Didactic Simulation of Historical Discoveries in Mathematics

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Abstract

The article describes a geometric context, the so called Trileg mini-geometry which can be used to introduce some deep mathematical ideas (such as the axiomatisation of geometry, the introduction of coordinates, non-solubility of some problems in synthetic geometry, models, etc) to secondary school and university students. Each idea arises naturally when students solve problems. The Trileg mini-geometry is described in a didactical way via stages which a student passes through when solving problems. When applicable, a historical note is given to provide an example of the parallel between phylogeny and ontogeny.

Keywords: geometry, axioms, ontogeny, phylogeny, axiomatic system, model

1 Introduction

It is generally agreed that many abstract concepts of university mathematics are very difficult to grasp and students often learn them by rote. We consider this to be an unfortunate situation mainly for our students, future mathematics teachers.

One of the ways to enable students to get an insight into some deep ideas of mathematics is to offer them a suitable mathematical environment which is simple from the technical point of view but rich in ideas.

In this paper, we will introduce an environment — the creation of axiomatic system in a geometric context developed by Milan Hejný and called Trileg mini-geometry which was successfully used in our experiments and experimental teaching at secondary and university levels.

2 Axiomatic system

The building of Euclidean geometry belongs amongst the most important discoveries in the history of humankind. Even though earlier Euclid's Stoicheia were used in many secondary schools as the standard geometrical textbooks, in the past fifty years this tradition has disappeared. The idea of an axiomatic building of a mathematical discipline is demanding and its geometrical presentation is far more complex than its arithmetic ones (such as Peano's axioms of the structure of natural numbers). That is why, if students are presented with an axiomatic system at all, then it is in arithmetic. In addition, the axiomatic system is given to them as a given one and they have no chance to participate in its creation.

We believe that there is a way to acquaint secondary students and future mathematics teachers with the axiomatisation of a geometric structure in a constructivist way, i.e., the axiomatic system is not given to students but they are required to find it for themselves. This will be done within Trileg mini-geometry.

3 Trileg mini-geometry

From now on, we will work in the Euclidean plane E^2. The starting point of our approach is a 'theory' which we called Trileg mini-geometry (or TMG) (more detail in Hejný, 1990).

TMG consists of one primitive notion 'a point' and theorems (axioms) which can be derived from the Euclidean plane by means of a special instrument, a trileg. It is a compass with an additional leg which points to the midpoint between the two outer legs. Using this instrument we can make two constructions (Fig. 1):

1. to given points A, B, find the midpoint $A - \circ - B$,
2. to given points C, D, find the point $s_C(D)$ symmetric to D with respect to C.

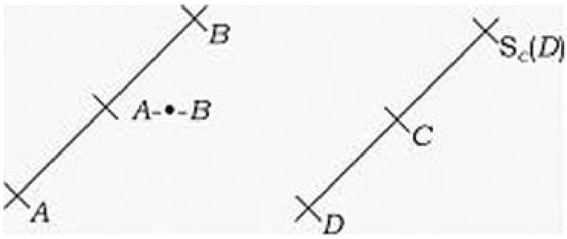

Figure 1

The trileg is the only tool available; students cannot use a ruler, compass or protractor.

A possible scenario of the implementation of TMG consists of 9 stages which will be presented in the way we use them with students. Each stage lasts at least one lesson.

3.1 Looking for relationships

In the first stage, we look for statements which can be posed within TMG about plane geometry. These statements are recorded via binary operations "s" and "$- \circ -$". It is obvious that if we construct point $C = s_A(B)$ to points A, B, it holds that $B - \circ - C = A$. This knowledge can formally be written in two ways:

a) for all $A, B \in E^2$, it holds $B - \circ - s_A(B) = A$,

b) for all $A, B \in E^2$, it holds $C = s_A(B) \Rightarrow B - \circ - C = A$.

Our task is to find as many similar relationships as possible and to record them formally via identity, or implication, or equivalence.

3.2 Solving equations

The statements discovered in the first stage by free experimentation are verified in geometric equations. For example, solving equation $A = s_X(B)$ means the following: given points A, B; we are to find all X for which the identity holds. From the diagram of the situation it is clear that $X = A - \circ - B$. Such a solution is not sufficient, though. It is necessary to solve the equation using only statements created in the first stage.[1]

For this a student uses the above implication and writes $A = s_X(B) \Rightarrow A - \circ - B = X$. Most of the class agrees with the solution but one student says that if we want to be precise, we have to write the result as $B - \circ - A = X$. Some students consider this unnecessary as it is evident that $A - \circ - B = B - \circ - A$. Others show that they have this identity in their list of statements.

This illustration shows the way in which students gradually learn to distinguish naive argumentation from the axiomatic argument.

[1]Some student solutions which come from our experiments will be presented in the paper via an imaginary student.

