MATHEMATICS IN THE SERVICE OF THE ISLAMIC COMMUNITY

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Abstract

The formation of Islamic culture was accompanied by a process of adoption and integration of the classical scientific tradition. Due to a long dispute over the role of mathematics and astronomy, a new category of disciplines emerged that supported the access of Muslim scholars to mathematics in general and to its applied branches in particular. Selected examples from different fields of applied mathematics (ar. hisāb) demonstrate the extant to which mathematics in the Islamic home-lands took root, developed and produced new practical disciplines the Islamic Community could benefit from.

S_n	=	number of pronunciations; n = number of characters
S_1	=	3 (three vowels)
S_2	=	12 ($=3 \cdot 4$: three vowels, one *sukūn*)
S_3	=	$4 \cdot S_2 - 3 = 45$ ('minus three' = three impossible double − *sukūn*)
S_4	=	$4 \cdot S_3 - 3 \cdot S_1 = 180 - 9 = 171$
		$[S_n = 4 \cdot S_{n-1} - 3 \cdot S_{n-3}$ or: $S_n = 3 \cdot S_{n-1} + 3 \cdot S_{n-2}]$
2) C_n^p	=	p different characters of an alphabet of n characters
C_{28}^5	=	$98 \cdot 280 = N_1$ (5 different characters like: ارسطاطالس)
N_2	=	$15 \cdot 120 = P_9^{1,1,2,2,3} = 9! : (2!\,2!\,3!)$ (permutations of combinations of 5)
N_3	=	$S_9 = 133 \cdot 893$ (pronunciations of word S_9)
N_4	=	$30 = P_5^{2,2} = 5! : 2!\,2!$ ("same" combinations of N_2)
A	=	$N_1 \cdot N_2 \cdot N_3 \cdot N_4 = 5\,968\,924\,232\,544\,000$

Figure 1 – Ibn Mun'im (Marrakech, 12th century)

Around 1207, a mathematician of Marrakech (Marocco), called Ahmad Ibn Mun'im, busied himself with a problem that seemed to be in his days just as unknown as futile: focused on the Arabic alphabet he wanted to find out the number of possible wordings produced by the different combinations of two, three or more Arabic root-consonants. The results of his attempt, the birth hour of combinatorics, are pretty discouraging for those of you who might want to study Arabic (figure 1). He found out, for instance, that in Arabic, which consists of an alphabet of 28 root-consonants each of which can either be pronounced with one of the three vowels a, i and u, or can be voiceless, or silent (the Arabs call it sukūn) which mustn't stand at the beginning of a word and not beside a second sukūn — he found out that the number of possible pronunciations of a word of n ($n \geq 2$) consonants with three vowels and one voiceless sukūn, amounts for $n = 2$ to 12, for $n = 3$ to 45 and for $n = 4$ to 171 possibilities. Finally, after having developped the necessary tools to refine his investigations, he was able to define the maximum number of possible pronunciations of any Arabic word. In one of his examples, he set forth that the number of possible pronunciations of a word that consists of nine characters, but only five distinct consonants — here Ibn Mun'im picked the Arabic name of Aristotle, Aristātālis — amounts to five trillions (5 068 924.232 544 000).

We only know of Ibn Mun'im's problem since 1980, and we still do not know precisely what made him tackle this particular one. Ibn Mun'm wasn't only versed in Mathematics, he also wrote on law and theology. By analyzing the Arabic language, the language of the Qur'ān, with the tools of combinatorics, he — implicitly — proved that God's revelation is — if only lexicographically — finite.

Quite evidently, the case of Ibn Mun'im demonstrates that the history of Mathematics in the Islamic lands had something to do with culture, or rather: with cultures. Not only Arabic and the Qur'ān could be involved, but also Aristotle was brought up.

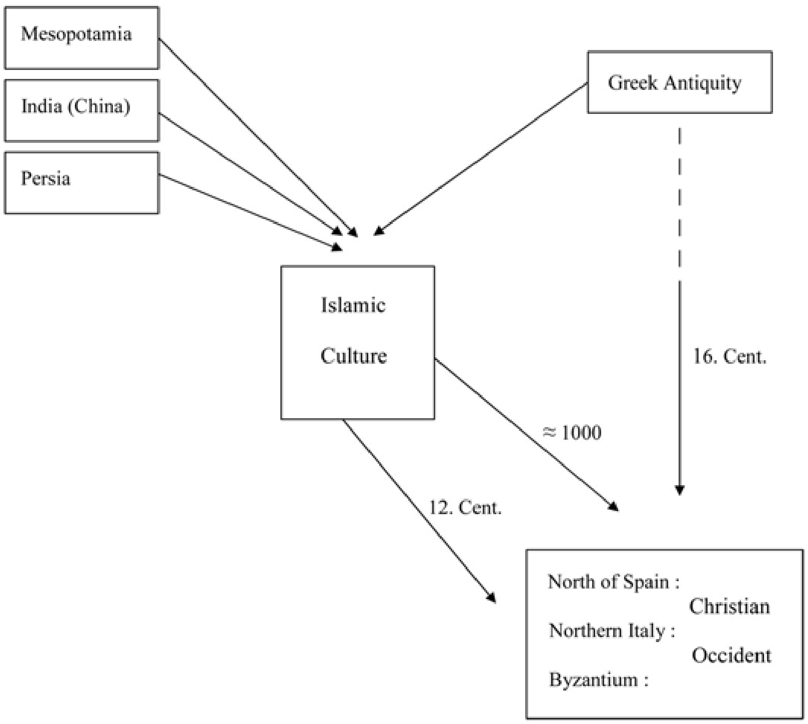

Figure 2 – Scheme of transmission

You all have heard about the leading role Islamic mathematicians played in transmitting Indian mathematics, and retransporting Greek mathematics into the West. But we also know, although much too less, that the Arabs did not content themselves with the role of mediation. Let us just take some terms, like, for example, 'sine', meaning 'pleat' or 'wrinkle', which is the simple and somehow misunderstood Latin translation of an Arabic translation of the Sanscrit word the Indians used for this trigonometric function. Or let's take the Arabic root of a term like 'algebra' which originally means 'to straighten' or 'to reset', one's dislocated shoulder, for instance, or the term 'logarithm' that, as you all know, stems back to the ninth century mathematician 'al-Khuwārizmī', the famous and often-cited first composer of an Arabic treatise on Algebra. Many well-known inventions in these fields were achieved in the Islamic orient, other outstanding ones only recently discovered. Thus, some decades before Ibn Mun'im, as-Samau'al, a Baghdadian Jew of Maroccan origin who later converted to Islam, had already laid the foundations for the famous triangle of Pascal and for the infinitesimal calculus of Leibniz, three centuries before their hitherto alleged founders.

Such achievements, undisputed and significant as they are, belong to a sphere of mathematics that could (and have been) called ‘scientific’ as opposed to the one I — from now on — call ‘practical’. Economically and socially developed societies, especially pre-modern and religiously orientated societies, could not afford to abstain from benefiting from mathematical knowledge, for various reasons. They made use of it in different fields and to different extent. It is this difference of incorporation of both, the theoretical and practical, modes of application of the so-called rational sciences into the mental organisation of societies that has led (or seduced) historians and anthropologists to explain, at least partially, the distinct process of cultural development and progress.

I am neither embarking on this somehow simplistic hypothesis, which could be given the shape of the equation mathematics equal development, nor am I starting to enumerate the respective peculiarities of the Islamic East and (or versus) the Christian West. I will rather, firstly, remain in the Islamic East and try to shed some light on mathematical disciplines that came into being there during the Middle Ages, in the specific context of Islamic societies and in an intellectual milieu that was inspired not only or not primarily by classical traditions; and I will, finally, give you some examples of how mathematics and the needs of the Islamic society interacted.

