several school's disciplines (phenomena in physics, functions as in calculus) all of which form part of the students' scientific culture.

In view of this, periodicity – as a quality of that which is repeated at certain intervals – should be part of a functional mathematical knowledge that would allow the student to travel between different areas of scientific knowledge. However, its current treatment in school limits its recognition and use as a result of the narrow analytic framework within which it is dealt. We give examples that illustrate two school situations related to the treatment of periodicity; the first is about a mathematical object belonging to the first school levels and the second is about objects belonging to higher educational levels.

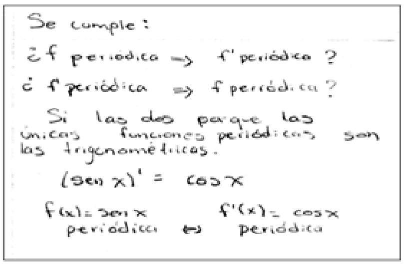
Example 1: Periodic Fractions

The initial learning in school for periodicity is related to patterns with special emphasis upon the observation of order. The tasks, aiding this, refer to completing sequences of drawings; nevertheless, these are not enough for the student to recognize the periodic behavior of a mathematical object, nor for him to take advantage of it, in order to perform some other type of tasks. As an example, figure 1 shows the procedure followed by a 21 years old college student when she was asked to find the number that ranks 120 in the periodic fraction 7/22: she began writing explicitly all numbers trying to reach the 120^{th} place (Buendía, 2006a).

Figure 1. Which is the number that ranks 120?

Example 2: Periodic Functions

At higher school levels, periodicity is treated as a property of functions, particularly trigonometric ones. This property is presented by using the equation $f(x) = f(x+p)$ placing x in the domain of the function and period p, which is institutionalized as *the* definition of periodicity and in many cases, is bounded to $\sin(x) = \sin(x+2\pi)$. Thus, the only reference that students often have, are trigonometric functions and especially, the sine one. Figure 2 shows the answer of a mathematics teacher at the double implication *f periodic ↔ f' periodic* (Buendía & Ordoñez 2009).



Yes, both (are true) because the only periodic functions are the trigonometric ones.

(sin x)'= cos x

f(x) = sin x   f'(x) = cos x

periodic ↔ periodic

Figure 2. *f periodic ↔ f' periodic*

Faced with this type of didactic phenomena, we tried to broaden the current educational schemes for the periodic, so that this property can be understood not only as the acquisition or application of the equation $f(x) = f(x+p)$, but as everything that is related to periodicity: the particular repetitive behavior of the mathematical object and how it can be significantly recognized. We use the history of mathematics as a source for information regarding the way men did what they did while developing mathematics related to the periodicity. In that sense, we will talk about the *periodic aspect*, as an expression that will allow us to characterize the periodic quality in different mathematical objects that live in school and to make it meaningful for each of them.

2 Theoretical and methodological aspects

According to Bell (1949), the mathematics of periodicity, as opposed to the mysticism with which natural periodic phenomena were treated, was originated in 1748 when Euler determined the values of circular functions. Whitehead (1983) mentions this property as a concrete example of the effect upon the abstract development of mathematics over science and even establishes that the birth of modern physics was based upon the application of the abstract of periodicity to a wide variety of concrete examples, and "when it became completely abstract, it was useful" (p. 334). The tasks performed by those people are totally dependent on their socio-cultural paradigms; this is the type of data that make up our significant basis for the periodic quality.

Socioepistemology, as a theoretical approach, recognizes mathematics as a human, cultural and historically determined activity; it points out that didactic phenomena cannot be understood or analyzed without reviewing the evolution of the object to be taught, (…) which leads us to question the contents and meanings proposed in the study plans (Ferrari and Farfán, 2008; p. 310). A socio-espistemological research is based upon the recognition of didactic phenomena and educational problems (figure 3) and undertakes a historical review, in those communities that build and use mathematical knowledge, evidencing those circumstances that are socio-culturally situated, which surround the scientific tasks of man (Buendía & Montiel, 2011). It is a search for practices, which, without being necessarily explicit, does foster, guide or even govern the generation of certain mathematical knowledge. As a product – never ending, always in constant configuration – of these reviews, it is feasible to propose an epistemology of practices that supports the role of these in the construction of mathematical knowledge.

