THE *Eléments De Géométrie* OF A. M. LEGENDRE

AN ANALYSIS OF SOME PROOFS FROM YESTERDAY AND TODAY'S POINT OF VIEW

Marta MENGHINI

Università di Roma "La Sapienza", Italy

marta.menghini@uniroma1.it

with the collaboration of Laura Micozzi

Abstract

The Eléments de Géométrie *of A. M. Legendre were largely adopted in many countries since their first edition in 1794. When around 1870 some countries, like Italy, decided to adopt Euclid's Elements much criticism raised against Legendre's book. The critics concerned mainly the use of algebraic means, and a general lack of rigor. Instead, no mention was made to Legendre's attempts to prove Euclid's 5^{th} Postulate.*

In the workshop we analysed the use of arithmetic and algebraic means on the part of Legendre, also having in mind today's didactical use of those means. We also confronted some proofs given by Legendre with those of the Euclidean tradition, and with Davies' American edition.

1 INTRODUCTION

A. M. Legendre (Paris 1752, Paris 1833), by undertaking the difficult task of writing a text for the teaching of geometry in schools, produced the *Eléments de Géométrie*, which was published for the first time in Paris in 1794.

This work was one of the most famous texts published during the French revolution, and it immediately had an uproarious success. The success was evidenced not only by the, at least, eleven other printings after the first edition (and these printings included additions and modifications made by the author himself, cfr. Schubring, 2004), but also because the text had an exceptional circulation outside of France. Legendre's manual was translated into every European language and Arabic. For a long time, it was used in French schools, as well as in Italian and American ones.

In Italy, Legendre's text was called into cause by the mathematicians Cremona, Betti and Brioschi during the drafting of new scholastic programs that were popularized by the Coppino reform in 1867 (Menghini, 1996). Luigi Cremona reproached Legendre's manual for having abandoned the purity of the geometry typical of the *Elements of Euclid*, "transforming geometrical theorems into algebraic formulas, i.e. substituting for concrete magnitudes (lines, angles, surfaces, and volumes) their measures".

In fact, in his *Eléments de Géométrie*, Legendre uses arithmetic *notations* and elementary algebraic rules. While this seems to the disadvantage of geometrical rigor, it makes the comprehension easier, and makes for a more fluent reading of the text. Even F. Klein (1909) spoke about how Legendre's approach differed from that of Euclid:

> The main goal [of Legendre] is, on the one hand, a system which is abstract and closed within elementary geometry; on the other hand, there are notable differences:
>
> 1. As for Legendre's text's expository style, it is continuous and easy to read [...].

2. As for the content, the essential point is that Legendre, contrary to Euclid, has a knowledgeable use of the elementary arithmetic of his time; for this reason [...] he is a "follower" of the fusion of arithmetic and geometry, and he even adds trigonometry to this fusion [...].
3. With respect to Euclid, Legendre's principle point of view shifts a bit from a logical perspective to an intuitive one. Euclid [...] places all weight on logical reasoning, which he attempts [...] not to mix with intuition; everything that must refer to intuition has already been declared in the axioms. For Legendre, however, this is not what is most important; even within a deduction, he often uses intuitive reasoning.

In this way, it becomes particularly interesting to trace the most significant points in the French mathematician's text — these points clarify the difference between the "Euclidean method" and "Legendrism".

The following texts that are cited are passages from a reprint of the book's fourteenth edition[1] (Legendre, 1957), which originates from Cremona's private library.

The English translations are taken from Davies' text (1852). This text's translation is not entirely faithful to Legendre's manual. For reasons of space, we will not insert the English translations of all the propositions.

2 The *Éléments de Géométrie*

The *Éléments de Géométrie* are subdivided into eight books. Each of these books begin by defining the geometrical subjects that the subsequent theorems refer to; the order of the propositions is chosen carefully. The arguments proposed are fairly simple and clear; moreover, by opportunistically inserting a few comments (scholia, corollaries, and notes), Legendre points out the importance of some results. This line is followed throughout the text, and it demonstrates the care that the mathematician dedicated to the didactic intent of his work.

A confirmation of what has just been asserted can be provided, for example, by the proof that the angles at the base of an isosceles triangle are equal[2] (Table 1).

From the very start, this proposition shows that Legendre's work is not a copy of Euclid's work, even if it is, without a doubt, inspired by Euclid. Indeed, Euclid proofs the same result (proposition V of his first book) by resorting to the prolongation of equal sides and constructing two triangles with two equal sides and a common angle. He thereby concludes the equality of the two triangles, as well as the equality of the angles at the base, by way of the SAS equality (proposition IV of Euclid's first book).