To make clear what I mean by ‘disciplines’ requires one further remark. In our usage we owe this term to the Latin founders of the Western academic curriculum. The Greek background of it, however, is ‘propaedeutic’, meaning ‘introductory learning’, and ‘gymnasía’, meaning ‘exercises’. In pre-Islamic Arabia no word existed that could be taken for that. The only word, the Qur‘ānic language offered for mathematics in the wider sense, was hisāb, reckoning, or rather: the reckoning of one’s sins in the hereafter. Therefore, these propaedeutic disciplines were translated from Greek into Arabic as riyādīyāt, meaning today, as 2.400 years ago in Athens, Sports as well as Mathematics. Thus, this type of mathematical discipline was regarded as an intellectual excercise that would provide the student with tools and methods by which higher knowledge in Metaphysics or Theology could be achieved. Geometry, Astronomy, Arithmetic, Music and sometimes Logic belonged to those introductory exercises. As we shall see, the Islamic culture made creative use of this classical tradition. But it added also new ones to it. It is these new ones, and in particular the ones that were regarded as proper mathematical disciplines, to which I want to draw your attention. For purely economic reasons, I explicity limit myself to these mathematical discipines and exclude astronomy and other related sciences where certainly similar developments could be followed up.

But what is a discipline? And more than that: when does a discipline once detected as such turn into being ‘practical’, that is: how can it be differentiated from what we called above the ‘scientific sphere’? Let me give you two examples in order to illustrate the outer limits of what I call the ‘practical’ sphere:

Abraham ben Ezra, a Jewish scholar of the 12. century who lived in the North of Spain and took part in the grand project of translating Arabic mathematical texts into Latin, transmitted the following problem, probably from Muslim Andalusia: Together with 30 of his students, among them 15 good-for-nothings, on a ship at sea in distress, he only saw one last resort to save their lives: 15 of them had to be thrown overboard (figure 3).

S	S	S	S	**G**	**G**	**G**	**G**	**G**	S	S	**G**	S	S	S	**G**	S	**G**	**G**	S	S	**G**	**G**	**G**	S	**G**	**G**	S	S	**G**
				14	**4**	**7**	**12**	**1**			**10**				**5**		**2**	**8**			**13**	**15**	**11**		**6**	**3**			**9**

Figure 3 – Abraham ben Ezra (Toledo, around 1150), Students (= S) and “good-for-nothings” (= G): “algebraic solution”

Of course, he knew the good-for-nothings among his students and so he ordered all of them to line up in a formation that seemed to be arbitrary and then applied a method of casting out each ninth of them. Miraculously, the 15 poor creatures who drew the terrible lot were all the good-for-nothings. The method by which he got rid of them he called ‘algebraic’. This type of problem belongs to the so-called ‘recreational’ problems. They circulated among specialists, were not directly applicable in social intercourse and were, in general, not studied, taught or commented upon.

$$
\begin{array}{lrcll}
\text{Assertion:} & x \cdot y & = & 10a + 10b + (5-a)\cdot(5-b) & [0 \leq a, b \leq 5] \\
\text{Be:} & (5+a)\cdot(5+b) & = & 25 + 5a + 5b + ab & \\
\text{Then:} & 10a + 10b + (5-a)\cdot(5-b) & = & 25 + 5a + 5b + ab & \\
 & 10a + 10b + 25 - 5a - 5b + ab & = & 25 + 5a + 5b + ab & \\
 & 25 + 5a + 5b + ab & = & 25 + 5a + 5b + ab & [a \text{ or } b < 0] \\
 & & & & [\text{q.e.d.}]
\end{array}
$$

Figure 4 – Mongolian “finger-multiplication”

On the opposite end of the scale we find another area of mathematical skills that equally does not belong to our investigation. The following example will make clear what I mean. At the end of the 19th century, the Russian traveler A. A. Ivanowski observed in Mongolia a particular method of finger-reckoning used by most of the Mongols he met: In order to multiply 6 by 7, for example, they bent in four fingers of one hand and three of the other hand, looked at their two hands and then added two numbers: 12 and 30, which gives 42, the correct product. He also observed that this method was only used when numbers between five and ten were involved. The mathematical proof of this method runs as you see here (see figure 4).

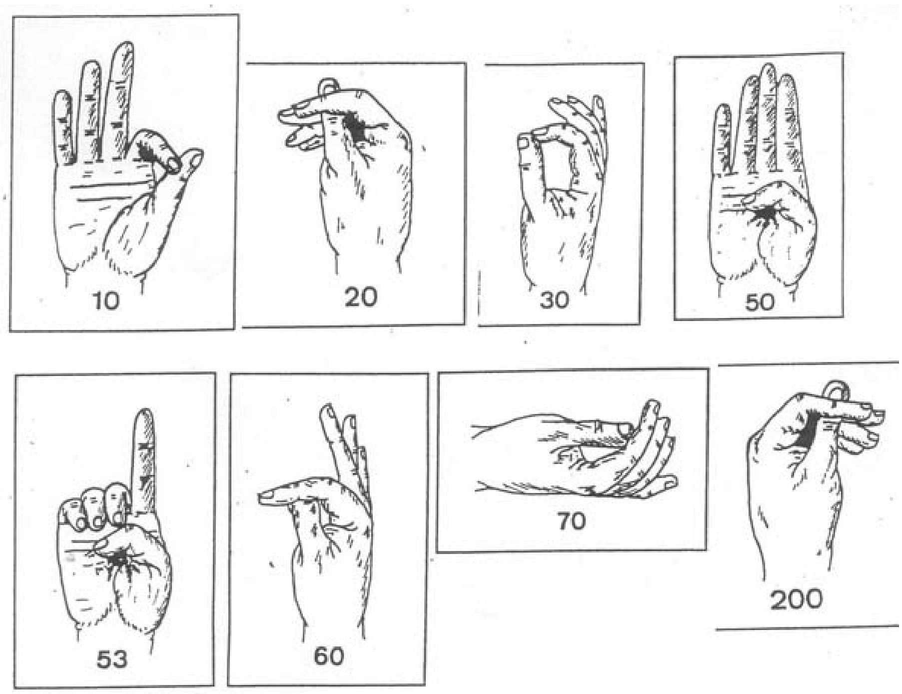

Figure 5 – Arabic finger-reckoning

In the Islamic East we still encounter today similar but much more elaborate methods of finger-reckoning (figure 5). This type of skills and tricks also belong to a stratum of mathematics that is different from what we are out for; but for different reasons. This type is, firstly, practised and transmitted locally; it is, secondly, not regarded as a kind of special knowledge that must be studied and taught; and it is, therefore, neither the object of an intellectual discourse, nor of technical improvements both of which are characteristic of any kind of scientific activity.

If we add up the pros and cons we get a rough idea of what makes the difference. In order to develop into something we can call a ‘practical mathematical discipline’ these skills, first of all, had to be written down — otherwise we would not know them at all. Then they had to be copied and circulated, that is be accepted by the community, and then they had to be commented upon and modified, that is integrated into the mathematical curriculum from where they could develop into something we would call a literature today.

We will now make use of this literary indicator, to fill the gap between the two fields excluded above and conclude: If the disciplines we are looking for produced texts there must exist other texts that used the former. Nothing is written — after all — about which nothing else was written. In all literatures and especially in Islamic literature, the most prolific of all pre-modern literatures, one genre stands for this law: the encyclopaedic literature, the genre of literature that claims to contain all others. If this is true, then our ‘practical’ disciplines too must have left traces in the Arabic-Islamic encyclopaedic literature, and in particular in encyclopaedias of sciences.

There, of course, we encounter ‘Mathematics’ or riyādīyāt, subordinated to Metaphysics or Philosophy, the highest of all sciences. By the end of the 9th century, the Arabs and their allied islamised nations had successfully integrated the cultural accomplishments of their earlier pagan and Christian enemies into their own culture. But there was one terribly dangerous aspect of this process of cultural assimilation: These accomplishments were all achieved in the sphere of cultures, prior to and outside Islam, and were now, possibly, infecting Islam with the virus of disbelief. Mathematics, in particular, threatened a fundamental dogma: How could the fact that a times b always and in all eternity yields the area of a rectangular figure with the sides a,b be reconciled with God’s omnipotence? Or, even worse, how could it be explained — what had already puzzled the Greeks — that even God could not know the exact value of the root of 2?