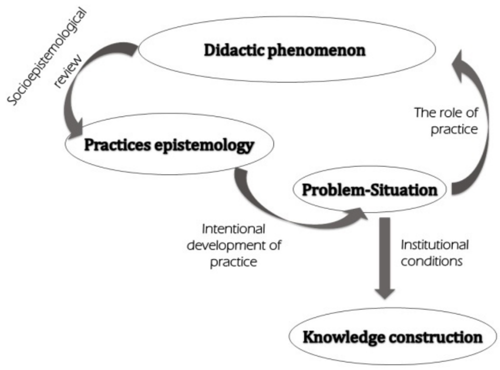


Figure 3. Methodology

The epistemology of practices is the base of the didactic designs for the mathematics' classroom; however, in order to achieve this, the practices must be re-interpreted (middle of the diagram figure 3). As a first step, the practices would have to be intentionally developed in a context based on the students' socio-cultural and institutional reality, because it is not the purpose to repeat activities or challenges in their historical version. Thus, Socioepistemology does not turn to history in order to incorporate it to the teaching of mathematics, but rather looks for what man did to generate knowledge, reinterprets those practices and develops them intentionally in didactic designs.

These didactic designs can be quite different: activities sequences, laboratory practices, proposed methods for class work. But what they all have in common is the previous work: a *socioepistemology* that supports them. And it is through them that a positive impact is achieved in the classroom.

The main objective of this paper is to show some of the results of the socio-espistemological review in its historical aspect to generate a significance base for the periodic quality; in order to do this, here are the methodological tools guiding this review.

2.1 The use given to periodicity

Under a socio-epistemological point of view, Espinoza and Cantoral (2010) state that scientific work must be understood as a production with history; belonging to an era, to a human being who has his own germinal ideas and his own means of significance. Additionally, it should be recognized that every mathematical work is different in regard to its intention to divulge, because there are differences between a didactic intention and the intention for publishing scientific knowledge.

What we have done is to try to distinguish the different *uses of periodicity* in various mathematical works which we felt were emblematic. This epistemological construction for *use* (Cordero, 2008; Cordero and Flores, 2007) refers to the different functions and forms that periodicity can take in mathematical works dealing with periodic situations, phenomena, movements or mathematical objects. Analyzing these different forms and functions of periodicity will allow us to evidence practices related to the significance of this property, or, to put it another way, practices related to the constitution of periodicity.

The chosen mathematical works are presented in three moments whose central point is the XVIII century. The reason for this selection is that during that era, trigonometric functions formally are included in the analysis and periodicity became a property of them. We consider then that the use of periodicity shows significant changes during this time. Therefore, analyzing what happens before and after the XVIII century is useful for the discussion we are proposing.

3 First Moment: Periodicity as a shared property that may be generalized

The works of Barbin (2006), Arnol'd (1990), among others, point out the scientific interest existing in the XVIII century related to time measuring, especially, the time having as its pragmatic purpose the creation of increasingly accurate clocks. At this time mechanistic thinking governs scientific work in such a way that science is the bearer of new developments, inventions, progress and break-ups. Techniques (inventions, machinery, and mechanical arts) are the tools that provide an insight into how nature works; phenomena become artificial issues that can be studied.

How to achieve a scientific relationship between the observable – let's say the swing of a pendulum - and the *know-how* sought by science — such as manufacturing increasingly accurate clocks? This was what society requires from scientists. In this paradigm, periodicity takes on the form of an internal quality characterizing different behaviours, something that is taken as given, intrinsic to the action and that allows the abstraction of the concept, in order to generate similarities between all those different behaviors.

Seemingly unrelated situations such as measuring the time, the nature of musical sounds, or the internal composition of bodies were addressed as special cases of vibratory motion. Hence then the relevance of studying and analyzing basic motions as the spring or pendulum and that meant finding relationships between components and laws. Hooke is a concrete example which illustrates the use of periodicity under this paradigm. His scientific work was guided by the search of laws, especially those governing elastic phenomena and materials, which make him go down in history (Cross, 1994).