On the other hand, in Legendre's proof, Legendre turns to the SSS equality (proposition XI). From a didactic perspective, his proof is both shorter and simpler than Euclid's.

Regarding the presentation of contents in the *Éléments de Géométrie*, Legendre first exhibits proofs of his theorems, and later makes use of the obtained results in order to resolve the various problems (that for the most part are constructions). This fact sets up one difference with respect to Euclid's text; in fact, as Euclid argues about figures of known construction, he mixes theorems and problems. In the *Éléments de Géométrie*, we find the first *problem* at the end of the second book (Table 2).

---

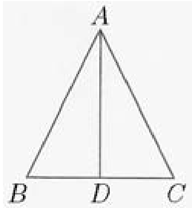
[1] The fourteenth edition is a reprint of the twelfth, which appeared in 1823.

[2] "PROPOSITION XII (BOOK I)
In an isosceles triangle, the angles opposite the equal sides are equal.
Let BAC be an isosceles triangle, having the side BA equal to the side AC; then will the angle C be equal to the angle B.
Join the vertex A, and the middle point D, of the base BC. Then, the triangles BAD, DAC, will have all the sides of the one equal to those of the other, each to each. For, BA is equal to AC, by hypothesis, AD is common, and BD is equal to DC by construction: therefore, by the last proposition, the angle B is equal to the angle C..."

Table 1

PROPOSITION XII (LIVRE I)

Dans un triangle isoscèle, les angles opposés aux côtés égaux sont égaux.

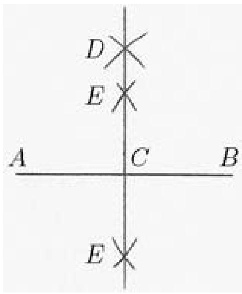


Soit lecôté $AB = AC$; je dis qu'on aura l'angle $C = B$.
Tirez la ligne AD du *sommet* A au point D, milieu de la *base* BC, les deux triangles ABD, ADC, auront les trois côtés égaux chacun à chacun; savoir AD commun, $AB = AC$ par hypothèse, et $BD = DC$ par construction; donc, en vertu du théorème précédent, l'angle B est égal à l'angle C.
Corollaire. Un triangle équilatéral est en même temps équiangle, c'est-à-dire, qu'il a ses angles égaux.
Scholie. L'égalité des triangles ABD, ACD, prouve en même temps que l'angle $BAD = DAC$, et que l'angle $BDA = ADC$; donc ces deux derniers sont droits; *donc la ligne menée du sommet d'un triangle isoscèle au milieu de sa base, est perpendiculaire à cette base, et divise l'angle du sommet en deux parties égales.*

Table 2

PROBLÊME I (LIVRE II)

Diviser la droite donnée AB en deux parties égales.



Des points A et B, comme centres, avec un rayon plus grand que la moitié de AB, décrivez deux arcs qui se coupent en D; le point D sera également éloigné des points A et B: marquez de même au-dessus ou au-dessous de la ligne AB un second point E également éloigné des points A et B, par les deux points D, E, tirez la ligne DE; je dis que DE coupera la ligne AB en deux parties égales au point C.
Car les deux points D et E étant chacun également éloignés des extrémités A et B, ils doivent se trouver tous deux dans la perpendiculaire élevée sur le milieu de AB. Mais par deux points donnés il ne peut passer qu'une seule ligne droite; donc la ligne DE sera cette perpendiculaire elle-même qui coupe la ligne AB en deux parties égales au point C.

For Euclid, the construction of geometric entities had a very important role in that it resolved the problem of the existence of those very objects. For this reason, Euclid did not make use of primitive entities until he had shown how to construct them. Legendre, however, did not submit to this same worry. Indeed, he contemplated the midpoint of the base of the isosceles triangle, but only afterwards would he show how to find it.

3 The First Definition of *Éléments de géométrie*

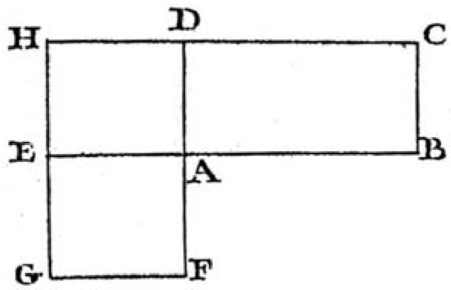
> DÉFINITION (LIVRE I). I. La Géométrie est une science qui a pour objet la mesure de l'étendue. L'étendue a trois dimensions, longueur, largeur et hauteur.[3]

This is the starting point for Legendre's work, and, in itself, it reveals the identity of all his work. Indeed, we see that with this definition he joins "practical" goals to geometry: "measuring" the "extension", that is calculating areas and volumes. The first explicit "fruit" of Legendre's work, is that the measure of a rectangle (and therefore, the area), can be calculated through the product of the base times the height[4] (Table 3).

