HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION:

Problems and Prospects

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ABSTRACT

Where educators are committed to goals determined by the need to teach students modern mathematics or mathematics as it is used in modern scientific or technological contexts, history of mathematics may be forced to serve aims not only foreign to its own but even antithetical to them. This "conflict of interest" must be confronted if one wishes to embrace the history of mathematics as something more than another tool in the mathematics educator's arsenal, but as an inquiry important in its own right. This paper will suggest that this may involve redefining our general goals for mathematics education more than the specific concrete ways in which we bring history of mathematics into the classroom. To this end, the paper suggests looking at the teaching of literature as a model.

1 Introduction

All of us who attend HPM conferences, receive its newsletter, and, generally support the organization, must, at some level at least, believe that history and pedagogy of mathematics do go together, that there is some deep harmony between them. I myself believe this no less than anyone else faithful to the HPM. Yet, for several years now I have been trying to show that the alignment between history of mathematics and mathematics education is neither obvious nor unproblematic (e.g. Fried, 2001, 2007). In particular, this is the case if one takes history of mathematics seriously, and not only as a tool to be used or cast away whenever it is found to be convenient or inconvenient. In the first—and longest—part of my talk, then, I shall give the general outline of where I have found difficulties in the incorporation of history of mathematics in mathematics education. It is this part that the word “problems” refers to in the title. But my intention here is not criticism. Rather it is to find the ground from which we may explore a deeper relationship between history of mathematics and mathematics education. For this reason, I shall give particular weight to the nature of the historical enterprise in general. In the second part, the part to which the word “prospects” in my title refers, I shall try not so much to solve the problems of the first part, but to see what it means to find a solution. And I might as well say here at the outset, that I think this means redefining our very goals in mathematics education. So my message will, in some ways, be a radical one.

2 Problems — A Quasi-Dilemma

I have used the word problems in the plural because there are, naturally, a variety of problems confronting mathematics educators who wish to bring history of mathematics into the classroom. Many of these problems are practical problems similar to those that burden anyone wanting to do something new or non-routine in mathematics education, or, any other kind of education. The point I wish to make in this first part of my talk is that these practical problems mask deeper, more fundamental difficulties, ones that have to do with the very identity of mathematics education and history of mathematics.

One immediately has a hint of this even with the expression “non-routine” just used: is the history of mathematics non-routine? In a factual way, of course, it is non-routine in the sense that it is not pursued in the classroom in the same way as algebra is. But is it non-routine in the sense of being non-essential? What about the very practical problem of time? This is generally high up in the list of practical problems: where can time be found for history of mathematics in an already very full curriculum? If history of mathematics is non-essential, then, indeed, making time for history must be justified. If it is somehow essential, essentially part of an ideal routine of mathematics, then time must be made for it. If the time given a subject in the curriculum is a measure of the priority given the subject, then whether or not time is given to the history of mathematics, or how much time is given it, reflects its priority in mathematics education—one might say its very legitimacy as a part of the routine. But if it is decided that it should be part of the routine, that, in turn, forces us to consider what it is the students will learn when they study the history of mathematics in their mathematics classes.

These kinds of questions, therefore, which begin as practical questions, leads us to ask about the nature of mathematics education, its aims and its priorities, and no less about the nature of the history of mathematics itself. I will begin then with history of mathematics; however, my remarks will mostly concern history itself. This is based on the assumption that to understand the nature of the history of mathematics we need to grasp what it means for the history of mathematics to be history. And it is taking the history of mathematics as history, what Uffe Jankvist calls “history as a goal” (e.g. Jankvist, 2009), that clashes with the history of mathematics as a tool for teaching mathematics.

2.1 History

It seems part and parcel of any intellectual pursuit that one thinks about the nature of the pursuit almost as much and as seriously as one pursues the actual business of it. This is plainly true regarding historians. So besides purely historical investigations, an historian such as G. R. Elton found time to write a book called *The Practice of History*; E. H. Carr wrote *What is History?*; R. G. Collingwood, *The Idea of History*; Marc Bloch, *The Historian's Craft*. Bloch's work, I might add, has a sharp poignancy about it, because Bloch wrote it as a prisoner of the Gestapo, and he was executed by them before he could finish it. This underlines the importance such works about history has for the historians who write them, which is not to say there is complete agreement among them on what history is and what is at the heart of the historian's craft. Still, there are some commonalities.