Above all, Hooke was a “man-laboratory”[1], so his interest in scientific issues had to be diverse; he addressed issues such as the composition of bodies, free fall and planetary motion. This *know-how* could surround that which, in our days, school succinctly presents as *Hook’s Law;* he had to begin by understanding the relationship between the force exerted on the mass of a chord and its position *x* because this relation would allow him to go further in all his other areas of interest. Periodicity was taken as the common property that allowed him to generate a more abstract way of thinking and to apply the main principles of the periodic quality on many other behaviors.

For example, he established that the internal movements of a solid body are vibratory and, therefore, those bodies whose particles oscillate harmonically will have a relatively stable form and volume (Gal, 2002). As for the fall of bodies, Hooke established that it was necessary to take into account whether the body was outside or inside the Earth; in this last case, the law is different since the layers traversed by the body will attract it in different directions. Therefore the law of motion inside is apparently similar to that observed in elastic oscillations (Arnol’d, 1990).

In the study of springs, school mathematics recognizes the historical fact of Hooke’s law but, in creating an epistemology of practices for periodicity, we are more interested on how Hooke used that property. In this first moment, periodicity was not yet assigned to a certain function, nevertheless the periodic quality favoured the development of scientific thinking. From the geometric and mechanic study of springs, periodicity allowed to study other vibrating situations. It was a shared quality that could be generalized.

4 Second Moment: Periodicity as a property of trigonometric functions

The XVIII century is characterized by the process of converting mathematical analysis into an autonomous scientific discipline: all the initial calculus’ concepts are gradually losing their geometric and mechanic shell, moving toward a geometric and algebraic formulation (Youschkevitech, 1976). The dominant paradigm of that time focuses on the mathematization of movement, and therefore, properties such as periodicity are being questioned from this angle.

The problem of a vibrating chord[2] perfectly reflects this new paradigm as it causes a strong reflection on the definitions given to several of the concepts addressed until then, among them that of function, its properties and representations. In fact, Grattan-Guiness (1970) states that most of these scientific controversies came to light only at the time when there was established a requirement for periodicity, as a condition for the solution of the problem. It seems that this property is the focus of discussion on the problem and the controversy lies on what was being understood as periodic at that time: in the XVIII century, this attribute is

[1] During the 40 years in which he was curator for the Royal Society (the English Academy for Science), he had to present between three and four experiments each week in order to demonstrate natural laws (either his own experiments or the experiments of other scientists). This characterizes his production – and his problems with other scientists of his time – because he did not always have time to mathematically support his experiments or findings or even to write them down (Arnol’d, 1990).

[2] The problem of the chord can be described in the following way: suppose that a flexible chord is pulled taut and its ends are fixed at 0 and a of the abscissa. Then the chord is slackened until it takes the form of a curve $y = f(x)$ and is then let go. The question is: what is the movement described by the curve?

intrinsically contained within an analytic expression, without reference to an independent domain; the analytic expression is the function itself (Farfán, 1997).

Euler, for example, accepted as a periodic function a parabola $f(x) = hx(a-x)$, since it can be made periodic by reflecting the arcs in relation to the straight lines $x = \pm na$. This seems to imply a return to geometric aspects so we can talk about a "geometric periodicity". Although this idea of assigning different intervals of definitions for a function was not consolidated at this time, for us is important to notice that periodicity is characterizing the repetitive behavior of the curve and not necessarily the function being analyzed.

In *Introductio in analysin infinitorum,* periodicity is a property that allowed Euler to generalize the properties in his newly raised geometric function, and, in his own words, *unravel many questions the use of which will be of advantage for the solution of the most difficult issues* (Durán, 2000). For a fuller discussion of the use of periodicity in *Introductio* see Buendía and Montiel 2011. Various authors as Durán (2000) consider the – implicit – recognition of periodic property for the sine and cosine.

In this second moment, periodicity takes sense in the mathematization of movement, especially as a tool for predicting. This use necessarily influences upon its formalization, as in Euler's *Introduction*, taking forms as $\sin(2\pi + z) = +\sin z$ or $\cos(2\pi \pm \varphi) = \cos\varphi$ which operate within an increasingly analytical framework for doing mathematics. These forms are the ones which in today's schools have become institutionalized.

In our socio-epistemological search for meaningful aspects of periodicity, this property can be seen now as qualifying the behavior of any given function, particularly in its graphic form. It also acquires meanings in the mathematical manipulation of periodic phenomena, for example, to predict on them.