Table 3

PROPOSITION IV (LIVRE III)

Deux rectangles quelconques $ABCD$, $AEGF$, sont entre eux comme les produits des bases multipliées par les hauteurs, de sorte qu'on a $ABCD : AEFG = AB \times AD : AEtimesAF$.



Ayant disposé les deux rectangles de manière que les angles en A soient opposés au sommet, prolongez les côtés GE, CD, jusqu'à leur rencontre en H; les deux rectangles $ABCD, AEHD$, ont même hauteur AD; ils sont donc entre eux comme leurs bases AB, AE: de même les deux rectangles $AEHD, AEGF$, ont même hauteur AE, ils sont donc entre eux comme leurs bases AD, AF, ainsi on aura les deux proportions, $ABCD : AEHD = AB : AE$, $AEHD : AEGF = AD : AF$. Multipliant ces propositions par ordre, et observant que le moyen terme AEHD peut être omis comme multiplicateur commun à l'antécédent et au conséquent, on aura, $ABCD : AEGF = AB \times AD : AE \times AF$.
Scholie. Donc on peut prendre pour mesure d'un rectangle le produit de sa base par sa hauteur, pourvu qu'on entende par ce produit celui de deux nombres, qui sont le nombre d'unités linéaires contenues dans la base, et le nombre d'unités linéaires contenues dans la hauteur.

4 The Measure in the *Éléments de Géométrie*

In reading the first definition of the *Éléments*, it seems that Legendre attributes a certain notoriety to the concept of "measure". The three terms that are used most often in the text — length, width and height — are also used by Euclid in his introduction to solid geometry.

---

[3] "DEFINITION (BOOK I)
I. GEOMETRY is the science which has for its object: 1st. The measure of extension; and 2dly. to discover, by means of such measure, the properties and relations of geometrical figures."

[4] "PROPOSITION IV (BOOK III)
Any two rectangles are to each other as the products of their bases and altitudes. [...]
Scholium. If we take a line of a given length, as one inch, one foot, one yard, &c., and regard it as the linear unit of measure, and find how many times this unit is contained in the base of any rectangle, and also, how many times it is contained in the altitude: then, the product of these two ratios may be assumed as the *measure* of the rectangle."

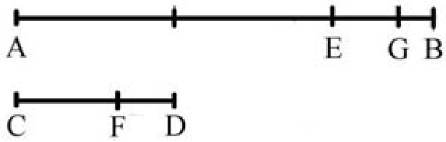
In general terms, we arrive at the concept of measure by exploiting the relationship between the magnitude itself and another magnitude taken as reference (the unit of measure). Assuming the postulate "of continuity" then, we can assert that the measure of a magnitude is *the positive real number* that expresses the relationship of that magnitude with respect to the unit of measure .

Legendre doesn't define the measure of a magnitude very rigorously, perhaps because, at the time, the bases had not been laid out so as to be able to express it in these precise terms. In spite of this, he was not that far away from it, and he illustrates the practical procedures that actively guide the reader to measure the segments and angles (Table 4).

Table 4

PROBLÊME XVII (LIVRE II)

Trouver le rapport numérique de deux lignes droites données AB, CD, si toutefois ces deux lignes ont entre elles une mesure commune.



Portez la plus petite CD sur la plus grande AB autant de fois qu'elle peut y être contenue; par exemple, deux fois, avec le reste BE. Portez le reste BE sur la ligne CD, autant de fois qu'il peut y être contenue; une fois, par exemple, avec le reste DF. Portez le second reste DF sur le premier BE, autant de fois qu'il peut y être contenu, une fois, par exemple, avec le reste BG. Portez le troisième reste BG sur le second DF, autant de fois qu'il peut y être contenu.

Continuez ainsi jusqu'à ce que vous ayez un reste qui soit contenu un nombre de fois juste dans le précédent.[5] Alors ce dernier reste sera la commune mesure des lignes proposées, et, en le regardant comme l'unité, on trouvera aisément les valeurs des restes précédents et enfin celles des deux lignes proposées, d'où l'on conclura leur rapport en nombres.

Par exemple, si l'on trouve que GB est contenu deux fois juste dans FD, BG sera la commune mesure des deux lignes proposées. Soit $BG = 1$, on aura $FD = 2$; mais EB contient une fois FD plus GB; donc $EB = 3$; CD contient une fois EB plus FD; donc $CD = 5$; enfin AB contient deux fois CD plus EB; donc $AB = 13$; donc le rapport des deux lignes AB, CD, est celui de 13 à 5. Si la ligne CD, était prise pour unité, la ligne AB serait $\frac{13}{5}$, et si la ligne AB était prise pour unité, la ligne CD serait $\frac{5}{13}$.