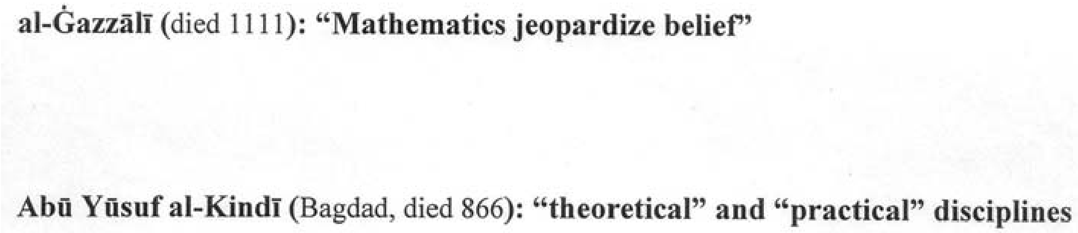

Figure 6

Al-Ghazzālī (figure 6), the most important religious philosopher in the Islamic Middle Ages (died 1111) expressed this fear with the following allegory: Someone who indulges too deeply in Arithmetic or Geometry is like a newly converted Muslim whose young belief is jeopardized when dealing with unbelievers. He must be protected against falling prey to them like a young boy at the river-side against falling in the water. But by the time of al-Ghazzālī something crucial had already happened to the religious assessment of the mathematical sciences. It had started with the earliest Islamic philosopher who had occupied himself with Greek mathematics and philosophy: Abū Yūsuf Ya’qūb al-Kindī, who died 866 in Baghdād and was the first to draw a distinction — which he borrowed, by the way, from Aristotle - between ‘theoretical’ and ‘practical’ mathematics (figure 6). According to him, for ex., measuring the depth of a well or the height of a mountain from a distant point must be differentiated from the what he called the 'speculative' branch of Geometry. Al-Kindī himself composed a treatise on this technique.

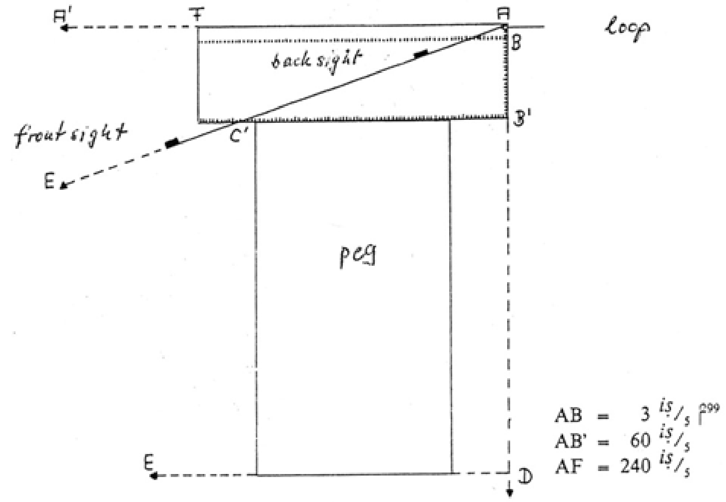

Figure 7 – Instrument of measuring width, depth and height of any kind of object (around 980):

– *'ilm al-'adad*	= science of numbers
– *ṣinā'at al-misāḥa*	= the art of measuring
– *ḥiyal*	= ruses
– *ḥisab al-hind*	= Indian reckoning
– *ḥisab al-mu'āmalāt*	= calculation of social affairs
– *al-ğabr wa'l-muqābala*	= Algebra

Figure 8 – al-Ḫuwārizmī (died 987): *Kitāb Mafātīḥ al-'ulūm*: the 'secretary's "practical branches"

And three generations later, Abū l-Wafā' al-Būzjānī, a mathematician of Baghdād of whom we will hear more later, inserted this squizze of an instrument in his handbook of Arithmetics & Geometry (figure 7). With this instrument, exactly this type of measurement could be operated. By sighting the object, the mobile tongue produces similar triangles by which the magnitude searched can be calculated. At the same time, a certain al-Khuwārizmī (not the al-Khuwārizmī whose name eventually mutated into our 'logarithm', but a later compatriot, a Persian speaking clerk and scholar from the central Asian oasis Khuwārizm) pushed this differentiation further (figure 8). He was the first to divide all sciences into foreign, non-Arabic ''ajam' sciences and into 'Arabic' or 'Islamic' sciences. Among the non-Arabic, the 'ajam, sciences of the first rank, the classical Greek mathematical sciences appear. But their practical branches carry names that are detached from the foreign origin of their theoretical sister-disciplines: 'ilm al-'adad (the science of numbers) instead of Arithmetics, sinā'at al-misāha (the art of measuring) instead of Geometry, hiyal (ruses, tricks) instead of Physics, and special terms like hisāb al-hind (Indian reckoning), hisāb al-mu'āmalāt (calculation of social affairs) and al-jabr wa'l-muqābala (our Algebra).

This tendency continued. I call it the 'domestication of sciences'. One generation after al-Khuwārizmī, somewhere south of Bukhara and Samarkand (in what is called today Uzbekistan) an exceptional book entitled "The Compendium of sciences" was written (figure 9).

The Compendium contains a striking example of this tendency. Next to nothing is known about the author, a certain Ibn Furai'ūn (or Farīghūn). Let us have a look at how this scholar

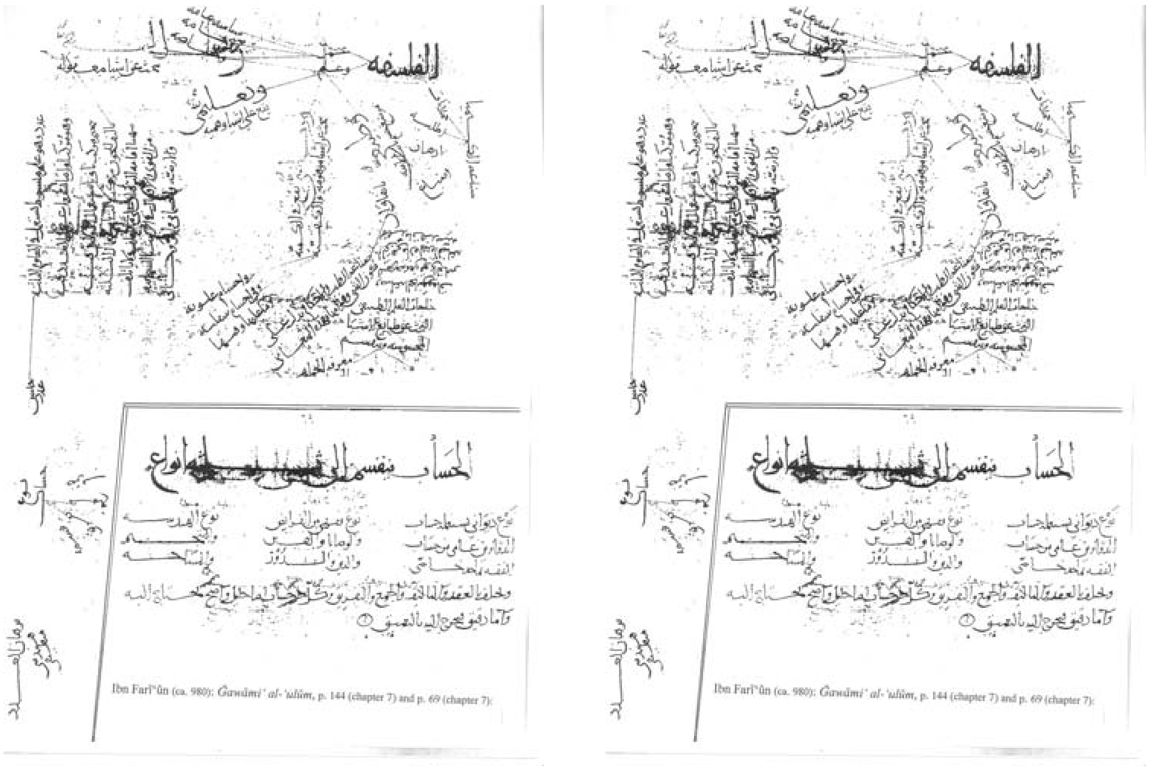

Figure 9 – Ibn Farīʿūn (ca. 980): *Ǧawāmiʿ al-ʿulūm*, p. 144 (chapter 7) and p. 69 (chapter 7)

proposed in a far Eastern province the division and ranking of the sciences we are interested in. A remarkable book, indeed, and not one we are used to read. All of its 171 pages are edited in ‘tree-form’ (ar. tasjīr). Close to 500 disciplines are arranged in 8 chapters, the first of which contains the Arabic philological disciplines, the last of which the occult and magic disciplines.