5 Third Moment: Periodicity and its variations

During the XIX century, the practical usefulness – understood as a scientific thought encompassing various areas – quickly gave way to purely mathematical interest. In this context, the use of periodicity becomes interesting as a property that can be exploited for the development of scientific thinking. This can be found in the work developed by Poincaré, who is considered as the last universal man. By broadening his studies on differential equations and on elliptic functions that have double periodicity[3], periodicity is a part of *invariance* under transformations forming a group. This property assumed a more complex form because, starting from the premise that a function is periodic with period p if $f(z) = f(z+p)$, it now emphasizes that for a complex function, what happens is that the function *f does not change* when z undergoes a transformation of the $z' = z + kp$ (k integer) group. Thus, the elliptic functions do not change when z is substituted by the group of transformations $z' = z + kp + k'p'$ (Collete, 1986).

Based upon this approach, he developed works such as the one that merited the prize offered by the King of Sweden, Oscar III, during the mathematical competition trying to determine the stability of the Solar System, as a variation of the three-body problem. This

[3] An elliptic function has two different periods p_1 and p_2 thus verifying that $E(z + p_1) = E(z)$ and $E(z + p_2) = E(z)$.

problem consists in determining at any given time, the positions and velocities of three bodies, of any mass, subjected to their mutual attraction and starting from given positions and velocities. The equations involved cannot be resolved in terms of known functions and since the problem even has countless solutions, Poincare concentrated on the relationship between these periodic solutions (Collette, 1986). Thus, he developed a new approach for finding solutions to differential equations governing periodic movements in which the positions and velocities are simultaneously involved.

According to Aluja (2005), the periodicity of movement is determined not only when the body goes through the same point, but at the same speed and in the same direction at a given time. The periodic quality is used in the so called Poincaré section: instead of following with a telescope a body's entire trajectory around the Earth, one focuses on a plane going from north to south, from one horizon to the other, and which is aligned with the center of our planet. One must take note of the place where it goes by for the first time, its speed and its direction, and one lays in wait, focusing only upon the plane. Periodicity causes the body to appear at the same point, with the same speed and direction, so the periodic quality and its variations can become a tool for prediction.

Thus, periodicity would seem to be made up by increasingly abstract generalizations, trying to understand and take advantage of the complexities of periodic movements. These descriptions must take into account the interrelations, but not only how between variables such as time and distance, but also in their variations. In this context, periodicity works as an important predictive tool because of its variations characteristics: how does a periodic function varies and how do their variations change.

Now is evident the poor treatment that the school gives to periodicity. In first place, periodicity is use to qualify a function – merely, a trigonometric one - but its teaching do not favor the recognition of the function behavior as the one that causes the periodic adjective. So, any sinusoidal function, including its graph or the motion modeled by it, inherited the periodic property; periodicity ends as "anything that repeats". This can cause didactic phenomena as shown in the initial examples. The third moment enriches this meaning by pointing out not only the repetitive character of a mathematical object, but also "how does it repeats itself".

6 Discussion: periodicity's uses

In the historical data we have presented, periodicity has been continually appearing in the development of the scientific thinking of humanity through different uses. To speak about the use of periodic property in the light of the – implicit – exercise of practices provides mathematics and especially periodicity with a different epistemological character.

Periodicity evolves from being an intrinsic property of several phenomena which cannot be questioned, because it does not make sense, but which allowed the generalization and finding similarities. This is, perhaps, a handling similar to the one found in school, especially in lessons on physics and differential equations that deduce laws and properties. However, recognizing the role of practices broadens our horizon.

Although there are moments where there is a formalization and generalization of scientific knowledge in which the periodic property takes its standard form as equality, it is important to recognize that the related tasks, as predicting in motions, give meaning and significance to those formalizations and generalizations. There is an implicit practice that gives meaning to what nowadays school presents as a pre-existing property which only has to be *applied.*

It should be pointed out that the scientific and meaningful use of periodicity describes not only the repetition of a movement, but also its successive variations. Or, it graphically describes the behavior of any type of curve: a translation in one direction.