Scholie. La méthode qu'on vient d'expliquer est le même que prescrit l'arithmétique pour trouver le commun diviseur de deux nombres; ainsi elle n'à pas besoin d'une autre démonstration.

Il est possible que, quelque loin qu'on continue l'opération, on ne trouve jamais un reste qui soit contenu un nombre de fois juste dans le précédent. Alors les deux lignes n'ont point de commune mesure, et sont ce qu'on appelle *incommensurables*: on en verra ci-aprés un exemple dans le rapport de la diagonale au côté du quarré. On ne peut donc alors trouver le rapport exact en nombres: mais en négligeant le dernier reste, on trouvera un rapport plus ou moins approché, selon que l'opération aura été poussée plus ou moins loin.

[5]If the quantities are incommensurable, this situation cannot be verified at all. As we'll see later on, Legendre is conscious of this fact even though he continues to deal with incommensurable quantities like commensurable ones.

This mention of the approximate ratio between incommensurable quantities allows us to observe how Legendre dealt with both rational and irrational numbers as something known; he did this without questioning their rigorous foundation.

In book II of *Elements of Geometry and Trigonometry* Davies, on the other hand, inserts the following text:

3. The ratio of magnitudes may be expressed by numbers, either exactly or approximately; and in the latter case, the approximation may be brought nearer to the true ratio than any assignable difference. Thus, of two magnitudes, one may be considered to be divided into some number of equal parts, each of the same kind as the whole, and regarding one of these parts as a unit of measure, the magnitude may be expressed by the number of units it contains. If the other magnitude contain an exact number of these units, it also may be expressed by the number of its units, and the two magnitudes are then said to be *commensurable*.

If the second magnitude does not contain the measuring unit an exact number of times, there may perhaps be a smaller unit which will be contained an exact number of times in each of the magnitudes. But if there is no unit of an *assignable* value, which is contained an exact number of times in each of the magnitudes, the magnitudes are said to be *incommensurable*.

It is plain, however, that if the unit of measure be repeated as many times as it is contained in the second magnitude, the result will differ from the second magnitude by a quantity less than the unit of measure, since the remainder is always less than the divisor. Now, since the unit of measure may be made as small as we please, it follows, that magnitudes may be represented by numbers to any degree of exactness, or they will differ from their numerical representatives by less than any assignable magnitude.

He then continues as Legendre does.[6]

5 Arithmetic and Algebra in the *Éléments de Géométrie*

In a geometry text, the introduction of *measure*, which is understood as a real number associated to a magnitude, implies the inevitable recourse to arithmetic and algebra. We have already observed that Legendre does not disdain the use of arithmetic and algebraic notations to explain geometric results [7].

In the initial part of his first book, in the paragraph dedicated to the *Explanation of Signs*, he introduces the arithmetic symbols. The *"Signs"* are, in fact, arithmetic symbols of equality, order, addition, subtraction and multiplication. However, in order to better understand Legendre's ease in adapting algebraic notations to geometrical facts, we must read the final part of the paragraph in question.

EXPLICATION DES TERMES ET DES SIGNES (LIVRE I)

L'expression $Ax(B+C-D)$ représente le produit de A par la quantité $B+C-D$. S'il fallait multiplier $A+B$ par $A-B+C$, on indiquerait le produit ainsi $(A+B)\times(A-B+C)$; tout ce qui est renfermé entre parenthèses est considéré comme une seule quantité.

Un nombre mis au-devant d'une ligne ou d'une quantité, sert de multiplicateur à cette ligne ou à cette quantité; ainsi, pour exprimer que la ligne AB est prise trois fois, on écrit $3AB$; pour désigner la moitié de l'angle A, on écrit $\frac{1}{2}A$.

Le carré de la ligne AB se désigne par $\overline{AB}^2$; son cube $\overline{AB}^3$. On expliquera en son lieu ce que signifient précisément le quarré et le cube d'une ligne.

Le signe $\surd$ indique une racine à extraire; ainsi $\sqrt{2}$ est la racine quarrée de 2; ...

Thus, we are aware of the possibility of carrying out operations with geometric magnitudes, but we find an even more explicit reference in the definition of *angle*:

[6] "4. We will illustrate these principles by finding the ratio between the straight lines CD and AB, which we will suppose commensurable..."

[7] See in § 3 proposition IV and in § 4 problem XVII.