Chief among the commonalities is an acute awareness of the tension between past and present or, at very least, the need to confront the question of past and present. It would be hard to think of history with no reference to the past of course. But is it just about the past? One must refer to the present to a certain extent since historians' materials, their objects of study, are things that have made their way into the present. For this reason Elton (1967) defines history as being "...concerned with all those human sayings, thoughts, deeds and sufferings which occurred in the past and have left present deposit; and it deals with them from the point of view of happening, change, and the particular" (p.23). The last part of Elton's statement makes it clear also that it is not just past or present that is essential but how these are treated, namely, "from the point of view of happening, change, and the particular." The historical mode of thinking demands treating these "survivals" from the past, as Michael Oakeshott calls them (see Oakeshott, 1999), precisely as survivals. A survival is a survival from another world. One interrogates survivors to understand where they came from—a world not conditioned by the existence of ours, yet one out of which ours grew.

Since I have mentioned Oakeshott, let me follow him a little further. To experience the past in the present *as history* — and for Oakeshott history is a mode of experience — one must view the past unconditionally. To describe a relationship to the past that depends on the present, in other words, that sees the past in terms of present values, needs, and ideas, Oakeshott uses the term "practical past." The historical past, accordingly, is defined in opposition to the practical past; it is a past understood in terms of its separateness from the present. Thus in his chapter on historical experience in *Experience and Its Modes* (Oakeshott, 1933), Oakeshott sets out the historian's task as follows:

> What the historian is interested in is a dead past; a past unlike the present. The *differentia* [emphasis in the original] of the historical past lies in its very disparity from what is contemporary. The historian does not set out to discover a past where the same beliefs, the same actions, the same intentions obtain as those which occupy his own world. His business is to elucidate a past independent of the present, and he is never (as an historian) tempted to subsume past events under general rules. He is concerned with a particular past. It is true, of course, that the historian postulates a general similarity between the historical past and the present, because he assumes the possibility of understanding what belongs to the historical past. But his particular business lies, not with this bare and general similarity, but with the detailed dissimilarity of past and present. He is concerned with the past as past, and with each moment of the past in so far as it is unlike any other moment" (p.106).

Historical experience is naturally an experience in the present and one belonging to a living and breathing historian. Still, that does not preclude the desideratum to view the past in its particularity, even though the past enters and informs the present in this way.

This desideratum takes its concrete form in historians' rule to avoid anachronism. It is not an easy rule to obey, since we are being who live in the present and whose immediate experience is not that of the historical subjects we study. The struggle with anachronism is at the heart of the tension between past and present, with which I began this section. It might be said, indeed, that the historical art is one that aims to keep that struggle alive. The dangers of submitting to anachronism and the subtle ways in which can subvert history was discussed most trenchantly and colorfully by Herbert Butterfield in his classic, *The Whig Interpretation of History* (Butterfield, 1931/1951). Although Butterfield's polemic was immediately directed towards specific historians such as Macaulay and Trevelyan, it was in fact a critical view, along the lines which we have been discussing, of all historiography. A definition, more or less, of Whig history is given by Butterfield right at the start of his book:

> What is discussed is the tendency in many historians to write on the side of Protestants and Whigs, to praise revolutions provided they have been successful, to emphasise certain principles of progress in the past and to produce a story which is the ratification if not the glorification of the present (p.v)

Whiggism creates a distortion of the past not only by reading modern intentions and conceptions into the doings and writings of thinkers in the past, which is anachronism in its most direct form, but also by forcing the past through a sieve keeping out ideas foreign to a modern way of looking at things and letting through those that can be related to modern interests. For example, in reading Proclus' *Commentary on Book I of Euclid's Elements*, a Whig historian would leave out Proclus' arguments in the "first prologue" about the nature of mathematical being and role of mathematics in the moral education of the soul and emphasize Proclus' comments relating to logical difficulties, missing cases, alternative proofs connected to the familiar geometrical propositions in the *Elements*. These things are truly to be found in Proclus, but a Whig historian would give the impression that these are the only things in Proclus, or the only things of any worth in Proclus.