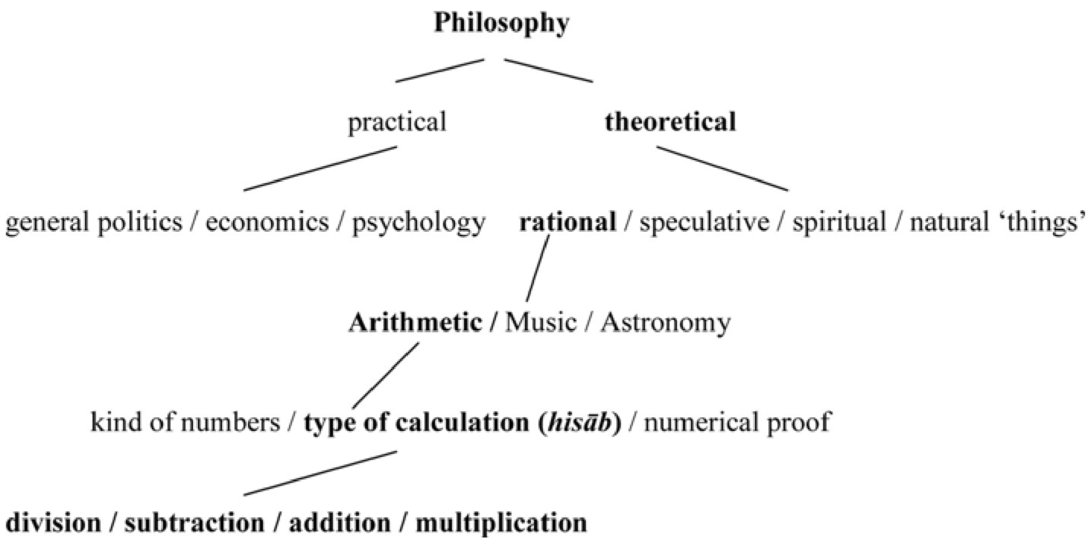

Figure 10 – Ibn Farīʿūn (ca. 980): *Čawšmiʿ al-ʿulūm*, p. 144 (chapter 7)

Figure 10 is an abridged version of the page you just saw, the first page of chapter seven, the chapter on Philosophy. Arithmetic is still regarded as a theoretical discipline of Philosophy, but classified as rational and, in addition, split up into various fields of application and techniques.

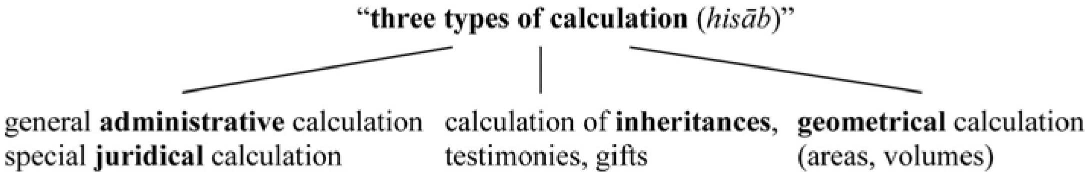

Figure 11 – Ibn Farī'ūn (ca. 980): *Ǧawāmiʿ al-ʿulūm*, page 69 (chapter 2: the "secretary's office")

The practical branches of mathematics, however, are already dealt with in chapter two, headed with "adab al-kuttāb", the art of the secretaries. Here, 'hisāb', reckoning or calculation, is classified as a complex discipline of various mathematical practices (figure 11).

We observe a significant change of perspective. Mathematics are not anymore regarded purely as a science of non-Islamic origin and of dangerous truths, but incorporated as far as possible into the range of disciplines without which an Islamic state could not exist. By the end of the 10th cent. two of the above mentioned terms had come to denote this area of culturally acknowledged practical mathematical sciences: 'ilm al-hisāb, the science of calculation, and hisāb al-mu'āmalāt, the calculation of social affairs. That by then the esteem of mathematics had put down roots in the Islamic society is clearly expressed by one of the most outstanding Islamic scientists of the Middle Ages, Ibn al-Haitham, known to the Latin West as Alhazen (the one who invented the first camera obscura and died 1040). He not only composed a treatise on the calculation of social affairs (hisāb al-mu'āmalāt) but introduced it with a provocative sentence: "The need of 'the calculation of social affairs' (hisāb al-mu'āmalāt) is natural; someone who has not mastered it is like someone who has lost one of the senses by which he is mastering his life."

Ibn al-Akfānī (died 1348): *Iršād al-qāʿid ilā asnš l-maqāṣid*, page 134ff.:

Manfa'a (benefit) of Geometry:

1. the science of the construction of buildings
2. the science of Optics
3. the science of rays/'burning mirrors'
4. the science of the centre of gravity
5. the science of measuring
6. the science of tapping of stretches of water
7. the science of pulling loads
8. the science of clocks
9. the science of military equipment
10. the science of pneumatic instruments

Manfa'a (benefit) of Arithmetics:

1. the science of 'open' calculation (without numerical notation)
2. the science of calculation with board and pencil (with 'Indian' numerals)
3. the science of Algebra
4. the science of the 'calculation with two faults'
5. the science of rotating bequests and legacies
6. the science of calculating with *dirham* and *dīnār*

Figure 12 – Ibn al-Haitham, "Alhazen" (**died 1040**): *Hisāb al-muʿāmalāt* = calculation of social affairs

Ibn al-Haitham's treatise was only recently discovered in a manuscript library in Istanbul. But it was known long before. It was mentioned by an anonymous coptic-christian writer of Mamluk Egypt and also by another Egyptian scholar, Muhammad b. al-AkfānI (died 1348) who inserted it into his encyclopaedia of sciences, the last one I want to present to you (figure 12). This book reflects the result of the process of what I called 'the domestication of sciences' that had started more than four centuries before. We still find there the so-called 'Quadrivium', the four Greek middle sciences of Geometry, Astronomy, Arithmetic and Music, in a premier position. This reverence for Hellenistic science whould not end in the Islamic East, nor in the West until the 18th century. However, long before the period of Enlightenment thoroughly rearranged the ranking of sciences in the West, in the Islamic East a key-word had appeared that stood for the new esteem for sciences. Ibn al-Akfānī calls it manfa'a, benefit. According to him any science is composed of two parts: of a — let us say — mother-discipline, and of its useful branches. The study of any science is fundamentally legitimized by the benefit of its branches to the Islamic society. If we look at Ibn al-Akfānī's list of 'useful branches' of Geometry and Arithmetic we get an impression of what had happened since and despite al-Ghazzālīs's warning two and a half centuries before:

Geometry now consisted of: 1) the science of the construction of buildings ('ilm 'uqūd al-abniya) 2) the science of optics ('ilm al-manāzir) 3) the science of rays/'burning mirrors' ('ilm al-marāyā) 4) the science of the centre of gravity ('ilm marākiz al-athqāl) 5) the science of measuring ('ilm al-misāha) 6) the science of the tapping of stretches of water ('ilm inbāt al-miyāh, irrigation) 7) the science of pulling loads ('ilm jarr al-athqāl) 8) the science of clocks ('ilm al-binkamāt) 9) the science of military equipment ('ilm al-ālāt al-harbīya) and 10) the science of pneumatic instruments ('ilm al-ālāt ar-rūhānīya).