DEFINITION IX (LIVRE I)

Lorsque deux lignes droites AB, AC, se rencontrent, la quantité plus ou moins grande dont elles sont écartées l'une de l'autre, quant à leur position, s'appelle *angle*, [...]. Les angles sont, comme toutes les quantités, susceptibles d'addition, de soustraction, de multiplication, et de division [...].

In Legendre's time, the possibility of mixing geometry with algebra was not an absurdity. Let us try to clarify this rather delicate matter.

The discovery of incommensurability between the side of a square and its diagonal sanctioned a preponderance of geometry on arithmetic. As such, the "power" of geometry was at its height with Euclid's *Elements*. Numbers were no longer sufficient for the description of magnitudes and their relations; therefore, elementary operations between magnitudes were carried out through the research of the fourth proportional with rigorously geometric procedures. It is a fact that then, just as now, a geometry that has been freed from practical use did not need an exposition in quantitative terms.

Legendre, on the other hand, made explicit reference to both practice and measure, therefore the symbols which expressed generic formulas became essential. Moreover, if we take into account the mathematical evolution that took place up through the eighteenth century, and we take into account this mathematician's studies, it would be absurd to imagine Legendre being reticent regarding numbers.

For Legendre, it's as if arithmetic — whose objects are numbers — was already known and acquired. In book III, immediately after the definitions, Legendre inserts an explicit note that clarifies his conduct.[8]

(LIVRE III)

N. B. Pour l'intelligence de ce livre et des suivants, il faut avoir présente la théorie des proportions, pour laquelle nous renvoyons aux traités ordinaires d'arithmétique et d'algèbre. Nous ferons seulement une observation, qui est très-importante pour fixer le vrai sens des propositions, et dissiper toute obscurité, soit dans l'énoncé, soit dans les démonstrations.

Si on a la proportion $A : B = C : D$, on sait que le produit des extrêmes $A \times D$ est égal au produit des moyens $B \times C$. Cette vérité est incontestable pour les nombres; elle l'est aussi puor des grandeurs quelconques, pourvu qu'elles s'expriment ou qu'on les imagine exprimées en nombres; et c'est ce qu'on peut toujours supposer: par exemple, si A, B, C, D, sont des lignes, on peut imaginer qu'une de ces quatre lignes, ou une cinquième, si l'on veut, serve à toutes de commune mesure et soit prise pour unité; alors A, B, C, D, représentent chacune un certain nombre d'unités, entier ou rompu, commensurable ou incommensurable, et la proportion entre les lignes A, B, C, D, devient une proportion de nombres.

Le produit des lignes A et D, qu'on appelle aussi leur *rectangle*, n'est donc autre chose que le nombre d'unités linéaires contenues dans A, multiplié par le nombre d'unités linéaires contenues dans B; et on conçoit facilement que ce produit peut et doit être égal à celui qui résulte semblablement des lignes B et C.

Les grandeurs A et B peuvent être d'une espèce, par exemple, des lignes, et les grandeurs C et D d'une autre espèce, par exemple, des surfaces; alors il faut toujours regarder ces grandeurs comme des nombres: A et B s'exprimeront en unités linéaires, C et D en unités superficielles, et le produit $A \times D$ sera un nombre comme le produit $B \times C$ [...]

Nous devons avertir aussi que plusieurs de nos démonstrations sont fondées sur quelques-unes des règles les plus simple de l'algèbre, lesquelles s'appuient elles-mêmes sur les axiomes connus: ainsi si l'on a $A = B + C$, et qu'on multiplie chaque membre par une même quantité M, on en conclut $A \times M = B \times M + C \times M$; pareillement si l'on a $A = B + C$ et $D = E - C$, et qu'on ajoute les quantités égales, en effaçant $+C$ et $-C$ qui se détruisent, on en conclura $A + D = B + E$, et ainsi des

[8] Davies dedicates an entire chapter to the theory of proportions, and therefore he does not translate Legendre's note.

autres. Tout cela est assez évident par soi-même; mais, en cas de difficulté, il sera bon de consulter les livres d'algèbre, et d'entre-mêler ainsi l'étude des deux sciences.

Having taken this position, Legendre intentionally excludes the content of Euclid's book V. Moreover, once he has established the link between magnitudes and numbers, the proportions between magnitudes become proportions between numbers, and therefore all algebraic properties expressed become valid for such proportions. In this "muddying", as Cremona put it, of geometry with algebra, one becomes aware of the modern didactic conception, according to which, algebra and geometry may integrate with each other.

Nevertheless, Legendre's text is geometric: algebra and arithmetic play only a supporting role so as not to weigh down the exposition of the theory of magnitudes. Legendre continues to prove geometric results in a synthetic way.