Whig historians treat the past, almost by definition, as a "practical past," adopting Oakeshott's term: they seek in the past what is useful for the present. The problem with this is that by using the present to determine what is useful for the present, one finally forfeits learning something from the past. More specifically, as Butterfield tells us:

> If we turn our present into an absolute to which all other generations are merely relative, we are in any case losing the truer vision of ourselves which history is able to give; we fail to realise those things in which we too are merely relative, and we lose a chance of discovering where, in the stream of the centuries, we ourselves, and our ideas and prejudices, stand. In other words we fail to see how we ourselves are, in our turn, not quite autonomous or unconditioned, but a part of the great historical process; not pioneers merely, but also passengers in the movement of things" (p.63)

In short, if the project of the Whig historians is to provide an enlightening view of the present, then their approach to this, according to Butterfield, is self-defeating.

2.2 Whig and non-Whig history of Mathematics

It can be argued that the problem of Whig history is particularly acute in the case of the history of mathematics. This is because mathematics enjoys a status of being a constant component of thought not only in the modern world, but also everywhere else and at all other times — even beyond the planet! Thus, when the question of how we should try and communicate to intelligent extraterrestrials was first seriously taken up, the widely accepted view "...was that pictorial messages based on science and mathematics would be universally comprehensible by technologically advanced civilizations" (Vakoch, 1998, p.698).[1] But to the extent that mathematics is continuous over time and place, a universal body of content, it is really a-historical and non-cultural, or, at best, its peculiarly historical and cultural aspects involve only trivial matters of form. With such an assumption in the background, it would be hard for a historian not to be Whiggish: present mathematical knowledge, short of logical errors, is mathematical knowledge *tout court*; past mathematical knowledge, to be understood, has merely to be translated into a modern idiom. What one learns from the history of mathematics, in its Whiggish form, is, in short, mathematics. And one feels fully justified in treating mathematicians of the past as Littlewood famously said of the Greek mathematicians, namely, as "Fellows of another

[1] Vakoch, the director of interstellar message composition at the famous SETI Institute, casts doubt on this still prevalent assumption.

college" (quoted in Hardy, 1992, p.81); mathematician's of the past, like one's colleagues, are useful for gaining insights into one's present mathematical research. Clifford Truesdell (1919-2000), who was a good example of this kind of historian, wrote unabashedly of this "practical past" as the goal of the history of mathematics:

> One of the main functions [the history of mathematical science] should fulfill is to help scientists understand some aspects of specific areas of mathematics about which they still don't fully know. What's more important, it helps them too. By satisfying their natural curiosity, typically present in everybody towards his or her own forefathers, it helps them indeed to get acquainted with their ancestors in spirit. As a consequence, they become able to put their efforts into perspective and, in the end, also able to give those efforts a more complete meaning" (in Giusti, 2003, p.21)

Truesdell brought to his own historical work immense and exacting mathematical and scientific insight, but his approach was generally to show where Euler, Lagrange, the Bernoullis, and the others he studied got it right and where they got it wrong—and right and wrong, in his view, were to be taken as absolutes: the same today as yesterday.

This is not to say that the history of mathematics can never make a judgment and never pronounce something right or wrong. There is certainly no lack of instances where mathematicians of the past pass judgment on their own contemporaries or on mathematicians of their own past: surely, historians must say something about that! But by what criteria can one pronounce something right or wrong? Except in the simplest cases, such criteria themselves must be taken as subjects for the history of mathematics. To remove them from history, to leave unquestioned the appropriateness of modern criteria, like those used to referee a mathematics research paper, for judging past mathematics is precisely the Whig position. A more sound approach, from an historical point of view, that is, a non-Whiggish one, is to show why writers of texts, which have made it to the present, thought their work or others' work right or wrong, to try and tease out their own presuppositions, not only regarding their criteria for correctness and incorrectness, but even more so for their way of conceiving a mathematical object or idea.

2.3 An Example: Similarity in Greek Mathematics

To give this last point some body and to begin my considerations of mathematics education, I would like to consider an example from Greek mathematics. I could have chosen one of several topics. For example, I could have chosen the word "mathematics" itself; its origin in words such as *mathein,* "to learn" (aorist infinitive) and *mathēma*, "a thing learned" or "a lesson," and its use in these senses by Plato, among others, makes one realize that even "mathematics" may not have had exactly the same meaning for Greeks as it does for us: the ancient *mathēmatikos* might have been a somewhat different creature than the modern mathematician. I could have chosen the problem of "geometric algebra." This was the focus of Sabetai Unguru's famous 1975 paper, which put on the table the issues I have been discussing in the specific context of Greek mathematics and its historiography. Instead of these, I will say a few words (too few, for sure, but time forbids more...) about the idea of similarity.[2] This is a good example of a mathematical idea that one might expect to be the same for us and Euclid if anything were. It is also a central idea

[2] See Fried & Unguru, 2001 and Fried, 2009 for a more detailed discussion.

in almost every school geometry curriculum, and many of the theorems students study can be found in Euclid.