3.3 Opening deep problems

So far each student has had his/her own list of statements. In this stage, we will agree on one common list φ consisting of twelve statements (see the table).

$P - \circ - P = P$	(1)	$R = P - \circ - Q \Rightarrow P = s_R(Q)$	(7)
$P - \circ - Q = Q - \circ - P$	(2)	$P = s_R(Q) \Rightarrow R = P - \circ - Q$	(8)
$s_P(s_P(Q)) = Q$	(3)	$P = s_R(Q) \Rightarrow Q = s_R(P)$	(9)
$s_P(P) = P$	(4)	$P - \circ - R = Q - \circ - R \Rightarrow P = Q$	(10)
$s_P(Q) - \circ - Q = P$	(5)	$s_P(R) = s_Q(R) \Rightarrow P = Q$	(11)
$s_{P-\circ-Q}(Q) = P$	(6)	$s_Q(P) = P \Rightarrow P = Q$	(12)

From now on, the theory of TMG is defined as a set of statements φ and all statements which can be derived from them. Our experience shows that TMG can be best explored via solving equations. An example set of equations is in Appendix 1.

The solver can solve the equations in a geometric way, or analytically (see below) or algebraically via a set of statements. Let us present an example of the equation solved in an algebraic way.

$$(X - \circ - F) - \circ - s_G(F - \circ - X) = s_{F-\circ-G}(X)$$

statement used	equation changed into
(2)+(2)	$s_G(F - \circ - X) - \circ - (F - \circ - X) = s_{F-\circ-G}(X)$
(5)	$G = s_{F-\circ-G}(X)$
(9)	$X = s_{F-\circ-G}(G)$
(6)	$X = F$

Next, we will look at two interesting equations: (a) $s_A(X) - \circ - B = X$, (b) $s_X(C) - \circ - X = A - \circ - B$. They can be solved in a geometric way and the results are:

a) Point X divides line segment AB in the ratio of 1 : 2.

b) Point X is the center of gravity of triangle ABC.

The problem is that we cannot find an algebraic solution to these equations. Moreover, we cannot write point X via symbols A, B, C, s, $- \circ -$. Is it our inability or can it not be done? If it cannot be done, why? Why is it not possible to divide a line segment using the trileg into three identical parts?

At this moment, we can tell students about the Greek problems of antiquity (cube duplicity, circle squaring and angle trisection) and point out the similarity of angle trisection and our problem of trisection. Angle trisection was algebraically proved impossible by a French engineer P. L. Wantzel in 1836. His approach was based on Descartes' and Fermat's discovery of the transfer between a geometric situation and an algebraic-arithmetic situation (see below). This leads us to the introduction of coordinates.

3.4 Operations of s and $- \circ -$ in coordinates

The fourth stage begins simply.[2] Students already know a coordinate system and so they have no problem with solving the following problems.

T1. Given points $A[a_1, a_2]$ and $B[b_1, b_2]$ in a plane. Find the coordinates of points $C = s_A(B)$ and $D = A - \circ - B$.

T2. Rewrite some of the previously solved equations into the language of algebra and solve them again.

[2] Of course, sometimes students come up with the use of coordinates at the beginning of the whole process.

By rewriting geometric equations into algebra, students acquire an important experience — what we have to find in geometry by insight, can be found in algebra by handling expressions. The problem, however, remains to interpret the algebraic statement and rewrite it back using operations s and $- \circ -$. That is the focus of the following task.

T3. Solve equation $s_A(X) - \circ - B = X$ analytically.

When we rewrite the equation as coordinates, we will get a system of equations:

$$\frac{(2a_1 - x_1) + b_1}{2} = x_1, \frac{(2a_2 - x_2) + b_2}{2} = x_2,$$

whose solution is $x_1 = \frac{2a_1+b_1}{3}$, $x_2 = \frac{2a_2+b_2}{3}$.

We can see that both coordinates are the same and thus we can easily limit ourselves to one coordinate (the task does not concern the plane but the straight line AB). It is important to see that we are not able to rewrite expression $\frac{2a+b}{3}$ using s and $- \circ -$.

3.5 The impossibility of trisecting a line segment with the trileg

The fourth stage brought about a problem of trisecting a line segment which is parallel to angle trisection. Let us look at the way in which Wantzel showed that only some constructions could be made with a pair of compasses and a ruler. The constructions are only those which after rewriting into the algebraic language lead to a system of liner and quadratic equations. With some degree of informality we can say that with a pair of compasses and a ruler we can construct only what we can calculate on a calculator with the four basic operations and square root operation and nothing else.

Wanzel's idea rests on three steps:

1. A geometric situation is changed into an algebraic one, that is to each geometric object an algebraic object is uniquely mapped and to each geometric construction step an algebraic operation is mapped. These operations will be called *permissible.*

2. A set Ω of all algebraic objects which can be received from given objects by the permissible operations is algebraically described.

