HEURISTIC GEOMETRY TEACHING: PREPARING THE GROUND OR A DEAD END?

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Abstract

In the 19th century, many geometry textbooks were written that claimed to support heuristic teaching. Were did these ideas come from and what did these mean in practice? The origins of the heuristic textbook, the difference with the traditional books and its significance for the teaching practice are discussed.

1 INTRODUCTION

In 1874, a Dutch math teacher by the name of Jan Versluys, published a book on the methodology and didactics of the teaching of mathematics, the first of its kind in The Netherlands (Versluys, 1874). One of the striking aspects of the book is the emphasis on what Versluys called *heuristic* teaching and learning. In a 19th century context, the meaning of the word heuristic is a bit different from today. While we use the word heuristic in connection with strategies for the solving of problems for which no standard algorithm or solution is available, for Versluys and his contemporary's heuristic stands for teaching and learning with a maximum of self-activity of the learner.

Versluys contrasted this heuristic teaching with what he called *dogmatic* teaching, a situation where the teacher is in command of the whole process and explains the subject matter to the learner, who just has to follow the master. To Versluys it is beyond any doubt that this dogmatic teaching and learning is of much less value than the heuristic method, and that therefore teaching and learning should be organized in a heuristic way as much as possible. His argument is mainly based on the strong motivation that heuristic teaching was supposed to give to the learner.

Versluys did not pretend that these were his own original ideas. In his book, he discusses two German authors, Karl Snell and Oskar Schlömilch, who wrote geometry textbooks in a heuristic form (Snell, 1841, Schlömilch, 1849). They were certainly not the first authors who claimed that their textbook was adapted to heuristic teaching and learning. Already in 1813, Johann Andreas Matthias published a mathematics textbook for the Prussian Gymnasia whose title explicitly stated that it was intended for heuristic school teaching (Matthias, 1813). There were many more. In an article on German mathematics textbooks of the 19th century, Erika Greve and Heinz Rau express their astonishment about the large number of geometry textbooks that do make the same claim (Greve & Rau, 1959).

Based on these sources one gets the impression that heuristic teaching and learning was, if not the dominant, at least an important aspect of geometry teaching in 19th century

Germany. However, was this really the case? Moreover, in so far it existed, how was this heuristic teaching organized in the classroom? What was the role of the teacher and the textbook?

There are other interesting questions to pose. Where did this idea of heuristic teaching and learning come from? In the second half of the 18th century a pedagogical reform movement came into being in which self-activity of the learner was an important element. Was heuristic math teaching just the result of the influence of this movement on mathematics education, or were there internal mathematical developments that also played a role? I mentioned Germany and The Netherlands, but what was the situation in other European countries? From the last quarter of the 20th century on, self-activity of the learner was much more in the focus of attention than before. Are there any connections between 19th century heuristic teaching and these modern trends?

At first, some remarks on the role of textbooks in the history of math teaching. (See also Schubring, 1999). That role is relatively new. Before the invention of printing, books were so expensive, that only very few people could afford a book. Math teaching, in so far it existed at all, was part of the liberal arts at the universities, where oral presentation of the subject matter was the dominant way of teaching. The professor read a fixed text, and the student made copies. That tradition of oral presentation influenced teaching for a long time.

The use of textbooks raised, in the end, a discussion about the role of the teacher. Should he be an expert on the subject he taught, as the university professor in the age of oral presentations? In that case, he could stick to oral presentation and the role of the teacher would remain dominant in the teaching process. Such an expert could write his own textbook. In such a situation, one could expect a large number of different textbooks being in print, most of them printed in a limited amount of copies and each in use in a small number of schools.

On the other hand, one could also argue that the availability of good textbooks does make it less necessary that the teacher is an expert himself. The teacher could rely on his textbook, which could guide him through the teaching process. In that case, one might argue that it was the responsibility of the state to provide for good textbooks, and that this could not be left to the teacher. In that situation, there will be only a limited amount of textbook titles in use, approved by the government, with a large number of copies of each. Interestingly, the two major powers in continental Europe in the 19th century, Prussia and France, took these two opposite positions.

