The Problem of the Dimensions of Space in the History of Geometry

Klaus VOLKERT

Universität zu Köln, Seminar für Mathematik und ihre Didaktik,
Gronewaldstraße 2, D-50931 Köln

k.volkert@uni-koeln.de

Abstract

In the following text we will study one aspect of the problem indicated in its title: How can we express the fact that space has exactly three dimensions using only the tools of classic synthetic geometry?

Space in Euclid's "Elements"

Solid geometry is dealt with in the books eleven to thirteen of Euclid's "Elements" (∼–300). A definition of space is missing in Euclid's text, we learn only the following:

"A solid is that which has length, breath, and depth.
An extremity of a solid is a surface."

(Definitions 1 and 2 of book XI, we cite from Heath's edition (Heath, 260)). It is even said that the classic Greek language had no term for our space. Thus it is not surprising that Euclid did not define it. But there is an obvious question: Can Euclid avoid any reference to space in his work? Because he is considering solids there must be at least three dimensions, but in principle there could be more! So we may ask: Are there propositions in Euclid's books which depend on the fact that space has exactly three dimensions? To be sure: this is a question asked from our modern point of view. In Euclid's work space remains negative[1] in the sense that it is only used implicitly.

The answer to this question is "yes" — we only have to look at the proposition 3 of book XI:

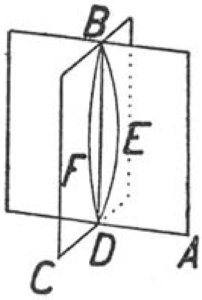
"If two planes cut one another, their common section is a straight line."

Obviously this is a statement about the position of two planes in space, so its proof rests not only on properties of the plane or the straight line (like "If a straight line and a plane have two points in common, the line is completely contained in that plane").

Many of the ideas contained in this article were developed during a stay at the Archives Henri Poincaré (Université Nancy 2) in the spring of 2007. I want to thank G. Heinzmann, Ph. Nabonnand and Ph. Lombard for their kind reception.

[1] This expression is taken from the history of arts, cf. Kern 183, 153.

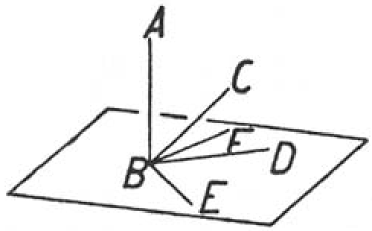
Euclid's proof goes like that:



Let the line DB be the section of the two given planes AB and BC. We want to show that DB is straight. "For, if not, from D to B let the straight line DEB be joined in the plane AB, and in the plane BC the straight line DFB.

Then the two straight lines DEB, DFB will have the same extremities, and will clearly enclose an area: which is absurd." (Heath, 276)

The most important consequence of XI, 3 deduced by Euclid is to be found in theorem 5 of book XI: "If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane." (Heath 1956, 281).



Here is the proof by reductio ad absurdum given by Euclid. Suppose that BD, BE are in the plane of reference but BC not. Because AB and BC meet in B there is a unique plane containing them (XI,2). So the two planes through BD, BE and AB, BC have the point A in common. By XI, 3 their section is a straight line passing through this point. Let it be BF. Because AB is orthogonal to the two straight lines BD and BE, it is orthogonal to every straight line in the plane of BD, BE passing through A. In particular it is orthogonal to BF (recall that this line is in the section of the two planes). So in the plane of AB, BC there are two straight lines — BC and BF — which are orthogonal to AB passing through B. In other words, the angle ABF would be equal to the angle ABC. That is not possible.[2]

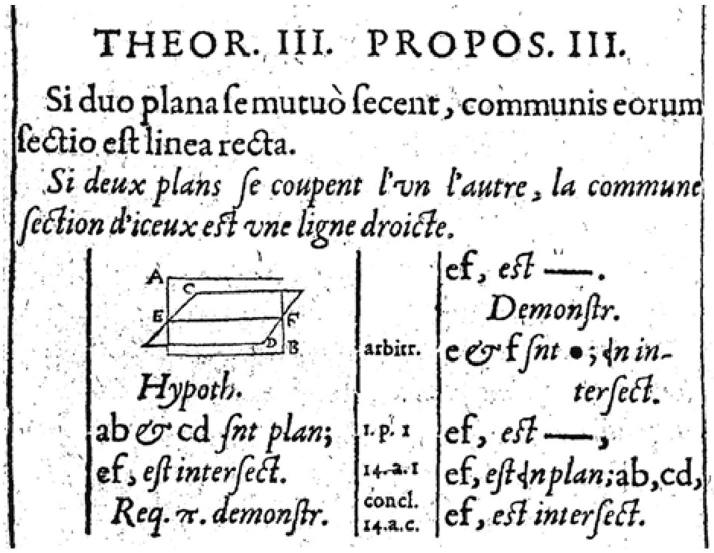
So we may state that the fact that space has three dimensions is equivalent to the fact that there are only three straight lines passing through a point and being orthogonal to each other. To us this seems to be a very natural characterization. But this is due to the fact that we are familiar with analytic geometry. From the point of view of classic synthetic geometry this characterization is not very useful because it is operational.