3. It will be shown that the algebraic object which corresponds to the unknown geometric object does not belong into Ω.

Exactly the same procedure will be simulated in TMG to show that we cannot trisect a line segment.

1) A geometric construction will be described in an algebraic language. To two points A and B, the line segment AB will be constructed and a coordinate system will be introduced so that number 0 corresponds to A and number 1 corresponds to B, that is $A[0]$ and $B[1]$. Now one real number corresponds to each point of line segment AB. Specifically, number $\frac{1}{3}$ corresponds to point X.

Next, if points $A[a]$ and $B[b]$ are given, algebraic operation h: $(a, b) \to 2a - b$ corresponds to construction $(A, B) \to s_A(B)$ and algebraic operation f: $(a, b) \to \frac{a+b}{2}$ corresponds to construction $(A, B) \to A - \circ - B$.

2) All algebraic objects from Ω which can be acquired from the given objects via permissible algebraic operations will be described algebraically.

With the repeated use of operation h, we can get all numbers $a + n(b - a)$, where $n \in \mathbf{Z}$, from numbers a, b. In other words, if $a = 0$, $b = 1$, all integers can be found by the operation h. This knowledge is not important for us because we have to get into the line segment AB itself.

The set of numbers which we can get by operation f will be created gradually. Without detriment to generality we suppose that the original numbers are $a = 0$ and $b = 1$. Let us label $\Omega_0 = \{0, 1\}$ a set of original numbers and see how it will grow if we use the operation f once, twice, three times, etc.

With one use of f number $\frac{1}{2}$ can be made. Let us label $\Omega_1 = \{0, \frac{1}{2}, 1\}$. If we use f at most twice, we will get $\Omega_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. If at most three times, we will get $\Omega_3 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\}$.

If we use f at most n times, we will get $\Omega_n = \{\frac{p}{2^n}; n \in \mathbf{N}, p = 0, 1, \ldots, 2^n\}$.

Thus we have proved that each number from the set $\Omega' = \cup\Omega_n$, which is the union of all sets Ω_n for all natural numbers n, belongs to the set Ω, too. On the other hand, the set Ω' is closed with respect to f, so if we limit ourselves to the interval $[0, 1]$, it holds that $\Omega' = \Omega$. If we work in the set of all real numbers, then $\Omega = \{\frac{p}{2^n}; n \in \mathbf{N}, p \in \mathbf{Z}\}$.

We have reached an important conclusion: Using the trileg, we can construct from points $A[0]$ and $B[1]$ only those points of the number line whose coordinates have the form of $\frac{p}{2^n}$, where $n \in \mathbf{N}, p \in \mathbf{Z}$.

3) We will show that number $\frac{1}{3}$ does not belong to Ω. Proof by contradiction: Suppose that $\frac{1}{3} \in \Omega$. Then there exists a natural number n and an integer p so that $\frac{1}{3} = \frac{p}{2^n}$. After simplification we get $2^n = 3p$ and that is a contradiction because the right hand of the equation is divisible by 3 while the left is not.

It is obvious that before the students reach the given conclusion, they will formulate other statements and hypotheses. The speed with which they will reach the conclusion depends on their mathematical ability.

3.6 Looking for relationships among identities

By solving equations in the second and third stages, students familiarized themselves with a set of identities φ. There exist more demanding and abstract problems which call for proofs of some other statements and the discovery of mutual dependence of the given statements. When solving such problems, students develop their ability to work with symbols in a structural way, without any visual anchoring. We will limit ourselves to several such problems here.

T1. Prove that from (2), it follows: $\forall\, U, V \in E^2 : U - \circ - (V - \circ - U) = (U - \circ - V) - \circ - U$.

T2. Prove that from (2) and (5), it follows: $\forall\, U, V \in E^2 : U = V - \circ - s_U(V)$.

T3. Find out one of the statements (1) to (6) from which we can prove that $\forall\, U, V \in E^2 : U = s_U(s_U(U))$.

T4. From (4) and (5) prove (1).

T5. From (1) and (8) prove (12).

T6. From (7) prove (6).

T7. From (6) prove (7).

The last two problems bring a strong result: Identities (6) and (7) are equivalent. Are there any other pairs of equivalent identities in the set φ?

T8. One of the implications (7) to (12) is equivalent to (3). Find out which one and prove the equivalence.

T9. Similarly for statement (5).

T10. Are (6) and (10) equivalent?

T11. Are (5) and (11) equivalent?

The last two tasks usually generate a discussion. Even though students prove quite easily that (6) $\Rightarrow$ (10) and (5) $\Rightarrow$ (11), they are not able to prove that (6) $\Leftarrow$ (10) and (5) $\Leftarrow$ (11).