2 Pott's School Edition of Euclid

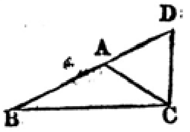
In order to appreciate the importance and relevance of the innovations in geometry teaching, I start with an example of a more traditional geometry textbook: Robert Potts School Edition of Euclid's first six books of the *Elements*, from 1850. It was "designed for the use of junior classes in public and private schools". In England, Euclid was still the standard for geometry teaching on schools. One of the characteristics of the Euclidean approach is the *synthetic* method. Very shortly summarized, in the synthetic method you start with things already known, and by putting them together, like pieces of a puzzle, you build or prove knew knowledge. New knowledge therefore is constructed by *synthesizing* old knowledge, hence the word *synthetic*.

In the 17th century, French mathematicians and philosophers began to propagate what they called *la méthode analytique*, or the analytical method. It was inspired by the developments within mathematics itself, the rise of algebra and analysis and the use of algebraic methods in geometry — which signified a breach with the Euclidean tradition. The idea of the analytical method is that you should not start with known facts, but should start with the problems you have to solve or theorems you want to prove. You suppose that you have solved the problem, or that the theorem is true, and then try to analyze it, that is to say by reasoning backwards and/or splitting up the problem in parts; you try to reduce the problem to facts already known. For example, when solving an equation, that is in fact the method followed. The advocates of this method called it *the way of the inventors*, and claimed that this was the way new knowledge was found in reality, and that the synthetic method was nothing more than a artificial make up after the invention was done.

After his treatment of the first six books of Euclid, Potts inserted a short chapter called *On the Ancient Geometrical Analysis.* In this chapter, he discusses the synthetic and analytical methods, but he does not refer to any contemporary discussions or influences. He suggests instead that Euclid in his lost *Porisms* had used the analytical method. He demonstrates that method in some additional construction problems. Figure 1 shows such a problem, with a combination of both methods: starting with an analysis, followed by a synthesis. Euclid himself never gives an analysis.

Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.

Analysis. Suppose BAC the triangle required, having BC equal to the given side, BAC equal to the given angle opposite to BC, also BD equal to the sum of the other two sides.



Join DC.

Then since the two sides BA, AC are equal to BD,
by taking BA from these equals,
the remainder AC is equal to the remainder AD.

Hence the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD.

But the exterior angle BAC of the triangle ADC is equal to the two interior and opposite angles ACD and ADC:

Wherefore the angle BAC is double the angle BDC, and BDC is the half of the angle BAC.

Hence the synthesis.

At the point D in BD, make the angle BDC equal to half the given angle,
and from B the other extremity of BD, draw BC equal to the given side, and meeting DC in C,
at C in CD make the angle DCA equal to the angle CDA, so that CA may meet BD in the point A.

Then the triangle ABC shall have the required conditions.

Figure 1

Potts edition is a genuine schoolbook, containing not only theory, but also notes, questions, exercises, hints and solutions. It shows, by incorporating examples of the analytical method, some modern influences. Nevertheless, its main part consists of more or less literally translations of books from Euclid's *Elements*, and therefore is displays the same characteristics. These can be summarized as starting with a torrent of definitions and axioms, a strict deductive-synthetic approach, the avoiding of arguments based on intuition or observation and the lack of any applications. Just those characteristics aroused the opposition of the author whose textbook I will discuss next, Alexis Claude Clairaut.

3 Clairaut's Éléments de Géométrie

Clairaut's geometry textbook was published in 1741. It contains a very interesting preface, in which he explains why he wrote the book the way he did. Although he mentions Euclid only once, he describes all the Euclidean characteristics I just summarized, and declares that just these deter and discourage the beginning student of geometry, confuse him on what is geometry all about, and make the study of geometry even boring for the more gifted student. There can be no doubt that Clairaut had Euclid in mind as a counter-example.

Therefore, his approach is different. Using surveying as a thread, he introduces concepts or theorems by practical examples, he omits definitions, axioms and theorems that are self-evident and uses intuition or observation whenever it seems appropriate. The way of reasoning is often more analytical than synthetic. Figure 2 shows a part from the beginning of the book, where a surveying problem is used to introduce the concept of a right angle.

III

Une ligne qui tombe sur une autre, sans pencher sur elle d'aucun côté, est perpendiculaire à cette ligne.