On this basis of significance for that which is periodic, visible activities such as *modeling, graphing, experimenting, measuring* stand out, all of which are performed in different situations and under very different mathematical tools. These activities could seem to be articulated, motivated or even regulated by practices such as *prediction* or *formalization.* The proposal is that these activities and practices, epistemologically related to the significant recognition of periodic property, sustain the didactic proposals.

7 Toward the mathematics classroom

7.1 Recognizing periodicity

Buendía (2006b, Buendía and Cordero, 2005) offers evidence that the intentional prediction on repetitive graphics supports the distinction between *it is repeated* and *how it is repeated.* In order to do this she presents a set of 8 diagrams (figure 5), asks for a description of how any type of mobile is moving in each of them, and asks for a prediction of the mobile's position in the 231 time (the diagrams show up to t = 12) and then to tell which of them are periodic. This promotes a distinction between the type of repetition present at each graph, because in order to be able to predict, it is necessary to distinguish the repetitive behavior of time as well as the repetitive behavior of distance. Thus, periodicals are those which comply, as was answered by an engineering student at the end of the interview, *with the condition of equal times and equal distances*.

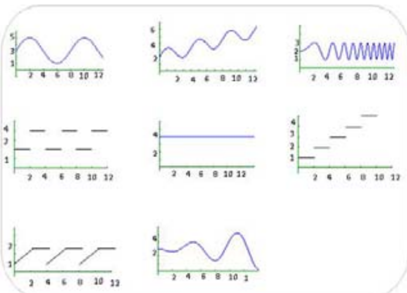


Figure 5. Predicting on diagrams

Vázquez (2008) states that the significant recognition of what is periodic requires two actions enhanced by the fact of predicting upon a repetitive object; first, identifying upon the object that part which will contain sufficient information to be able to predict (analysis unit) and, then, to be able to apply some procedure in order to perform said prediction. It is important to point out that these actions are performed using the mathematical tools that students acquire according to their age and educational level. The example in figure 6 shows these two actions performed by Dulce, a 7 year old student, who at the beginning of the interview said that she knew how to add from 10 to 10, even though she had not yet been taught this in school.

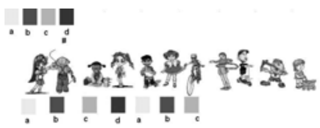
To end the week, I want to give different flavored chewing gum to my friends in school, following that order: (a) mint, (b) tutti-frutti, (c) peppermint, and (d) cinnamon. My friends are 23. I want to know, what flavor will correspond to the last person in line? 	First, she matched with her fingers the printed images of the chewing gums with the children standing in line, but then the researcher gave her several boxes of chewing gum. There were yellow boxes for the mint flavor, pink boxes for tutti-frutti, green for peppermint and red for cinnamon. Dulce realized that, in order to be able to predict, the easiest thing would be to first do her *units of analysis* in the form of small piles of colors, which would make it easier for her to take chewing gums from each group.  Once she had identified the four flavors, she discovered that with 8 chewing gums she was able to repeat them by adding from 8 to 8. She added three times 8 and then took away one chewing gum saying that the mint flavor would correspond to him.

Figure 6. Prediction activity for children

These researches suggest that the significant recognition of the periodic aspect is linked not only to the completion of sequences, but also to activities involving long term predictions; they do not depend on whether a student can be able to predict or not, because it is in the use of his own mathematical tools that the way in which an object is repeated comes into play. This poses a much more significant backdrop for the discussion of periodicity.

7.2 Periodicity and its variations

Buendía and Ordoñez (2009) performed an analysis on Miranda's (2003) didactic material, in which, faced with the need to make a prediction, it becomes necessary to analyze not only the repetition of a movement, but also its variations. This material is about cam design (figure 6); the analysis of how the movement changes, according to the cam's form, is a key point that must be taken into account to ensure that the mechanism will not cause any problems.

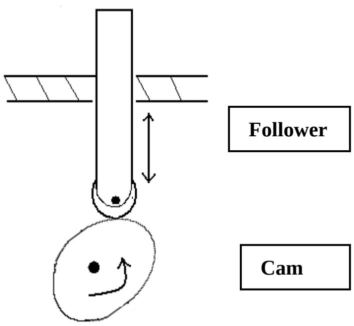


Figure 6. The follower regularly pushes the cam

Miranda points out that, although there are cams that cause seemingly soft movements, such as the parabolic (see figure 7), its derivatives may actually not be thus. Being that derivative is a measurement of the speed with which the cam's movement changes, it will help to check that the follower's movement is *soft* and maintains its periodic behavior. Therefore it is important that, for high speeds, we check that the cam profiles do not show abrupt changes in speed or acceleration, so the diagrams must illustrate that there are no abrupt changes in slope or discontinuities.