In the fourth proposition of the third book, we have seen how Legendre made use of the theory of proportions to demonstrate a geometric theorem:

As we have read, the proof is very simple. We obtain two proportions, which are the result of the previous proposition, and from their product the desired result follows. On the other hand, the analogous proposition present in Euclid's *Elements* is anything but simple:

PROPOSITION 23 (BOOK VI) OF EUCLID

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

The terms of this theorem already prove complicated. The proof is very complex. Euclid is forced to introduce the *compounded ratio* (without even having defined it) which represents the product of two ratios. As he does not consider the ratios between magnitudes as numbers, he cannot resort to multiplication between ratios.

In spite of the immense admiration for Euclid's rigor and consistency, the concision and ease of Legendre's proof is evident.

6 Axioms in the Éléments de Géométrie

Legendre displays his topics of geometry by following the classic axiomatic method. The axioms are listed in the initial part of book I[9] and are the following:

AXIOMES

1. Deux quantités égales à une troisième sont égales entre elles.
2. Le tout est plus grand que sa partie.
3. Le tout est égal à la somme des parties dans lesquelles il a été divisé.
4. D'un point à un autre on ne peut mener qu'une seule ligne droite.
5. Deux grandeurs, ligne, surface ou solide, sont égales, lorsqu'étant placées l'une sur l'autre elles coïncident dans toute leur étendue.

In neither the *Éléments de Géométrie*, nor in the notes to its appendix are there any comments on the above axioms by the author. The number of Legendre's axioms correspond

[9] Not everything in this paragraph which is taken from Legendre's text has been translated into English. Davies does not suppose the same axioms that Legendre does and, in fact, the formulation resembles Euclid's, even if not completely. For example, Davies does not cite the proposition in which Legendre shows that all right angles are equal, because he postulates the equality of right angles. Moreover, he postulates that given a point and a line, there will only be one parallel through that point. As a result, he then demonstrates the fifth postulate.

to the number of Euclid's postulates, but as far as their contents are concerned, the two of these differ notably.

In Legendre, the distinction between Postulates and Common Notions no longer existed; indeed, three of his axioms (the first, the second, and the fifth) are propositions analogous to the ones that Euclid inserts in his common notions.

One first obvious observation, is that his five axioms are not sufficient to infer all his theorems of elementary geometry. The absence of a postulate of continuity, and the absence of Euclid's Postulate V (or an equivalent one) stands out.

The imperfection linked to continuity might also be "justified". In many of his proofs, Legendre makes quiet recourse to Archimede's Postulate. Such a recourse allows us to think that the mathematician retains continuity to be a manifest property.

The other flaw, however, is connected to Legendre's firm conviction that he'd resolved the problem of parallels. The mathematician's intentions are not conjectured, but rather they are perfectly expressed in the memoir, *Réflexion sur différentes Manières de démontrer la Théorie des Parallèles ou le théorème sur la somme des trois angles du triangle*[10], which was edited by Legendre himself in the very year of his death. His intention was to demonstrate a property equivalent to postulate V, or rather that *the sum of the angles in any given triangle is of 180 degrees.* He provided various proofs of this proposition, some of which were quoted in different editions of the *Éléments.* In his memoir we can find some of the original proofs, as well as some of the reasons that compelled Legendre to modify the new editions of the *Éléments.* Most interesting, however, are the memoir's conclusions where Legendre explicitly states that he has rigorously proved and concluded the theory of parallels:

> *Quel que soit au reste le jugement qu'on en portera, j'aurai toujours à me féliciter de l'espèce de hasard qui m'a permis de présenter au choix des géomètres, deux démonstrations également rigoureuses de la Théorie des Parallèles (car avant que je publiasse mon ouvrage, il n'existait aucun livre élémentaire où la démonstration de la théorie de parallèles pût être regardée comme absolument rigoureuse)*[11]*; l'une (celle de la 12e édition) plus directe et plus conforme aux méthodes ordinaires; l'autres, fonde sur un principe nouveau, mais dont l'application rentre dans les formes élémentaires les plus simple.*

Therefore, why should he have inserted an axiom that would have solved the problem of the theory of parallels?

As is known, in the concluding section of his proof, Legendre lost control of the procedure, which was too based in intuition. After the proof, Legendre states the direct consequences of the proposition in six appendixes. Among these, he states the theorem of the exterior angle, Euclid's fifth postulate, and the uniqueness of a parallel to a line through a point.

In reference to Hilbert's axiomatic formulation, the absence of Axioms of Order in the *Éléments* must also be noted. In formal terms, with such an absence one might suspect that the line that Legendre intended was not infinite; and in fact, definition III of book I states: "III. La *ligne droite* est le plus court chemin d'un point à un autre [12]". According to this definition, in reality, the straight line is a segment. Legendre does not postulate, as Euclid does, the indefinite possibility of extending the line, but rather he uses it.