And arithmetic, consisted of: 1) the science of 'open calculation' ('ilm al-hisāb al-maftūh) 2) the science of calculation with board and pencil ('ilm hisāb at-takht wa l-mail, i.e. with Indian numerals) 3) the science of Algebra ('ilm al-jabr wa l-muqābala) 4) the science of the calculation with two faults ('ilm hisāb al-khata'ain) 5) the science of the rotating bequests and legacies ('ilm ad-daur wa l-wasāyā) 6) the science of calculating with dirham and dīnār (that is algebraic equations with more than one unknown quantity).

If we remember now our fundamental law of literature (in short: "no text is not based on an other text") we can conclude that Ibn al-Akfānī did not just put forth a theory of the structure of sciences, but rather assessed the literary shape these disciplines had taken by his time. And, indeed, he adds to each of them a three-part list of texts that could be recommended to the beginner, to the advanced student and to the professional reader, or — to put it in modern terms — to the bachelor, to the doctorand and to the professor. This reading-list has much of a scientific 'who's who' of the Islamic Middle Ages. Most of the famous scholars are mentioned, their texts listed. But there also appear names that remain unknown until today, and titles of texts that do not betray more than the disciplines they treat. The sheer number of the texts, the names of the disciplines they are assigned to and the wide range of famous and forgotten professional writers indicate a decisive change: beyond the venerable (and still somehow dubious) sciences of Geometry and Arithmetic certain fields of applied mathematics had become subjects of the academic milieu. They were taught and studied, between Uzbekistan and Andalusia, and written upon. Constantly, books appeared that proposed to the reader to make use of new methods and techniques. Most of the (known) texts contain a peculiar introductory element: they are addressed not only to professionals, such as jurists, in particular to Qādīs and their officials, to clerks in the customs and other administrative sections of the state (tax office, military office, office of public constructions etc.) — but they were also addressed to the general public (ar.: 'āmm), to the private taxpayer, money-changer, day labourer or employer.

What we have here are the two facets of the same coin. The fact that the strict border-line between theoretical and practical mathematics was given up by the Islamic mathematicians reflects the cultural esteem that was paid for their professional contributions to the amelioration of social conditions. But when responding to the social need for their art they had to adapt their standard scientific methods to the particular demands of the public. Thus, mathematics were enriched with disciplines most of which did not exist before and outside the Islamic period, were diffused into different segments of society and were, finally, 'domesticated', regarded as skills useful for every single believer and for the community as a whole.

This surview seems to confirm our investigation. Due to a long process, the 'practical' disciplines we were looking for had not only come into existence but had also branched out in numerous different scientific fields. But what exactly did they offer to the Islamic community? Which social need domesticated them? Most of what I have been — and will be — talking about was neglected hitherto by the historians of mathematics — too simplistic for them — and, on the other hand, by the historians of Islamic culture — too mathematical for them. In Oriental libraries, however, several hundred texts on the art of calculation of administrative, social and juridical affairs have been registered; only a few, perhaps a dozen, are edited and/or translated. And each of them contains dozens or even hundreds of problems of very different nature and scientific niveau. My way out of this dilemma of quantity will be to present to you some selected examples of the major fields were mathematics and Islamic needs met.

[t = tax quota; T = total tax; = unit of area; G = total area; A = total tax collector's share; R = total tax officials' share; Kh = total *ḫarāǧ*]

(1) $t/g = T/G \rightarrow T = G \cdot (t/g)$

(2) $G = T : (t/g)$

(3) $G_2 : G_1 = T_2 : T_1 \rightarrow G_2 = (G_1 \cdot T_2) : T_1$

(4) $T + A = (t/g + a/g) \cdot G \rightarrow A = (t/g + a/g) \cdot G - T$

(5) $T + A + R = (t/g + a/g + r/g) \cdot G$

(6) $Kh = T + A + R$ then

(7) $T : (T + A) = t/g : (t/g + a/g)$

⋮

etc.

Figure 13 – Abū l-Wafā' al-Būzǧānī (died 998): *Kitāb fīm'a yahtāǧ ilaihi* (page 287ff.)

Let me start with a timeless problem: taxation, and with an author of the 'Abbāsid period: our Abū l-Wafā' al-Būzjānī, a high official in the BaGhdad administration and at the same time outstanding mathematician of the late 10th century. In his book "What the mathematicians and the officials need to know about the art of calculation" (a book of 350 pages) he points to a problem that is addressed to both the tax payer and to the tax collector. The former is offered methods to protect himself against abuse and exploitation; the latter is warned not to treat the former unjustly. Abū l-Wafā' proceeds as follows (figure 13): In the agricultural milieu of Iraq the basis of the procedure of taxation is the tax-quota, that is the quantity tisq per unit of area jarīb that has to be paid on the total area, capital G. Thereof, the total tax, capital T, can be calculated in dirham (1). Reversely, if quota and total tax are known one can calculate the total area, capital G (2); equation (3) depicts the operation necessary if, for example, a farmer wants to conclude from his neighbour's total tax his own tax owing. But tax credits are not — as we all know — that simple and were not in 'Abbāsid Iraq - as we shall see. Beside the proper tax, the Tisq, a proportional expense allowance, called ayīn (hence capital A), had to be paid to the tax collector. This produces the equation (4); and finally a third tax, called rawāj (capital R), the proportional share that had to be deducted for the official in the dīwān, the central tax administration, complicated the affair (5). This seems to be evident so far. But Abū l-Wafā‘ demonstrates the mathematical ambivalence of this formula that is generated by a loophole in the Islamic tax law. The procedure chosen operates on the basis that both, ayīn and rawāj, have to be paid in proportion and — this is the crucial point — in addition to the actual tax quota fixed by the law. This is clearly to the disadvantage of the farmer. The just solution, however, would require to include the various additional taxes, A and R, into the legally prescribed total tax T. The total gross tax then, the kharāj, must not exceed the sum of the legal quotas (6). And the additonal taxes must be deducted proportionally and step by step by way of the following proportion (7). The solution results in a second degree equation and differs from the first, ‘unjust’ method by 6 to 9 %. Abū l-Wafā’s treatise remained well-known all through the Islamic Middle Ages. Not because he had gained the grateful respect of the tax payers let alone of the tax collectors’; but because he was the first mathematician to compose a full compendium of what our encyclopaedists called ‘hisāb al-mu’āmalāt’.

[Turkish bath: 30 visitors, 3 Jews; fees: Muslims $\frac{1}{2}$, Christians 2, Jews 3 *dirham*]

Be: $3x + 2y + \frac{1}{2}z = 30$ and: $x + y + z = 30$

For: $x = 3 \rightarrow z = 21 - y$ and: $y = 5$

Figure 14 – aš-Šaqqāq (12th cent., Syria)

From now on treatises of this type contained a first chapter with an intensive introduction into the basic operations of Arithmetic, Algebra and Geometry; then a second chapter on their application in social affairs; and, finally a third chapter on ‘curiosities’, meaning mathematical riddles dressed up as everyday problems like the following one (figure 14): The attendant of a Turkish bath that demands different entrance fees (for Muslims half a dirham, for Christians two dirhams and for Jews three dirhams) finds 30 dirhams in the day’s takings. He had registered 30 visitors, three of them Jews. But who were the remaining 27? Here, we are crossing over an invisible border-line to the ‘recreational’ problems mentioned above. But beyond the clear algebraic procedure in this case we realize another motive of the author, his playful but sincere pedagogical request: mathematics are everywhere; look around and practice!