The modern notion of similarity is colored by the idea of a transformation. For example, one speaks of a "dilation" or "contraction" which is a transformation taking a point (x,y) to a point (x',y') where $x'/x=y'/y=k$, k being the "magnification factor." This view of similarity has great power. For one, congruence can be thought of as a special case, namely, when the magnification factor is equal to 1. But more than this, by conceiving similarity in terms of transformations, one relates similarity to the entire space containing geometrical objects. This priority of space to objects is also, of course, what a coordinate system establishes. The effect of this is that similar figures will be similar in virtue of being in a space transformed in a certain way so that similarity becomes one thing for all figures. We still say that similar figures are figures having the same shape though perhaps differing in size; however, because it is the space rather than the object that is the immediate referent for the transformation, we do not have to trouble ourselves too much about what it means say a figure has a certain shape that is the same as another.

Assuming the modern notion of similarity, one can easily show how statements concerning similarity in Euclid, Archimedes or Apollonius are consistent with it, statements such as these:

> Euclid, *Elem*.VI.8: "If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another."
>
> Euclid, *Elem*.III.24: "Similar segments of circles on equal straight lines are equal to one another."
>
> Apollonius, *Conics* VI.11: "Every parabola is similar to every parabola."
>
> Archimedes, *On Plane Equilibriums*, Postulate 5: "In unequal but similar figures, the centers of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides."

Indeed, it is the power of the modern notion of similarity that allows one to think of these statements in a perfectly unified way.

But the Greek approach to similarity was not unified. This was because, in that geometry, to be similar, *homoios*, meant *primarily*, and not derivatively, to be of the same shape. With that, the question of what it means to be the same shape becomes central. And since there are different kinds of shapes, there are, accordingly, different definitions of being similar, different answers to the question what makes two figures have the same shape: similarity of rectilinear figures is one thing; similarity of circular segments is another; similarity of conic sections, yet another. Thus we have from Euclid alone:

> *Elem. III (def. 11), Similar circular segments*: "Similar segments of circles are those which contain equal angles or in which there are angles equal to one another."
>
> *Elem. VI (def. 1), Similar rectilineal figures*: "Similar rectilineal figures are such that they have each of their angles equal and sides about the equal angles proportional."
>
> *Elem. IX (def. 9), Similar solid figures*: "Similar solid figures are those contained by similar plane areas (*epipedōn*) equal in number."

Elem. IX (def. 24), Similar cones and cylinders: "Similar cones and cylinders are those of which the axes and diameters of the bases are proportional."

Add to these, other definitions from Apollonius and Archimedes:

Apollonius, Conica VI (def. 2), Similar conic sections: "...similar [conic section] are such that, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates to the lengths they cut off from the axes from the vertex of the section are equal to one another, while the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios."

Archimedes, Conoids and Spheroids (Introduction), Similar obtuse-angled conoids (i.e. hyperbolas of revolution): "Obtuse-angled conoids are called similar when the cones containing the conoids are similar."

One can see that even the notion of ratio and proportion, though generally a component in the various definitions is not essential: specifically, it plays no part in the definition of similar circular segments. In fact, Euclid's book on ratio and proportion is Book V of the *Elements*, whereas the definition of similar circular segments appears in Book III.

The connection between congruence and similarity that is so natural for us was much more subtle in Greek mathematics. For one, there is no mathematical term "congruence" in Euclid or anywhere else in the mathematics of his time. There is "equality," *isos*, and figures are equal if they are congruent, that is, if one can be made to coincide *epharmozein* with the other. And so we have Euclid's common notion 4: "*Things which coincide* (*ta epharmozonta*) with one another are equal to one another." "To be congruent" is a good translation for *epharmozein*, but the latter is not a basic relation in Euclid but a basic criterion for equality. For this reason, our congruence theorems for triangles are theorems about equality in the *Elements* where equality is proven by proving congruence. On the other hand, for Apollonius, conic sections are equal if and only if they are congruent, that is, if and only if they can fit on one another: two ellipses of the same area are not called equal. Far from being a special case of similarity, equality and similarity a kept apart in Apollonius' *Conics*. Thus, in Book VI of the *Conics*, where Apollonius speaks about similarity and equality, we have pairs of propositions such as *Conics*, VI.2, which says that ellipses and hyperbolas are equal whenever the "figures" on their axes are equal and similar,[3] and *Conics*, VI.12, which says that hyperbolas and ellipses are similar when the same "figure" is similar. From a modern perspective, VI.2 ought to be a trivial corollary of VI.12, which, needless to say, does not refer to or require VI.2 in its demonstration.