For example, he proves, ab absurdo, that (Table 5):

In conclusion, Legendre's system of axioms "is not complete". Nevertheless, rather than persist about the absence of a few important arguments, it is preferable to analyze those that have been dealt with effectively by the mathematician.

[10] *Mémoires de l'Académie des sciences de Paris* — Volume XII — 1833.

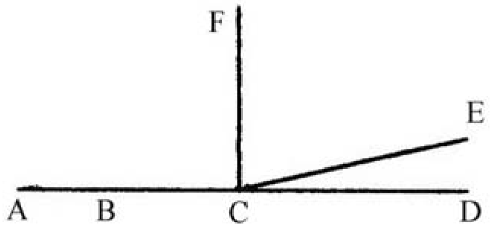
[11] In the original text, everything that is here written within parenthesis is cited in a note.

[12] In this definition the idea of distance between two points, and therefore an ulterior reference to measure, clearly emerges.

Table 5

PROPOSITION III (LIVRE I)

Deux lignes droites qui ont deux points communs coïncident l'une avec l'autre dans toute leur étendue, et ne forment qu'une seule et même ligne droite.



Soient les deux points communs A et B; d'abord les deux lignes n'en doivent faire qu'une entre A et B, car sans cela il y aurait deux lignes droites de A en B, ce qui est impossible. Supposons ensuite que ces lignes étant prolongées, elles commencent à se séparer au point C, l'une devenant CD, l'autre CE. Menons au point C la ligne CF, qui fasse avec CA l'angle droit ACF. Puisque la ligne ACD est droite, l'angle FCD sera un angle droit; puisque la ligne ACE est droite, l'angle FCE sera pareillement un angle droit. Mais la partie FCE ne peut pas être égale au tout FCD; donc les lignes droites qui ont deux points A et B communs, ne peuvent se séparer en aucun point de leur prolongement.

6.1 The Equality of Triangles (the first and fifth axiom)

The first axiom, *"TWO QUANTITIES EQUAL TO A THIRD ARE EQUAL AMONG EACH OTHER"* establishes the transitive property of equality. In the presentation of geometry, *equality* plays a very important role, so much so that one must speak about it well before developing theorems. *Rigid Movement* is intimately linked to *Equality*, and Legendre implicitly makes reference to it in his fifth axiom, *"TWO MAGNITUDES, LINES, SURFACES OR SOLIDS, ARE EQUAL WHEN, BEING SITUATED ONE ON TOP OF THE OTHER, THEY COINCIDE IN ALL OF THEIR EXTENSION"*. Euclid also makes reference to movement in his proofs of the equality of triangles, while Hilbert does not employ such a concept. With a rational treatment, we introduce "equality" axiomatically, or "rigid movement" by deducting the other accordingly. This procedure, however, proves complex. Legendre admits as primitive concepts both equality and movement (even if he doesn't openly mention it). The first axiom guarantees the transitive property of equality, while with the last axiom, two figures are declared equal when they can be coincided point by point. Legendre also proposes the criteria for the equality of triangles (Table 6).

Proposition VI is really a consequence of Legendre's fifth axiom, and more than a proof, it is a justification based on the superimposition of the two triangles. The proof is analogous to the one from current texts, as well as to Euclid's, but what differs are the terms of this last proof (proposition IV of book I):

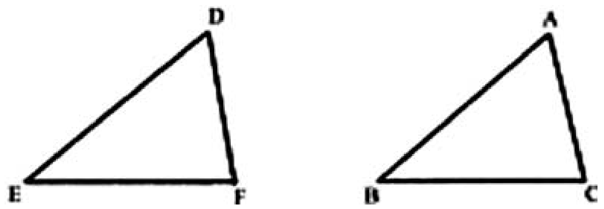
> « If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equals the remaining angles respectively, namely those opposite the equal sides. »

The different layout of this formulation is in the Euclidean concept of "equal". In fact, in the *Elements*, the term "equal", if applied to polygons, assumes the meaning of our "equivalent". This fact explains why, in Euclidean terms, the equality of elements of triangles may be specified even if those very triangles are declared equal. These problems no longer exist for Legendre because he differentiates the two concepts by using both the terms "equal" and "equivalent".

Table 6

PROPOSITION VI (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont un angle égal compris entre deux côtés égaux chacun à chacun[13].



Corollaire. De ce trois choses sont égales dans deux triangles, savoir, l'angle $A = D$, le côté $AB = DE$, et le côté $AC = DF$, on peut conclure que les trois autres le sont, savoir, l'angle $B = E$, l'angle $C = F$, et le côté $BC = EF$.