Let us turn now to a second field of Islamic law, the so-called ‘calculation of inheritances’ (ar.: hisāb al-farā‘id). On no other legal field the Qur‘ān is more explicit and precise than on the law of inheritance. In 10 verses, exact prescriptions are reveiled on who inherits what share of the deceased’s property. Here is an example of how, mathematics effected different interpretations of these divine prescriptions. We probably have all heard that according to the Qur‘ān (Sura 4, “The Women”, verse 11) the female inheritance share is half of the male share. But what happens to an hermaphrodite? A person who’s sex cannot be decided ultimately? Who is neither male nor female, or — to put it positively — both at the same time? The problem turned out to be much more complicated than expected (figure 15). In fact, it is the first reported case of Islamic jurist-mathematicians to deal with probability. At first, the jurists had to find the criteria by which a person could be declared to be an hermaphrodite. They finally agreed on a definition by exclusion: As long as the anatomy, the social behaviour and the individual articulation of the dubious person could not be clearly assigned to one of the two sexes the person had to be regarded as an hermaphrodite (or androgyne!). The second stage of the solution now required the mathematician to translate this intermediary position into fractions. The problem was that the legal prescription could not be simply translated into mathematics. A ‘middle’ position was something else than an arithmetic mean. So, not amazingly, the major Islamic law-schools put forth quite different

1/2 (male + female)

	Shares		Shares	
	Son	*khunthā*	Son	*khunthā*
Abū Hanīfa	2	1	56/84	28/84
Abū Yūsuf II (+ variant) "case 1": male "case 2": female	2 (1) 2 (2)	2 (1) 1 (1)		
	4 (3)	3 (2)	48/84 (50 2/5 /84)	36/84 (33 3/5 /84)
ash-Sha'bī ash-Shaibānī 1: male 2: female	1/2 2/3	1/2 1/3		
	7 [= 7/6 × 2]	5 [= 5/6 × 2]	49/84	35/84
Shāfi'ites 1: male 2: female	1/2 2/3	1/2 1/3		
 a: male b: female	*mauqūf* (rest) = 1/6		 56/84	rest = 14/84 42/84

Figure 15 – The hermaphrodite's (*khunthā*) share: solutions of the classical law-schools

solutions. By adding up differently the two probabilities: that the hermaphrodite child is of (1) male or (2) female sex they all arrived at different solutions. [The right column gives the different solutions in the highest common denominator, in eighty-forths.] In fact, it was the arbitrariness of mathematical alternatives that made them find their solution, not the letter of the law. An early jurist of Baghdād, Jābir b. Zaid, seems to have realized this danger and tried to get rid of the entire problem by proposing: "Put the hermaphrodite in front of a wall; if he urinates onto it and it gets wet — he is a man; if not — she is a woman!"

$x_1, x_2, \ldots, x_6 =$ qur'ānic quotas 1/2, 1/3, 2/3, 1/4, 1/6, 1/8
$a, b, \ldots, f =$ number of heirs of equal quality
$q =$ quota
$R =$ remainder of inheritance
$T =$ testamentary legacy
Then: $q(ax_1 + bx_2 + \ldots + fx_6) + T + R = 1$

Figure 16 – Simplified formula for the division of inheritances

But this was not the only problem the Qur'ānic inheritance law posed to the jurists. [Among them a saying circulated that runs: "Half of all legal knowledge belongs to the farā'id, the law of inheritance, and half of the farā'id is hisāb, Arithmetic."] In fact, the mathematical mantraps of the Islamic inheritance law proved to be so numerous that a special discipline developed: our hisāb al-farā'id. The main difficulty was to interpret God's word unambiguously. Contradictions had to be ruled out and the solutions had to be applicable to all possible cases. And these stipulations could not be fulfilled without mathematics. In order to give you an idea of the basic elements of the division of inheritances I have sketched for you a simplified structure (figure 16). From this equation, you may imagine the influence mathematicians gained on this discipline. In fact, most of the specialists of the inheritance law had a mathematical formation. I will now skip several centuries of the remarkable career of this interdisciplinary marriage.

Definitions:

Heirs: Husband (= H), Mother (= M); 2 daughters (= D);

Inheritance: 21 *dīnār*, one slave, a garden;

Division: Husband = money; mother = slave; daughters = garden.

Representation:

91		12/13	
21	3/13	1/22	H
14	2/13	1/23	M
28	4/13	1/3	D_1
28	4/13	1/3	D_2

Result: Husband = 21, Mother = 14, Daughters = 2 × 28; Inheritance = 91

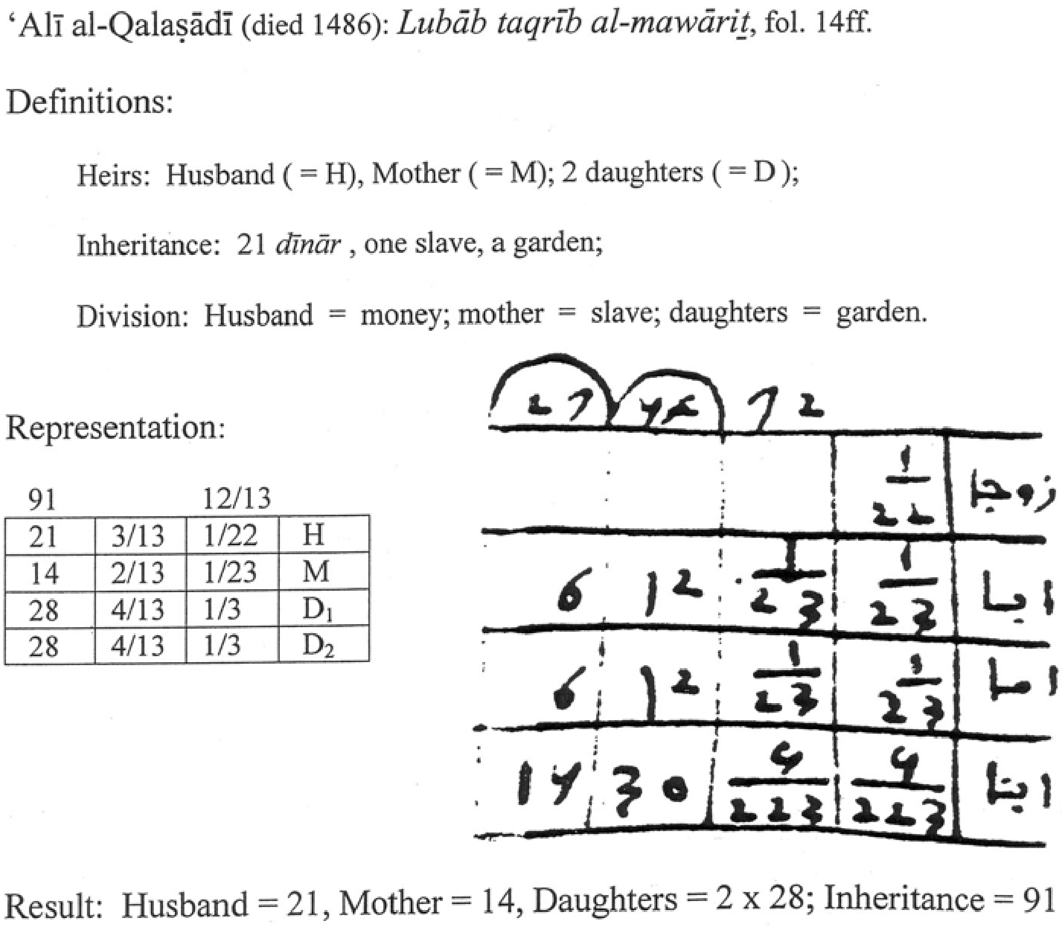

Figure 17 – 'Alī al-Qalaşādī (died 1486): *Lubāb taqrīb al-mawārith*, fol. 14ff.