I could say more about the curious role the phrase "equal and similar" has in the semiotics of Greek mathematical discourse (see Fried, 2009), but I think what has been said suffices to show the radical difference between the modern conceptualization of similarity, which flows from the idea of transformations, and that of Euclid and Apollonius, which derives directly from the consideration of the nature of particular mathematical objects. I have tried to argue that, regardless of one's interpretation of the matter, as an historian, one is obliged to confront a conceptual difference such as this; one is obliged to face the divide between past and present. This is what I have called the historian's commitment. It is a commitment that, I believe, is ultimately at odds with the Whig perspective, but at very least it is one that requires the historian to look intently at

[3] The "figure" or *eidos* of an ellipse or hyperbola is the rectangle whose sides are the diameter and latus rectum of the section.

the past as the past and to adopt, even as a sort of null hypothesis, the position that the past is different than the present. This brings us to mathematics education and the nature of its own commitments.

2.4 Mathematics Education

So what kinds of considerations must mathematics educators bring to bear on an historical discussion like the one above concerning similarity in Euclid and Apollonius? First of all, while the history of mathematics can bracket the present in order to understand the past, mathematics education typically justifies itself by the power and necessity of mathematics in modern contexts, in science, engineering, economics and industry. This is certainly consistent with the spirit of the American *Principles and Standards for School Mathematics* (NCTM, 2000). There, we read:

> The level of mathematical thinking and problem solving needed in the workplace has increased dramatically.
>
> In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors.
>
> ...More students pursue educational paths that prepare them for lifelong work as mathematicians, statisticians, engineers, and scientists.
>
> ...Today, many students are not learning the mathematics they need. In some instances, students do not have the opportunity to learn significant mathematics. In others, students lack commitment or are not engaged by existing curricula. (NCTM, 2000, Introduction)

Regarding the specific question of similarity, the emphasis of the NCTM, as one might expect, is the modern one of transformations. Starting with preschool (!), instructional programs are supposed to enable students to "Apply transformations and use symmetry to analyze mathematical situations" (NCTM, 2000, Overview of the geometry standard for Pre-K-2). And by the middle school years, this standard is matched by the expectation that students be able to:

- Describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling;
- Examine the congruence, similarity, and line or rotational symmetry of objects using transformations. (NCTM, 2000, Overview of the geometry standard for Grades 6-8)

This emphasis on modern mathematics in the school program—and in the case of geometry, on transformations—is hardly unique to the NCTM program.[4] Nor does the emphasis on the societal and scientific needs of modern mathematics belong only to the

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[4] See for example the Italian program for mathematics as presented in the European Mathematical Society (EMS, 2001) document, Reference Levels in School Mathematics Education in Europe, Italy, where again students in the first years of high school (ages 14-16), according to a "widespread experimental curriculum," the Brocca curriculum, are to learn:

- Euclidean plane and space and its isometric transformations. Geometric figures and their properties. Equidecomposable polygons; Pythagorean theorem.
- Dilatations and similarities. Thales theorem (i.e.: In a parallel projection between two lines, the lengths of corresponding segments are proportional) (p.3).

Needless to say, the idea of transformations can be found in the national curricula of many of the countries described in the EMS document.

NCTM, although it is perhaps more explicit there than elsewhere. This emphasis is not at all unreasonable: the ideas and methods of modern mathematics are, as I have already said, truly powerful and deep.