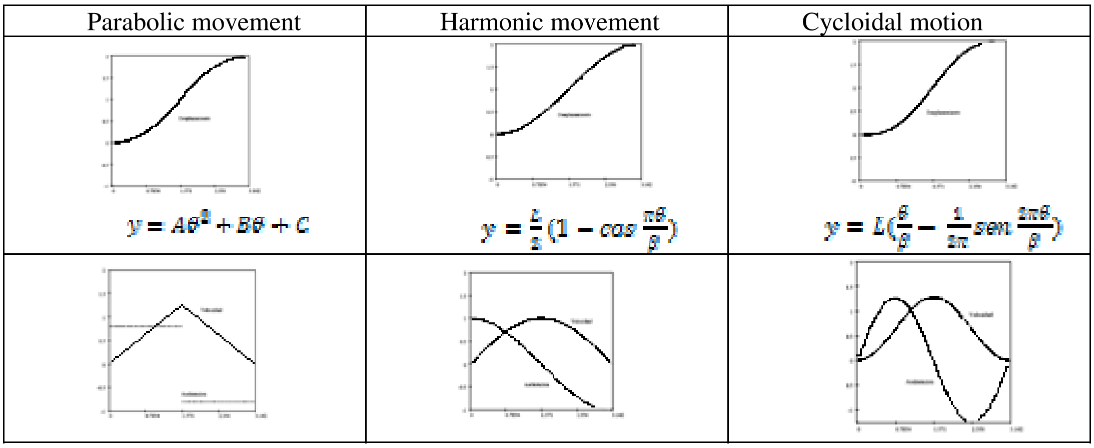


Figure 7. Displacement, speed and accelerations of possible movements

In this treatment, the mathematical concepts of successive derivatives and the physical concepts of speed and acceleration acquire meaning in the mathematization of the periodic variation.

7.3 The periodicity in graphics

The most common presentation of periodicity in text books compares point by point two states of one function by means of the equation $f(t+k) = f(t)$. However, we have discussed how the periodic quality, in the behaviour of graphics, can be identified comparing *equal pieces*. This is often present in visual characterizations involving mathematical terms such as the displacement (Courant and John, 1982):

> *Geometrically interpreted, f(t) has period p (p = b-a) if a displacement of its diagram in p units toward the right hand side again leads us to the same graphic.* (p. 337)

Viewing periodicity through graphics may be done point by point or by pieces; both are present in the didactic system and the teacher's explanations should take into account these different uses for that which is periodic.

8 Final comments

School deals with a finished mathematical knowledge and its goal is to find a way for students to acquire that knowledge. By proposing a socio-epistemology of the mathematical knowledge, we are questioning that mathematical knowledge. From being a finished mathematical result that we have to learn, it turns out to be the product of human activity while developing scientific knowledge; its nature is recognized as totally situated and dependant of the context.

In each of the epistemological moments presented, there is a set of historical data about scientific studies in which the periodicity was somehow involved. The basis of meanings for periodic property has considered not only those historical facts, but also why and how the men got involved. We can recognized that the mathematics developed has been guided or even governed by underlying practices such as predicting; from them, periodicity gets meaning and significance.

We are problematizing the periodic property knowledge confronting school mathematics with the use of knowledge in different historical settings. The purpose is to recognize those significations that belong to knowledge but usually become diluted, altered or lost when setting up a school discourse.

Periodicity turns out to be so much more than the application or the testing of equality. Through history, periodicity was used as an argument that qualifies a certain repetitive behavior and from there, under a more analytic use, was a property for functions that modeled oscillatory movements. School mathematics usually recognizes this last moment, but we think the very nature of periodic property is shaped also by its other socio-epistemological moments shape. All of them should have space in the mathematical curricula in the form of didactic interventions that intentionally develop the practices that constitute the socio-epistemology for periodicity.

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