Outre la nécessité de mesurer la distance d'un point à un autre, il arrive souvent qu'on est encore obligé de mesurer la distance d'un point à une ligne. Un homme, par exemple, placé en D sur le bord d'une rivière (fig. 1), se propose de savoir combien

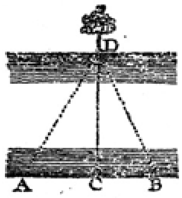


Fig. 1.

il y a du lieu où il est à l'autre bord AB. Il est clair que, dans ce cas, pour mesurer la distance cherchée, il faut prendre la plus courte de toutes les lignes droites DA, DB, etc., qu'on peut tirer du point D à la droite AB. Or il est aisé de voir que cette ligne, la plus courte dont on a besoin, est la ligne DC, qu'on suppose ne pencher ni vers A, ni vers B. C'est donc sur cette ligne, à laquelle on a donné le nom de perpendiculaire, qu'il faut porter la mesure

Figure 2

XVII

Faire un carré égal à deux autres pris ensemble.

Supposons présentement qu'on veuille faire un carré égal à la somme des deux carrés inégaux ADC *d*, CFE *f* (fig. 61), ou, ce qui revient au même,

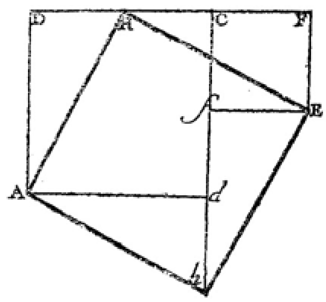


Fig. 61.

qu'on se propose de changer la figure ADFE *f d* en un carré.

En suivant l'esprit de la méthode précédente, on cherchera s'il n'est point possible, de trouver dans la ligne DF, quelque point H, tel

1° Que tirant les lignes AH et HE, et faisant tourner les triangles ADH, EFH, autour des points A et E, jusqu'à ce qu'ils aient les positions A *d h*, E *f h*, ces deux triangles se joignent en *h*;

2° Que les quatre côtés AH, HE, E *h*, *h* A, soient égaux et perpendiculaires les uns aux autres.

Figure 3

However, the *Éléments* of Clairaut mainly contains pure mathematics. In figure 3 a part of the proof of the Pythagorean Theorem is shown. The style is wholly different from that of Euclid. Clairaut starts with demonstrating how to double a square, and then he poses the question how to find a square equal to two squares that are not equal to each other. In solving this problem he explicitly refers to the problem he just solved before. Then in the end the Pythagorean Theorem appears as a result of these procedures, it is not formulated beforehand.

As Clairaut had already foreseen in his preface, his use of practical examples, of intuition and observation and of the analytical method resulted in reproaches that he maltreated mathematics, even today. However, with his book Clairaut, a first class mathematician and physicist himself, had put on the agenda the problem how to avoid "scaring off the beginner", and how to motivate and arouse the interest of the pupils. In the 18^{th} century that problem could perhaps be discarded as unimportant, but with the introduction of mathematics teaching as a compulsory topic in secondary education, this problem became highly relevant. In that sense, Clairaut could be described as the first geometry textbook author who took modern didactics seriously.

4 From France to Prussia

The ideas's of *la méthode analytique* and *la route des inventeurs* were intensely discussed by the *philosophes* of the French Enlightment, especially by d'Alembert in a contribution to the *Encyclopédie*. However, avoiding scaring off the beginners was not one of d'Alemberts concerns. He stressed that a student of mathematics should really "grasp the genius of the inventors in order to be able to master them and to be more creative"(Schubring, 1999, pag 43). One of the conclusions he drew was that textbooks should be written by the inventors of modern times, which is to say by first class scientists.

During the French revolution, the education of the masses was a major concern and the teaching of mathematics had to play major role in this. The succeeding revolutionary administrations saw the development of good textbooks as their responsibility. They undertook several efforts to develop elementary math textbooks written by top class scientist, but with no great success. One of the reasons might have been that, unlike Clairaut, most eminent scientist did not automatically make eminent textbook writers. Overmore, the concept of *elementary* were interpreted, as we can see already with d'Alembert, literally: an *elementary* textbook should treat the *Elements*, that is to say the basic concepts and building blocks of the science concerned. However, such a textbook was most likely not easy at all, and not "elementary" in the modern sense, and not apt for the education of the masses.[1]

As it turned out, in the first half of the 19$^{\text{th}}$ century the textbooks by E. S. Lacroix acquired almost a monopoly in the French educational system. Lacroix, although being a competent mathematician, was not a first class researcher. In the beginning of his career, Lacroix used parts of Clairaut's textbooks and adhered to the idea to follow *the way of the inventors*. However, in the end he abandoned these ideas. Lacroix's textbooks, interesting in their own right, are not a topic of concern within the context of heuristic math teaching.