But accepting this kind of emphasis also means that mathematics educators cannot bracket the present, as historians can and must. When mathematics educators — even those with real historical sensitivity and knowledge — confront a chapter in the history of mathematics, like ours on similarity in geometry, they must heed, to some extent at least, the counterweight of their obligation to teach mathematics in a modern spirit. They must consider how relevant the chapter is to the modern mathematical ideas they need to convey, how well it fits the subjects required by their curriculum. Their considerations of time and scheduling, as I mentioned in the introduction, are only signs that history of mathematics in the classroom must be subordinated to such standards as I have described in the example of the NCTM. There may be some historical topics for which a happy medium can be found, some cases where chapter in history of mathematics fits snuggly in the curriculum without requiring too great a compromise as to its historical character. This may be; but it is not the point. The point is that when mathematics education emphasizes mathematics as it is understood and practiced today, as it is needed in science and engineering, it will be predisposed to treat the history of mathematics in a Whiggish spirit, it will have that sieve in hand which separates relevant from irrelevant ideas.[5] This predisposition is not an injunction to be Whiggish; it is, rather, a kind of internal pressure at work in any attempt to introduce history of mathematics into mathematics education, where the latter is directed, as it generally is, towards modern mathematics.

One might be tempted to compare this situation with that of the history of mathematics itself. After all, it too, as remarked above, struggles with the problem of anachronism. However, engaging in that struggle is part of what it means to do the history mathematics: historians are derelict of their duty if they not live in the tension between past and present. But in the case of mathematics education the problem is one of conflicting demands, a kind of dilemma: keep modern mathematics as one’s main end and thus make history serve modern mathematics, that is, adopt a Whig version of history of mathematics, or keep history of mathematics as history and put aside the perfectly legitimate emphases of programs that train students to use and understand the modern mathematics essential for all the pure and applied sciences.

This is a dilemma; but it is a quasi-dilemma because the force of the dilemma derives from accepting ends like those described in the example of the NCTM. Those ends built as they are on the power of modern mathematics to address societal and scientific needs are legitimate and not easily dismissed; however, they are not absolute. One can entertain other ideas about what it means to teach mathematics or what it means to be considered mathematically educated. With this we can begin to consider the prospects of a mathematics education shaped by history of mathematics as a form of knowledge rather than a mathematics education that only uses history of mathematics as a tool to promote ends not necessarily in line with those of history.

[5] A student of mine, Abdelrachman Affan, will soon be examining how presuppositions of mathematics teachers weighing the inclusion of history in mathematics education differ from those of history teachers weighing the inclusion of mathematics in history education.

3. Prospects

3.1 History as a goal

In considering the prospects for making history of mathematics part of mathematics education, I ought to reiterate what I mean by this. Again, to use Uffe Jankvist's phrase, the problem is how history of mathematics can become a *goal* in mathematics education. When we teach our students the idea of a derivative of a function or the meaning of similar figures or the Pythagorean theorem, we may have in mind some applications of these things: in teaching the Pythagorean theorem, for example, we may have in mind the application of that theorem in deriving the equation of a circle in analytic geometry. I doubt any mathematics educator, however, would seriously weigh the option of not teaching the Pythagorean theorem if some other way of deriving the equation of a circle were shown or if it happened that the equation of a circle were dropped from the curriculum. We teach the Pythagorean theorem because we believe that ignorance of it is unforgivable in a person deemed mathematically educated. This, obviously, is what we mean first of all when we say that we teach something as a goal. But it also means more than that. For knowing the Pythagorean theorem cannot mean the mere ability to recite it. It must be understood in relation to other things one learns. Therefore, one is taught, for example, in what ways the Pythagorean theorem is equivalent to the vanishing of the scalar product of orthogonal vectors—and, one hopes, one is told also how it differs from this. One might say that knowing the Pythagorean theorem is learning to see it as part of one's mathematical landscape.

Asking what things ought to be focal points in a mathematical landscape is a way of framing the entire challenge of forming a curriculum. Thus, one might ask — and this, minus the metaphor, is generally the way the question is put — where does history of mathematics fall in this mathematical landscape of ours or where should it be placed? One asks, for example, in conjunction with what topics should historical highlights be brought in, what historical episodes are to be presented to students and when? These questions have a seeming reasonableness, yet I believe they hide a categorical error. For asking where history of mathematics is located, or should be located, in the mathematics curriculum, just as one might ask about the Pythagorean theorem, is a little like asking where among the buildings here — the library, the administration offices, the laboratories — should we place the Vienna University of Technology? Inquiring where topics and ideas are placed in the mathematical curriculum assumes, to use a term from Saussurean semiotics, a synchronic view of mathematics: one sees mathematical concepts, techniques, theorems as part of a coherent harmonious whole where everything has a definite place and immutable relationship with everything else. In such a view it makes sense to say that the Pythagorean theorem truly is equivalent to $\mathbf{u}\cdot\mathbf{v}=0$ whenever $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. History, on the other hand, is not just another *component* in this harmonious whole; rather it is another *view* of the landscape of mathematical concepts, techniques, and theorems. Again appealing to Saussurean terms, this other view is a diachronic one, which Saussure himself said is as incomparable with the synchronic view as a longitudinal section of a plant is with its cross section: both explain the whole but from different perspectives and according to different principles of order.