Some later improvements

For the following we notice that Euclid presupposes that the section of two planes is a line. If asked why he did so he could quote the second definition above: The extremity of a plane is a line.

But it is possible to simplify Euclid's argument in this respect. A first possibility is indicated in the following citation of Pierre Hérigone (1634):

[2]The theorem XI, 3 is used in book XI in the proofs of the following theorems: 5, 6, 7. 13, 14, 16, 17 und 38 (cf. Neuenschwander 1974, 93f).


THEOR. III. PROPOS. III.

Si duo plana se mutuò secent, communis eorum sectio est linea recta.

Si deux plans se coupent l'vn l'autre, la commune section d'iceux est vne ligne droicte.

		ef, *est* —.
		Demonstr.
	arbitr.	e & f *snt* •; ∢*n intersect.*
Hypoth.		
ab & cd *snt plan*;	1.p.1	ef, *est* —,
ef, *est intersect.*	14.a.1	ef, *est* ∢*n plan*; ab, cd,
Req. π. *demonstr.*	concl. 14.a.c.	ef, *est intersect.*

Hérigone used a special symbolism to write down his proofs. It is not to difficult for us to understand it. The points E and F are in the section of the two planes so is the straight line EF joining them (EF is in the plane AB because E and F are in that plane, EF is in the plane CD because E and F are in that plane too; cf. above). Hence the section is a straight line.[3]

Another type of argument is to be found in Legendre's "Eléments de géométrie" (1794): Let's suppose that the points E, F and G are in the section and that they are not situated on a straight line. Then the intersecting planes must be identical because three points which are not collinear determine exactly one plane.[4]

We may use this to answer the question raised in footnote 4: if there is a point in the section outside the straight line EF, then the two intersecting planes are identical and every point of them is in the intersection. So in combining the argument given by Hérigone with that given by Legendre we get the following theorem: If the section of two non-identical planes contains two points, then this section is exactly a straight line.

This is nice. But there is an obvious question: Can we reduce the hypothesis of our theorem to "there is one point in the section"? The answer is "yes": Christian von Staudt was the first (to my knowledge) to formulate this. In his "Geometrie der Lage" (1847) he states:

"20. …two planes, which pass through one and the same point, cut one another in a straight line which passes also through that point and outside of it there are no common points of the two planes."[5]

Von Staudt gives no proof of his nice theorem. We find such a demonstration about 20 years later in a book written by Richard Baltzer "Elemente der Mathematik". Baltzer's book was a widely used compendium of the contents of school mathematics in his time (school is here to be understood as "German Gymnasium"); it is valuable not only for its mathematics but also for its historical remarks. In particular Baltzer introduced non-Euclidean geometry to the German public by his book.[6]

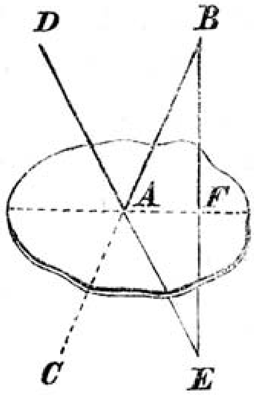
[3] It would be more precise to state that the section contains that straight line. It is not proven that there are no points in that section outside the straight line EF. We come back to that question soon.

[4] Cf. Euclid XI, 2.

[5] "20. …zwei Ebenen, welche durch einen und denselben Punkt gehen, schneiden sich in einer Geraden, welche ebenfalls durch jenen Punkt geht, außerhalb aber die beiden Ebenen keinen gemeinsamen Punkt mit einander gemein haben." (von Staudt 1847, 8)

[6] To be precise we must state that this is true for the second edition of its second part treating geometry (1867). For more details of the importance of Baltzer's book one may consult the book Voelke 2005, 56–57.

Here is Baltzer's proof[7]:



Let A indicate the point of intersection of the planes p and p'. In p' we take two straight lines passing through A with points B and C, D and E (cf. the drawing above). Now the points B and E are both in p', so we can join them by the straight line BE in p'. Because B is above p and E below the straight line BE has to intersect the plane p in a point F. So F is an another point of the section of the two planes. Therefore that section contains two points and we can continue the argument as above.