At the end of this career we find a North-African exile from Spain, the mathematician and jurist 'Alī al-Qalasādī (died 1486), who composed his book "Pearls of coming close to the testimonies" in an amazingly clear and logical spirit (figure 17). The case I picked out from this book is neither spectacular nor problematic: a woman died and left her husband, her mother and two daughters all her property. It consisted of 21 dīnār in cash, a slave and a garden. The partition yields that the husband's share equals the cash, the mother's share the slave and the two daughter's share the garden. What is the value of the slave and the garden? Two things are remarkable: First of all, the formalized procedure. At first, the heirs are assigned their relative shares: one forth to the husband, one sixth to the mother, and then a third to each of the daughters. This adds up to 13/12 (twelveths); therefore, the denominator has to be increased by one, that is to 13. Now, the value of the slave and of the garden can be fixed. The case, clearly, is not real. al-Qalasādī divided his treatise into numerous sections each dominated by a specific combination of juridical prescriptions and mathematical procedures. He then constructs an abstract case that would allow the reader to solve all cases of the same type. Here, the inheritance case is an algebraic problem in disguise. The second remarkable thing refers to the notation. To the best of my knowledge, this is the first indication of the fraction line in the history of Arabic mathematical notation. The Italian Fibonacci who was the first European to use this fraction line had probably copied it when studying in North-Africa with fore-runners of our al-Qalasādī.

A last field which I previously called the 'calculation of social affairs' remains. This is somehow misleading. After all, the examples until now have not been unsocial. But they stem back to — more or less — professional milieus. By social I mean, more precisely, the general need and access of common folk to the knowledge offered by these disciplines. Pre-modern societies, and especially the intercontinental Islamic societies were loaded with systems of measurements. Every item that had to be measured was measured in a particular unit. Depending on the object to be measured: time, distances, money, crops, spices, land, textiles and so forth, the dimensions had to be considered: lenght, width, area, volume and weight, and finally often the monetary value had to be computed. In addition, these systems were useless once you crossed the border of your province or even your home-city. The money-changer was the back-bone of the Oriental economy. No Oriental sūq could do without him, no long-distance-trade without reliable tables of the equivalence of units. Again, it was the mathematicians who took care of that.

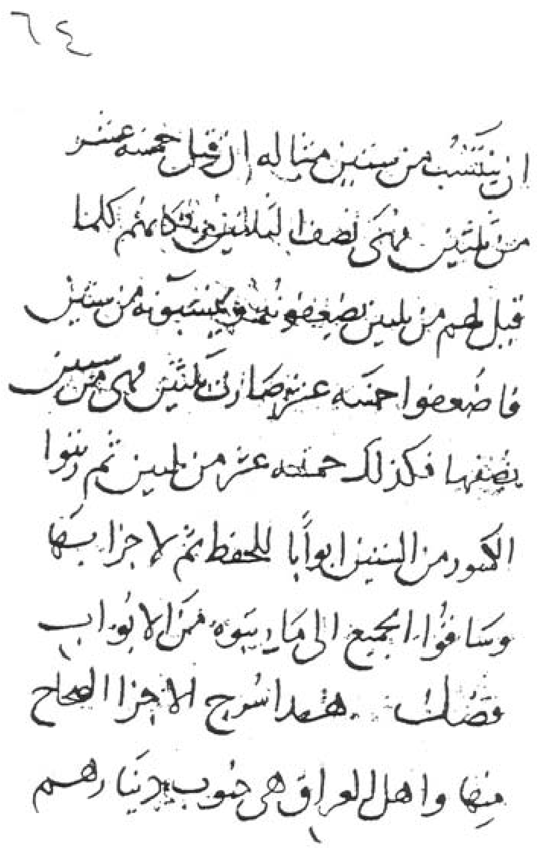

Figure 18 – 'Alī b. al-Khidr al-Qurašī (died 1067, Damascus)

Here is an example of one of the computations everybody who dealt with merchandise had to know: the conversion of sexagesimal into decimal numbers. The page in figure 18 belongs to a book I recently edited, translated and commented upon. As you see, no numerals are

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1 bezogen auf 60 ergibt $\frac{\frac{1}{6}}{10}$: 2 → $\frac{\frac{1}{3}}{10}$: 3 → $\frac{\frac{1}{2}}{10}$: 4 → $\frac{\frac{1}{3}}{5}$: 5 → $\frac{\frac{1}{2}}{6}$: 6 →
$\frac{1}{10}$: 7 → $\frac{1}{10} + \frac{\frac{1}{6}}{10}$: 8 → $\frac{1}{10} + \frac{\frac{1}{3}}{10}$: 9 → $\frac{1}{10} + \frac{\frac{1}{2}}{10}$: 10 → $\frac{1}{6}$: 11 → $\frac{1}{10}$ +
$\frac{\frac{1}{2}}{6}$: 12 → $\frac{1}{5}$: 13 → $\frac{1}{6} + \frac{\frac{1}{2}}{10}$: 14 → $\frac{1}{6} + \frac{\frac{1}{3}}{5}$: 15 → $\frac{1}{4}$: 16 → $\frac{1}{6} + \frac{1}{10}$: 17
→ $\frac{1}{5} + \frac{\frac{1}{2}}{6}$: 18 → $\frac{1}{5} + \frac{1}{10}$: 19 → $\frac{1}{4} + \frac{\frac{1}{3}}{5}$: 20 → $\frac{1}{3}$: 21 → $\frac{1}{4} + \frac{1}{10}$; 22 →
$\frac{1}{6} + \frac{1}{5}$; 23 → $\frac{1}{3} + \frac{\frac{1}{2}}{10}$: 24 → $\frac{2}{5}$; 25 → $\frac{1}{4} + \frac{1}{6}$; 26 → $\frac{1}{3} + \frac{1}{10}$; 27 → $\frac{1}{4}$
+ $\frac{1}{5}$ (65) ; 28 → $\frac{1}{5} + \frac{\frac{1}{6}}{10}$; 29 → $\frac{1}{4} + \frac{1}{6} + \frac{\frac{1}{3}}{5}$; 30 → $\frac{1}{2}$; 31 → $\frac{1}{4} + \frac{1}{6}$ +
$\frac{1}{10}$; 32 → $\frac{1}{3} + \frac{1}{5}$; 33 → $\frac{1}{2} + \frac{\frac{1}{2}}{10}$; 34 → $\frac{1}{2} + \frac{\frac{1}{3}}{5}$; 35 → $\frac{1}{3} + \frac{1}{4}$; 36 → $\frac{1}{2}$
+ $\frac{1}{10}$; 37 → $\frac{1}{4} + \frac{1}{6} + \frac{1}{5}$; 38 → $\frac{1}{3} + \frac{1}{5} + \frac{1}{10}$; 39 → $\frac{1}{2} + \frac{1}{10} + \frac{\frac{1}{2}}{10}$; 40 →
$\frac{2}{3}$; 41 → $\frac{1}{3} + \frac{1}{4} + \frac{1}{10}$; 42 → $\frac{1}{2} + \frac{1}{5}$; 43 → $\frac{2}{3} + \frac{\frac{1}{2}}{10}$; 44 → $\frac{2}{3} + \frac{\frac{1}{3}}{5}$; 45
→ $\frac{1}{2} + \frac{1}{4}$; 46 → $\frac{2}{3} + \frac{1}{10}$; 47 → $\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$; 48 → $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$; 49 →
$\frac{1}{2} + \frac{1}{4} + \frac{\frac{1}{3}}{5}$; 50 → $\frac{1}{2} + \frac{1}{3}$; 51 → $\frac{1}{2} + \frac{1}{4} + \frac{1}{10}$; 52 → $\frac{2}{3} + \frac{1}{5}$; 53 → $\frac{1}{2}$ +
$\frac{1}{3} + \frac{\frac{1}{2}}{10}$; 54 → $\frac{1}{2} + \frac{1}{3} + \frac{\frac{1}{3}}{5}$; 55 → $\frac{2}{3} + \frac{1}{4}$; 56 → $\frac{1}{2} + \frac{1}{3} + \frac{1}{10}$; 57 → $\frac{1}{2}$
+ $\frac{1}{4} + \frac{1}{5}$; 58 → $\frac{2}{3} + \frac{1}{5} + \frac{1}{10}$ (66) ; 59 → $\frac{2}{3}$ + [$\frac{1}{4}$][87] + $\frac{\frac{1}{3}}{5}$; 60 → 1;

Figure 19 – 'Alī b. al-Khidr al-Qurašī (transcription of page 64)

used. By the life-time of its author, a certain 'Alī b. al-Khidr al-Qurashī, who died 1067 in Damascus at the age of 37, the Indian numerals, our Arabic numerals, had not yet found their way into this type of hisāb-treatise. And in figure 19 you see the modern transcription of the table, one of several dozens put together in this book. Verily no text of this type can get by without a thorough introduction into the basic mathematical methods required for such an operation. And it is only then that they proceed to more complicated computations like the calculation of interest, of profits or of labor costs.