Like in France, education was intensely discussed in the German states in the second half of the 18$^{\text{th}}$ century. An important didactical movement, the *Reform Movement*, had originated there. The ideas of teaching by illustration, of a natural way of learning in which for instance axioms and general theorems are not the starting point, but at most an endpoint, and of self-activity, or *Selbsttätigkeit*, were important elements in this movement.

In the first half of the 19$^{\text{th}}$ century, in the work of pedagogues such as Pestalozzi and Diesterweg these elements played also an important role. Diesterweg formulated these ideas as follows: *"The so called scientific method is the deductive, synthetic, (...) and often, in the worst case, the purely dogmatic, the elementary on the other hand, is the inductive, analytical, (...) heuristic one"* (Diesterweg, 1970) To be sure, Diesterweg advocated strongly this "elementary" way of teaching, not only for the elementary school, but also, in his own words, "in all schools, even in universities". And clearly, for Diesterweg the meaning of "elementary" was different than in France.

Educational policy on textbooks in Prussia was the complete opposite of that in France. Under the influence of the neo-humanistic movement, the Prussian government accomplished a reform of the universities, and one of the important tasks of the universities became the training of the teachers of the famous Prussian *gymnasia*. The government considered these teachers as fully competent in their field of teaching, and it saw no urgent need to prescribe to them what textbooks they should use.

Concluding, there was a strong didactical pressure towards more self-activity in math teaching, and the government would pose no obstacles. Teachers were competent in their field of teaching and in the course of the first half of the 19$^{\text{th}}$ even a system of teacher training was developed, in which the ideas of modern pedagogues were advocated. So, one

[1] This remained a part of the French tradition: nobody will consider the series *Éléments de mathématique* by Bourbaki to be elementary textbooks!

might expect that heuristic math teaching became an important aspect in the secondary schools. On paper it did.

5 Heuristic geometry books in Prussia

And indeed, already in one of the first math textbooks after the Napoleontic era, the word *heuristic* appears in the title. This title is *Leitfaden für einen heuristischen Schulunterricht über die allgemeine Größenlehre, Elementare Geometrie, ebene Trigonometrie und die Apollonische Kegelschnitte* (Magdeburg, 1813), by Johann Andreas Matthias (1761–1837). In translation: *Guideline for a heuristic schoolteaching on the general theory of magnitudes* (in fact algebra), *elementary geometry, plane trigonometry and Apollonian conic sections.* Matthias was teacher, later head of the *Domgymnasium* in Magdeburg, and director of the teacher training college, attached to that school. His textbook was in print until 1867, having then its 11^{th} printing.

However, if one starts to read this little book, one is surprised, or even disappointed. The booklet resembles in no way Clairauts book. It is in fact no more than a compendium, containing in a very compact way all the subjects to be taught in the Prussian gymnasium. For instance, this is the way Matthias gives the proof of the Pythagorean Theorem. (translation by the author).

Matthias' proof of Pythagoras:

"To learn: In a rectangular triangle is the square of the hypotenuse equal to the squares of the two other sides.

Proof: The application of § 102, of § 50, 1 and of § 66 with 67 demands for the auxiliary construction according to § 89. One has the compare parallelogram *ah* with the square on *ab* and parallelogram *hc* with the square on *bc*, and one has to take into account § 45, because of § 102."

So, this *Leitfaden* responds in no way to our idea what a heuristic textbook should be. How to explain is this strange contradiction? In many ways, the university background of the Prussian gymnasium teachers and their thorough knowledge of their teaching subject was of course an important step forward for the professional position of these teachers. However, it had, from a didactical point of view, also a backward effect. Gymnasium teachers were inclined to behave as university professors and relied more on oral lecturing than on the use of textbooks. A report of 1838 contains many complaints about the lack of textbooks and the *unnötigen Vielschreibei* — the unnecessary and frequent writing down — that was the consequence (Thiersch, 1838). Perhaps, the problem was not so much the lack of textbooks — there were enough textbooks in print - but more the way the gymnasium teachers used them, or did not use them. Their focus was on oral presentation and that had also its impact on the way German textbooks were written. The usual form of a German textbook was not the extensive handbook, containing a complete introduction into the subject, like the French textbooks of Lacroix, but a *Leitfaden*, a guideline, containing only a condensed treatment of the subject, intended to support the teacher, not to direct or to replace him. It is easy to see that this preference for a dominant role for the teacher was in fact in conflict with the pedagogical Reform Movement. In this Movement, a strong emphasis was laid on the self-activity of the learner.