PROPOSITION VII (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont un côté égal adjacent à deux angles égaux chacun à chacun.

PROPOSITION XI (LIVRE I)

Deux triangles sont égaux, lorsqu'ils ont les trois côtés égaux chacun à chacun.

Soit le côté $AB = DE$, $AC = DF$, $BC = EF$, je dis qu'on aura l'angle $A = D$, $B = E$, $C = F$. Car si l'angle A était plus grand que l'angle D, comme les côtés AB, AC, sont égaux aux côtés DE, DF, chacun à chacun, il s'ensuivrait, par le théorème précédent, que le côté BC est plus grand que EF; et si l'angle A était plus petit que l'angle D, il *s'ensuivrait* que le côté BC est plus petit que EF; or, BC est égal à EF; donc l'angle A ne peut être ni plus grand ni plus petit que l'angle D; donc il lui est égal. On prouvera de même que l'angle $B = E$, et que l'angle $C = F$.

For the proof of the Proposition VII, Legendre does not appeal to the previous proposition. Instead, each time Euclid must declare the equality of two triangles, he turns to the fourth proposition, and therefore this last one carries out the role of postulate. For Legendre, thanks to the fifth axiom, the two triangles ABC and DEF are equal for the simple fact that they can be brought to coincide.

In the appendix, analogously to the previous proposition, the equality of the respective elements of the two triangles is highlighted.

Proposition XI represents the SSS equality. Euclid proves this by using the *theorem of the isosceles triangle.* Legendre proposes a proof *ab absurdo* by alluding to the previous proposition, the tenth.

6.2 The Second and Third Axiom

The terms of the second and third of Legendre's axioms are: *THE WHOLE IS GREATER THAN ITS PART*, and *THE WHOLE IS EQUAL TO THE SUM OF THE PARTS IN WHICH IT HAS BEEN DIVIDED.* These propositions are particularly intuitive and, as one can well imagine, they are also intuitively used in proofs. As shown, for example, in Table 5,

[13] "PROPOSITION VI (BOOK I)
If two triangles have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, the two triangles will be equal.
Cor. When two triangles have three things equal, viz., the side $ED = BA$, the side $DF = AC$, and the angle $D = A$, the remaining three are also respectively equal, viz., the side $EF = BC$, the angle $E = B$, and the angle $F = C$."

or to prove that *"Les angles droits sont tous égaux entre eux"* (Proposition I, Livre I), or in the Proposition II, Livre I *"Toute ligne droite CD, qui rencontre une autre AB, fait avec celle-ci deux angles adjacent ACD, BCD, dont la somme est égale à deux angles droits"*[14]

6.3 The Fourth Axiom

Finally, Legendre's fourth axiom states that *FROM ONE POINT TO ANOTHER, YOU CAN DRAW ONLY A SINGLE STRAIGHT LINE.* As observed earlier, Legendre already made use of this axiom in proposition I.

Of the five axioms, this is the only one that makes explicit reference to geometric entities, i.e. the point and the line. If we want to be meticulous about it, the terms guarantee, grammatically, the uniqueness of a line through two points, but they do not guarantee its existence. For Legendre, it's as if the existence of geometric objects were an absolutely intuitive matter.

References

- Davies, C., 1852, *Elements of geometry and trigonometry*, from the works of A. M. Legendre, New-York.
- Klein, F., 1909, *Elementarmathematik von höheren Standtpunkte aus*, Teil II: Geometrie, Teubner Leipzig.
- Legendre, A. M., 1857, *Éléments de Géométrie*, quatorzième edition, Paris.
- Menghini, M., 1996, "The Euclidean method in geometry teaching", in Jahnke, H. N., Knoche, N. & Otte, K. (Hrsg.) *History of Mathematics and Education: Ideas and Experiences*, Vanderhoeck & Ruprecht, Göttingen, 195–212.
- Schubring, G., 2004, "Neues über Legendre in Italien", in W. Hein, P. Ulrich (eds.) *Mathematik im Fluß der Zeit*, Algorismus 44, 256–274.

[14] *"If one straight line meets another straight line, the sum of the two adjacent angles will be equal to two right angles.*
Let the straight line DC meet the straight line AB at C; then will the angle ACD plus the angle DCB, be equal to two right angles. At the point C suppose CE to be drawn perpendicular to AB: then, $ACE+ECB =$ two right angles. But ECB is equal to $ECD + DCB$: hence, $ACE + ECD + DCB =$ two right angles. But $ACE + ECD = ACD$: therefore, $ACD + DCB =$ two right angles."