Although somehow formalized in style and structure, each one of these texts is highly individual. Between the lines, valuable and unexpected information is transmitted. In the above mentioned treatise one paragraph investigates the difficulty to handle the different religious calendar systems of the Persians, Arabs, Copts, Byzantines and Jews. Think of holidays or the duration of contracts in sun- or moon-based calendars and you can imagine where the problems started. In another paragraph where the binomial formula $(a-b)^2$ is explained I found the first explicit use of a negative number in an Arabic text. Side by side, social and mathematical skills are trained. Mathematics were accepted as a tool to make things run smoothly and to ease the burden of life. Even in more sensitive areas. In areas that touched upon the piety of the believer. This is where my second-to-last example is taken from.

[W_0 = original quantity = 10 *raṭl*; W = remaining third = ?; W_1 = evaporated quantity = 1 *raṭl*; W_2 = quantity skimmed off = 3 *raṭl*]

$$W = \left[\frac{1}{3}W_0 \cdot (W_0 - W_1 - W_2)\right] : (W_0 - W_1) = 2\frac{2}{9} \ \textit{raṭl}$$

Figure 20 – Ibn al-Humām (died 1457): *Fath* VIII, page 168/10

As so many before him, Ibn al-Humām (died 1457), a hanafī jurist of great reputation, struggled with the prohibition of alcohol (figure 20). The central problem had always been: when did any pressed juice turn into an alcoholic, intoxicating drink? After all, the prophet Muhammad himself had been offered every morning a fresh drink of pressed dates his wife 'Ā'isha had prepared the evening before. Had his drink already started to ferment and if yes, to which extent? Evidently, Ibn al-Humām was familiar with the art of preparation of wine (ar.: khamr) and similar alcoholic drinks. In order to prevent the fruit juice from fermenting it had to be brought to the boil, that is pasteurized. Only if reduced to one third of its original quantity was the juice regarded as legally admissable. This process of condensation could be measured. Four quantities were involved: the original quantity of pressed juice ($= W_0$), the remaining third ($= W$), the quantity evaporated during the process of boiling ($= W_1$) and the quantity of foam that had to be continually skimmed off from the boiling surface ($= W_2$). Therefore, our jurist Ibn al-Humām put up this equation.

$$\text{If: } W_1 = 6 \ \textit{ra\d{t}l};\ W_2 = 4 \ \textit{ra\d{t}l};\ W = 8\frac{1}{3} \ \textit{ra\d{t}l};\ W_0 = x;$$

$$\begin{array}{ll} \text{Then:} & (W_0 - W_1 - W_2) : W = (W_0 - W_1) : \frac{1}{3}W_0 \\ \rightarrow & (x-10) : 8\frac{1}{3} = (x-6) : \frac{1}{3}x \\ \rightarrow & x - 10 = 3 \cdot \left(8\frac{1}{3} - 50\right) : x \\ \rightarrow & x^2 - 10x = 25x - 150 \\ \rightarrow & x = 17\frac{1}{2} + \sqrt{1\,225 - 600} : \sqrt{4} = 30 \end{array}$$

Figure 21 – 'Abdalqāhir al-Baġdādī (died 1037): *at-Takmila fī l-hisāb*, page 283ff.

He apparently, however, benefited from the considerations of a well-known fellow-scholar of the same law-school, the Jurist and mathematician 'Abdalqāhir al-Baghdādī from Isfahan who had already dealt with this problem four centuries before Ibn al-Humām. In figure 21 you have his algebraic procedure how to find out how much juice had been pressed to produce eight and one third ratl, about 4 litres, of harmless grape juice.

The important question, however, remains: Did this simple formula affect the drinking habits of Muslims? Was it applied by the wine-growers, predominantly Christians? Or, was it used by the municipal authorities to check the legality of sour drinks consumed in public and privately? In fact, we only have indirect historical evidence of the application of such or similar methods to make sure that this or that drink was prohibited or not. If considering the role of practical mathematics in the Muslim society we must be aware of the simple fact that this role was not recorded. Practice ends, so to say, in the realm of the oral and is genuinely non-literary. Therefore, direct evidence of the application of mathematically inspired solutions of everyday problems are rare. But if, on the other hand, as we were able to observe, certain practical mathematical disciplines came into being, were developed and standardized in terms of teaching and instructional texts — this could not have happened out of nothing. I understand this phenomenon as indirect evidence of the social demands on experts and of their commitment to respond to it; as indirect evidence of a circulation of needs and devices that was inspired by a growing readiness to accept methods other than and outside of the literal interpretation of the holy texts and — on the other side — other than traditional customs. At least until the 15th century, this area where calculable methods replaced arbitrary ones spread.

My very last example is meant to underline this conclusion. The example (figure 22) belongs to a shiite author of the 15th century, a scholar of the religious law, no mathematician

Prayers/Duties	possible combinations	minimum of duties
2 (B,C)	2 (B,C,B|C,B,C)	3 (C,B,C)
		[or: B,C,B]
3 (A,B,C)	6 (A,C,B,C|C,A,B,C|C,B,A,B)	7 (C,B,C,A,C,B,C)
	(A,B,C,B|B,A,C,B|B,C,A,B)	[or: ——]
4 (A,B,C,D)	24 (——)	15 (——)
5 (A,B,C,D,E)	120 (——)	31 (——)
————!!!		
6	720	63
7	5 400	127

Figure 22 – Miqdād b. ʻAbdallāh as-Suyūrī (died 1423): *Nadad al-qawāʻid al-fiqhīya*, page 168.ff.

nota bene, a certain Miqdād b. 'Abdallāh as-Suyūrī, from Persia, whose only concern was to improve the piety of the believers. In his book, *The frame of the legal principles*, he investigates the situation of a believer who — for whatever reason — was not sure about which one of the daily five prayers he had just performed invalidly because he had forgotten the distinct prescriptions for each of the obligatory 5 daily Islamic prayers. According to Islamic law and for the good of his own spiritual welfare the believer is obliged to compensate for such a neglect of his duties — in the first place by making up for the prayer left out, by repeating it correctly. The solution seems to be perfectly obvious. But once you look closely at it it becomes tricky. If one of the prayers at noon and in the afternoon was affected he is obliged to perform at least three prayers since the possible combinations of the noon prayer (let's call it B) and the afternoon prayer (let's call it C) renders two possibilities (B–C–B and C–B–C). Either one adds up to a minimum of three prayers. Miqdād now adds a third prayer, the Morning Prayer A, to the problem. If the invalid prayer was one out of three duties, it renders six possible combinations and exactly a minimum of seven prayers to be performed in a row. He then goes on to explain the development with four and five prayers, as he is supposed to do, but then — he gets carried away with his idea, adds a sixth and seventh duty. "And", he says, "you can continue this ad infinitum". Miqdād, obviously, had not only discovered the arithmetic rule — the combinatoric one he did not grasp — of the legal prescription, but he had also acknowledged the universality that underlies the divine prescriptions.

We see here a particular and familiar, proto-modern mind at work. A mind that had been set into motion by the same sprit that had, long before and at the other end of the Islamic world, inspired our first mathematician, Ibn Mun'im in Marrakech, to use combinatorics for his linguistic analysis. In between the two of them, in time and space, a largely unknown history of practical every-day mathematics waits to be discovered. You are welcome to contribute to it. You may for that omit the prayers, and even have a glass of wine, but you have to learn Arabic. Thank you for your patience.