However, according to most pedagogues and teacher trainers, this problem, the contrast between the dominant role of the teacher and the demand for self-activity of the learner, could be solved by the introduction of a special form of teaching. Oral lecturing in the classroom should have the form of a *Socratic Dialogue*, a kind of discussion inspired by the dialogues written by Plato. The teacher should not give a traditional lecture, but he should, by asking suitable questions, engage his pupils in a dialogue. By means of this dialogue, the pupils learn their mathematics. A classical example in mathematics of course is the dialogue Menon, written by Plato, in which Socrates engages in a dialogue with a slave, and in the course of this dialogue, the slave learns how to double the square. The idea behind this way of teaching is that the learner engages actively in the discussion and has the feeling that he more or less finds the solution to the question himself — or could at least have done so. The Greek verb for finding out is $\upsilon\rho\iota\sigma\kappa\omega$ (heurisko), hence the word heuristic. Interestingly, Karl Weierstrass, one of the founding fathers of modern analysis, wrote a paper on Socratic teaching and its applicability in the classroom to get his teaching license. (Weierstrass, 1841)

One can imagine that within the framework of the Neo-Humanistic movement, that was inspired by the Greek civilization and that heavily influenced German teaching; the idea of combining oral teaching with classical Greek methods seemed very attractive. Although the heuristic teaching method was certainly more than a modern version of the Socratic dialogue, it is important that there was, at least within a circle of math educators and teacher trainers, a consensus that it was possible to combine oral teaching with heuristic teaching, and that therefore self-activity of the students was possible in a classroom where the teacher held a dominant position.

However, one can have its doubts about the real impact of these ideas in the classroom. The head of a Prussian gymnasium wrote around the middle of the 19$^{\text{th}}$ century: *It is a remarkable phenomenon that, while the system of elementary schools went in the last thirty year, regarding didactics and methodology, through an enormous reform, the gymnasia remained in this period almost motionless* (Rethwisch, 1893). The Thiersch report points in the same direction.(Thiersch, 1838) And also Karl Weierstrass expresses in his paper doubts about the applicability of the method. Like the Dutch author Jacob de Gelder wrote already: it is certainly the most difficult method for the teacher. One might suspect that, in spite of the abundant use of words like *heuristic* and *self-activity*, in reality math teaching in the 19$^{\text{th}}$ century was much more traditional and *dogmatic* than heuristic.

6 Snell's Lehrbuch der Geometrie

As to be expected, not every textbook author was satisfied with this situation, and some tried to write books that were more in accordance with heuristic ideas. One of them was Karl Snell (1806–1886) He studied mathematics and philosophy, and was active in both fields. He was a math teacher at the municipal gymnasium in Dresden, published some books about more general pedagogical issues, and in 1841 appeared the first edition of his *Lehrbuch der Geometrie*. In 1844 he became professor in mathematics and physics at the University of Jena.

Snells book contains a lengthy foreword, in which he explains why his book is so different from the usual geometry books. He discusses mainly mathematical matters: the material he left out of his book and the way he arranged the remaining mathematical content. His main objection to the usual books is their lack of coherence, the missing of a central point of view. The theorems are presented in an order that facilitates only the way they are deduced from each, without taking into account the way there are connected concerning their content matter.

Snell not only discussed German books, he also made some interesting remarks on English and French books. On English geometry teaching, he remarked that Euclid has become "a sort of mathematical orthodoxy that is likewise unfruitful as the orthodoxy of their high church". On the book of Lacroix he remarked "that it is mainly admirable in that it is really a really work of art to drag the matters so much from their natural coherence, and still letting intact a complete coherence of deduction".

To Snell two things were important: that content matter, not logic, should determine the structure of the book, and that the mathematics should be presented in a natural way. Only then, he said, mathematics can become “einen humanischen wissenschaftlichen Bildungsmittel”, a means of education both humane and scientific. These two facets of a good textbook are linked up with each other, and this makes it possible for the learner to grasp a general understanding of geometry. Only on that basis, he can develop self-activity, finding theorems and proofs by himself.