So how one understands history as a goal for mathematics education must be different from how one understands, say, the Pythagorean theorem as a goal. Both are understood as being valuable in themselves as knowledge, that is, not as a tool only for something else more important; but while the latter fits into a kind of system as a key component, part of the puzzle, as it were, the former involves changing how we look at mathematics education altogether. I recognize this is a strong statement, but it is, I think, an inescapable one in light of what was said in the first part about the nature of the historical outlook. Indeed, it is inescapable in light of the simple fact that history *has* a definite outlook; as Sabetai Unguru has said, "The history of mathematics is history not mathematics" (Unguru, 1979, p.563): history of mathematics is not just another mathematical topic.

3.2 What kind of curriculum would be consistent with history of mathematics as a goal?

I put this as a question and put it this way because the moment one reorients oneself towards mathematics viewed historically I believe that more than one way of realizing an educational program for it will present itself, that there is no single definite curriculum answering the requirements of history of mathematics as a goal. That said, one aspect of such a program is surely unavoidable. I have in mind the presence of original mathematical texts in one form or another, abridged or unabridged, translated or un-translated, singly or conjointly with other texts. In this connection, it is worth recalling the remark by G. R. Elton quoted above about history being "...concerned with all those human sayings, thoughts, deeds and sufferings which occurred in the past and have left present deposit..." (Elton, 1967, p.23). For the general historian, that "present deposit" includes written texts; for the intellectual historian, like the historian of mathematics, it is *primarily* texts. It is true that the written texts, as has been so often pointed out, are only end products and hide a long, often private and tacit process of thought; however, more than anything else in our possession, mathematicians' own texts nevertheless provide hints of that process and do show how the mathematicians themselves sought to present their thought. For this reason Eva T. H. Brann (1979) has contrasted original texts and modern textbooks in the following way:

> Textbooks, then, are opposed to works that are original in both senses of the term, in being the discoveries or reflection of the writer himself, and in taking a study to its intellectual origins, using the original language of discovery. In sum, textbooks follow primarily a scheme of presentation; texts convey the order of inquiry (p.100).

In an obvious way, original mathematical texts also reflect how mathematicians have sought to engage other mathematicians in their thought; they represent communicated thought. And that places original texts at the center of what we should call mathematical culture and tradition. Tradition, which in some sense is nothing more than a collection of mathematical texts,[6] has, accordingly, come into the arguments of those who promote the use of original sources in teaching mathematics (and here I should underline that I am quite far from the first to stress the importance of original works: indeed, we shall hear more about this with Michael Glaubitz's plenary). For example, Laubenbacher, Pengelley, & Siddoway (1994) write this in defense of using original texts

> For a novelist, poet, painter or philosopher such observations would be old news, since their disciplines have long recognized the importance of studying the original work, techniques and perspectives of classical masters. And in so doing, they are never removed from an

[6] I am paraphrasing Brann (1979, p.64) here.

> understanding of how people have struggled, and have created works of art. Young artists thus see themselves as part of a creative tradition. Unfortunately, we have lost this sense of tradition in our discipline, and, ironically, we can perhaps blame much of this loss on the dazzling explosion of mathematics in this century. It is time we step back from our accomplishments and recapture a historical perspective.

Becoming absorbed in a tradition, however, is not a matter of nostalgia, nor of slavish loyalty, nor yet of chauvinism. A tradition always involves a kind of tension that, as Brann (1979) has pointed out, is built into the very etymology of the word. For *tradere,* from which "tradition" is derived, means both "to pass on" and "to betray." This doubleness in the meaning of the word captures a doubleness in our attention when we read texts seriously: we strain to understand the meaning and intention of our authors, what questions they are asking and how they are responding to them; however, we are also attentive to our own thoughts and our own desire to move beyond those whom we read, our betrayal of them. In this way, reading historical mathematical texts, on the one hand, forces us to try and understand past mathematicians in their own voice without imposing our own conceptions, while, on the other hand, our own self is kept in view; we see our mathematical selves in the act of confronting alternatives to them. Becoming involved in a tradition, in this corpus of mathematical works, brings us, therefore, to a true historical consciousness, which, as argued at the outset, is not so much living in the past as it is living the tension between the past and the present.