Let us pause for a moment and think about the history we have learned. There was a considerable progress in sharpening the hypothesis of our theorem reducing it from the existence of a whole line to that of a single point. But there was no real progress in the axiomatic foundation of solid (nor of plane) geometry. Baltzer, Legendre and all the other geometers used Euclid's axioms and postulates without completing them — or even worse![8]

The solution

The first mathematician doing so was Moritz Pasch (1882). In his "Lectures on recent geometry" (1882) Pasch gave an axiomatic base for projective geometry. In particular he formulated for the first time in the history of geometry a complete set of axioms of incidence, order and congruence[9]. From our modern point of view his treatment is complicated by his empiristic philosophy of geometry forcing him to built up the projective space by enlarging step by step a finite range. In the section devoted to planes Pasch introduces the following axiom (he called it "Kernsatz"): "If two planes P, P' have a point in common, one can designate another point which is in one plane with all the points of P and with all the points of P'." (Pasch 1926, 20)[10] Following Pasch this is a simple matter of fact — we learn it by our experience. The idea of Pasch was taken up by Hilbert in his now famous "Foundations of geometry". He uses two axioms to characterize the three-dimensional space: I,7. "If two planes α, β have a point A in common, then they have a least one other point B in common." and I, 8 "There are at least four points which are not in a plane." (Hilbert 1972 , 4)[11] He comments on these two axioms: the first expresses the fact that space has not more than three dimensions, the second that it has not less than three dimensions. It is possible to state that Hilbert solved the problem to characterize three-dimensional space with the means of synthetic geometry.

[7] Heath ascribes the proof given here to Killing (1898), 43.

[8] Legendre's axioms are far less complete than Euclid's for example.

[9] The axioms of incidence and the axioms of order are more or less the same as the "graphic" properties which were discussed by Poncelet (in difference to the metric properties).

[10] "III. Kernsatz. — Wenn zwei ebene Flächen P, P' einen Punkt gemeinsam haben, so kann man einen anderen Punkt angeben, der sowohl mit allen Punkten von P als auch mit allen Punkten von P' je in einer ebenen Fläche enthalten ist." (Pasch 1976, 20).

[11] "I 7. Wenn zwei Ebenen α, β einen Punkt A gemein haben, so haben sie wenigstens noch einen weiteren Punkt B gemein. I 8. Es gibt wenigstens vier nicht in einer Ebene gelegene Punkte." (Hilbert 1972, 4)

There is still a little problem: the axiom I, 7 is not very convincing — it is not obvious. Thus the question is: Can we replace Hilbert's axiom by a statement which seems to be evident and obvious? We can do that and the answer was proposed implicitly by Baltzer's proof of von Staudt's theorem. This proof uses the "fact" that space is separated by any of its planes. For this reason the two resulting half-spaces are disjoint and the straight line joining points in different half-spaces cut the plane in a point whereas the straight line through two points in the same half-space doesn't meet the plane. We find this axiom in a slightly modified form in A. N. Whitehead's "The axioms of descriptive geometry" (1907):

"For three-dimensional geometry two other axioms are requested: XV. A point can be found external to any plane. ... XVI. Given any plane p, and any point A outside it, and any point Q on it, and any point B on the prolongation AQ, then, if X is any other point [on the straight line through A and B], either X lies on the plane p, or AX intersects the plane p, or BX intersects the plane p..."

Axiom XVI secures the limitation to three dimensions, and the division of space by a plane. It can also be proved from the axioms that, if two planes intersect in at least one point, they intersect in a straight line." (Whitehead 1907, 6)[12]

As we have seen it is possible to proof XI, 3 on the base of this axiom. So our history has come to an end in the sense that we have found the place of Euclid's theorem in a complete axiomatic system including a satisfying formulation of the axiom. To the formalistic mathematician the last remark is meaningless but in real history of mathematics it is important. Once again we get a hint that the formalistic point of view is not adequate to understand history!

References

- Baltzer, R., ^{2}1867, *Die Elemente der Mathematik. Zweiter Band. Planimetrie, Stereometrie, Trigonometrie*, Leipzig : Hirzel.
- Heath, Sir T., 1956, *Euclid. The thirteen books of the Elements. Vol. 3 (Books X–XIII)*, New York : Dover.
- Hérigone, P., 1634, *Cursus mathematicus .../Cours mathématique...*, Paris : le Gras.
- Hilbert, D., 1972, *Grundlagen der Geometrie*, Stuttgart : Teubner [11. Auflage].
- Kern, St., 1983, *The Culture of Time and Space 1880–1918*, London : Weidenfeld and Nicolson.
- Killing, W., 1889, *Einführung in die Grundlagen der Geometrie. Band II*, Paderborn : Schöningh.
- Neuenschwander, E., 1974, "Die stereometrischen Bücher der Elemente des Euklid", *Archive for History of Exact Sciences* **14**, 91–125.
- Pasch, M., 1976, *Vorlesungen über neuere Geometrie*, Berlin : Springer — Reprint of the edition Berlin : Springer, ^{2}1926.
- Staudt, Chr. von, 1847, *Geometrie der Lage*, Nürnberg : Korn.
- Voelke, J.–D., 2005, *Renaissance de la géométrie non euclidienne entre 1860 et 1900*, Bern : P. Lang.

[12] This is not an axiom of descriptive (that is projective) geometry but of Euclidean geometry. In projective geometry two planes always cut in a straight line (there are no planes which are parallel one the another).

– Whitehead, A. N., 1907, *The Axioms of Descriptive Geometry*, Cambridge : The University Press.

An extended version of this article will be published in the “Festschrift für Werner Ast”, ed. by J. Schönbeck in 2008.