From his book, we can see what Snell had in mind when speaking about presenting mathematics in a natural way. He composed his text like an ongoing story. He formulates definitions and theorems at the end of an explanation, where they emerge in a natural way, as a summing up of what just has been learned. They are not set apart from the main text; they just are a part of the story Snell want to tell us. Even typographically, they are hard to find in the text. An other consequence is that Snell does not treat construction problems in the main text; they are set apart as applications in separate chapters. Unlike Clairaut, Snell does not discuss the value of intuition and observation, in his proofs he does not rely on this kind of reasoning.

As an example, I give a short summary of Snell’s treatment of Pythagoras. He treats this theorem in the context of similarity, in a more general discussion of the relations between the sides and angles of triangles. He proves the similarity of the triangles that are created when drawing the perpendicular on the hypotenuse, dwells on the ratios of the resulting line segments and proves the theorem as just a application of these similarities. Then he continues: “This theorem, which is known as the Theorem of Pythagoras, allows calculating, when the length of two sides of a rectangular triangle is known, the length of the third.” He closes this section with a discussion on the incommensurability of numbers and line segments.

Snell’s book differs greatly from the “guideline-type” books of Matthias; in the way it is organised and structured, and in the style it is written. That does not mean however that Snell had also a different opinion about how to organise teaching. In his foreword, he remarked that his book could be used in two ways: as a schoolbook, to support the teaching of the schoolmaster, and as a book for self-study. He adds that if his intention was mainly the use as a schoolbook, it should have had another form: more concise, more sketching only the outlines of the subject, and containing more problems. From this remarks it becomes clear that for Snell the oral presentation of the teacher remained the principal part of the teaching; the schoolbook should be read afterwards. Self-activity in the classroom for Snell did not mean self-study.

7 Conclusion

However, that brings us to a principal question. If the focus is on the oral presentation by the teacher, why should the textbook be written in a heuristic form anyway? As we have seen, the ideal was that the teacher did not hold a monologue, but that he used the Socratic dialogue. If the textbook should have the same style as the lessons of the teacher, the utmost consequence would be a textbook written in the form of a dialogue. Such textbooks do exist, but mostly for primary education. They were not in use in secondary education.

Both Schlömilch and Versluys raised the question why to use a textbook in a heuristic style. Schlömilchs argumentation is that in order to consolidate the learning matter the students should write essays on the theorems and their proofs in a dogmatic style. When they have dogmatic textbook, they can learn all by head, and just copy the book, without any real understanding. That is impossible with a heuristic textbook. (Schlömilch, 1848)

Versluys however, drew a different conclusion. He argued: *“It follows from the foregoing that it is not wrong to use with heuristic teaching a dogmatic textbook. In this respect, I see no large difference in the value of a textbook. Overall, a dogmatic textbook is easier for the pupil; on the other hand, a dogmatic textbook gives more occasion to mechanical learning.* [Schlömilch’s point] *One should not forget that one that uses a dogmatic textbook, does not* *automatically teach in a dogmatic way".*(Versluys, 1874, pag. 29) It is no surprise then that, although Versluys advocated heuristic teaching, the textbooks of Versluys himself are purely dogmatic.

I think this was a much too optimistic view. In theory, heuristic teaching combined with a dogmatic textbook might be possible, but in practice, it was not. It is simply impossible for a teacher to conduct *Socratic dialogues* in all his lessons and classes. Teachers rely on their textbooks, and dogmatic textbooks invite to dogmatic teaching. Teachers that that in their own mathematical education at the university were used to dogmatic teaching, are even more likely to use that form.

Self-activity today is based on the use of voluminous and extensive textbooks and workbooks, which forces the teacher into the role attendant and coach of the learning process that occurs when the pupil is working through his textbook. That creates new didactical discussions and problems, but that is not within the scope of this paper.

The innovations in the textbooks of Clairaut, Snell and many others did pave the way to the modern textbooks we use nowadays. Only when the crucial role of the textbook was fully appreciated, modern forms of self-activity were possible. The idea of combining heuristic teaching with oral lecturing and dogmatic textbooks however was leading into a dead end street.

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