So, what I am proposing is that we view mathematics education, or rather becoming educated mathematically, not so much as the mastery of certain techniques in mathematics or even certain concepts in mathematics such as a function or derivative, but as reading and learning to read a collection of mathematical texts. Although certain texts, such as Euclid's *Elements*, would always be included, the particular choice of texts is precisely one of the things that will distinguish those different ways of realizing a program making history as a goal, which I alluded to at the beginning of this section. The reading of texts, the care given to authors' modes of presentation and their points of attention, the cultural context of works, and so on, would make mathematics education into a kind of literary education.

The comparison between a mathematics education informed by historical mathematical texts and literature education seems to me natural and potentially fruitful. To start, there are connections between mathematics and literature despite the modern habit of placing them on two sides of an impassable fence: Paul Valéry, for example, found endless inspiration in mathematics, and the great literary critic Northrop Frye said explicitly that, "The pure mathematician proceeds by making postulates and assumptions and seeing what comes out of them, and what the poet or novelist does is rather similar" (Frye, 1964, p.126). More directly relevant to mathematics education, the comparison between literature and mathematics forces us to turn away from justifying mathematics education on the basis of utility and towards justifying on the basis of culture and our own human identity as makers of mathematics.

But what I would like to stress most of all in this connection is the way literature sees the study of literature as a matter of dwelling in a world of texts, just as I would like to suggest for mathematics education in the light of history of mathematics. It is a different kind of landscape than the synchronous one described above in which there was a harmony of universal and immutable mathematical ideas; it is a landscape of texts whose place can be understood only in terms of other texts, and, as texts, these are creations of the human imagination. This, among other places, was stressed in Frye's book *The Educated Imagination* (Frye, 1964), which he addressed to teachers. Frye also makes the point that this kind of landscape is a landscape also for writers, not just learners:

> This allusiveness in literature is significant, because it shows...that in literature you don't just read one poem or novel after another, but enter into a complete world of which every work of literature forms a part. This affects the writer as much as it does the reader. Many people think that the original writer is always directly inspired by life, and that only commonplace or derivative writers get inspired by books (p.69).

In a little essay called "The Prerequisites," the poet Robert Frost writes something very similar to Frye:

> A poem is best read in the light of all the other poems ever written. We read A the better to read B (we have to start somewhere; we may get very little out of A). We read B the better to read C, C the better to read D, D the better to go back and get something more out of A. Progress is not the aim, but circulation. The thing is to get among the poems where they hold each other apart in their places as the stars do (Cox & Lathem, 1968, p.97).

Returning to history of mathematics and mathematics education, what these comparisons with literature point to is the prospect of a mathematics education in which students engage in a mathematical version of what Robert Hutchins called "the great conversation" (Hutchins, 1952). Because this means students must read authors carefully and try to understand what those authors were trying to communicate and why, because they must place themselves in another time and try to reenact the voice of another, to use Collingwood's phrase, because they must pay attention to the nuances of the text, its form, its particular way of putting things, participation in this great mathematical conversation is truly an historical enterprise. Being mathematically educated, in this light, means becoming at home in this conversation. Mathematics education conceived this way would make history a goal. But because these texts are indeed mathematical, students are also doing mathematics in precisely the way Frye says that writers, by reading writers, are involved in something essential to writing. This of course is a very different view of a mathematics education than one is used to — certainly it is different than the mathematics education suggested by the quotations above from the NCTM Principles and Standards — yet it is a mathematics education that does not neglect mathematical thinking and, more, it is one that brings one into a greater mathematical world. So, with that, I should perhaps end by repeating what Butterfield said about what we lose without a truly historical approach to things, or, put in the present terms, without being a part of this mathematical conversation, without knowing the diverse mathematical voices of the past and only seeing things through the present:

> ...we lose a chance of discovering where, in the stream of the centuries, we ourselves, and our ideas and prejudices, stand. In other words we fail to see how we ourselves are, in our turn, not quite autonomous or unconditioned, but a part of the great historical process; not pioneers merely, but also passengers in the movement of things" (Butterfield, 1931/1951, p.63)

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