Some think that it is only their inability, others start having doubts whether it is possible at all. After some time someone formulates a key question:

How can I prove that something cannot be proved? (*)

From the didactic point of view, T10 and T11 played a very important role in all our experiments. They represent a problem which exceeds Greek mathematics and even the mathematics of the 18th century. This problem was brought into the history of human knowledge by the problem of the fifth postulate and its solution required, as we know, an enormous effort.[3] We are convinced that similarly to phylogeny, in ontogeny the discovery of such a deep idea must be preceded by a long period of looking for a solution to a seemingly insolvable problem.

We will discuss the students' reactions to question (*) in the next section. Now, we will continue with solving other tasks toward the discovery of an axiomatic system of TMG.

T12. From (6) and (11) prove (5), (7), (8) and (10).

T13. From (3), (6) and (11) prove (2), (5), (7), (8), (9) and (10).

The last task provides us with a good insight into the structure of φ. Its solution is a series of proofs which can be depicted in a graph (fig. 2).

$$\begin{array}{ccccccccc} 8 & \leftarrow & 5 & \leftarrow & \boxed{6} & \rightarrow & 7 & \rightarrow & 10 \\ & & \uparrow & & \downarrow & & & & \\ & & \boxed{11} & \rightarrow & 2 & \leftarrow & \boxed{3} & \rightarrow & 9 \end{array}$$

Figure 2

The arrow in the graph represents a 'partial' implication which will be explained in the examples. Only one arrow leads to number 8 (from number 5). That is (5) $\Rightarrow$ (8). Two arrows lead to number 5 – from 6 and from 11. It means $(6) \wedge (11) \Rightarrow (5)$. Three arrows lead to number 2 – from 3, 6 and 11. It means $(3) \wedge (6) \wedge (11) \Rightarrow (5)$.

T14. Add one more statement to (3), (6) and (11) so that all the 8 remaining statements from φ could be proved from these four. Describe the structure in a graph.

3.7 Building an axiomatic system

We know from the previous stage that from four statements (3), (6), (11) and (12), all statements from φ can be proved. We have a feeling that there are more such quartets. We will find some of them.

T1. Prove that from (1), (3), (5) and (6), all statements from φ can be proved.

T2. Similarly for (3), (4), (5) and (6).

T3. Similarly for (3), (6), (11) and (12).

T4. Add two more statements to (1) and (11) so that all statements from φ can be proved from them. Find two different solutions.

T5. Add three more statements to (4) so that all statements from φ can be proved from them. Find six different solutions.

T6. Find at least 30 different quartets of statements from φ so that all statements from φ can be proved from them.

Each solution can be briefly described in a graph. For example, the graph in fig. 3 represents the solution to T3.

[3]Let us remember how close Giorlamo Saccheri (1667–1733) was to the discovery of non-Euclidean geometry and thus to the solution of the problem of parallels and how his conviction of the impossibility of two geometries prevented him from reaching the goal.

$$\begin{tikzcd}
& 1 & 4 \arrow[l] & \boxed{12} \arrow[l] & \\
8 & 5 \arrow[l] \arrow[u] & \boxed{6} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & 7 \arrow[r] & 10 \\
& \boxed{11} \arrow[u] \arrow[r] & 2 & \boxed{3} \arrow[l] \arrow[r] & 9
\end{tikzcd}$$

Figure 3

Students trying to solve T6 soon find out that equivalences $(3) \Leftrightarrow (9)$, $(5) \Leftrightarrow (8)$ and $(6) \Leftrightarrow (7)$ enable us to generate solutions. For example, from the solution $\{(3), (6), (11), (12)\}$ we immediately have three more solutions $\{(9), (6), (11), (12)\}$, $\{(3), (7), (11), (12)\}$ and $\{(9), (7), (11), (12)\}$. Thus we will consider only one of each pair of equivalent statements. In other words, the set of 12 statements will be reduced to 9. Let us agree that we will omit statements (7), (8) and (9) and work with TMG based on the set of statements $\Psi = \{(1), (2)(3), (4), (5), (6), (10), (11), (12)\}$.

In Appendix 2, 16 different quartets of statements from Ψ are given from which all statements from Ψ (and thus all statements from φ) can be proved.

3.8 PROOF OF NON-PROVABILITY

By solving T6, the student approaches the identification of an axiomatic system. He/she knows many quartets of statements from which all statements from φ can be deduced. Nevertheless, we cannot say that he/she has found an axiomatic system. He/she is not sure whether any of the 4 statements can be proved from the other 3. If so, it would be possible to find an axiomatic system from 3 statements. This doubt returns us to question (*).

In our experience, students take weeks before they fully understand question (*). At first, they think (similarly to phylogeny) that the implications $(10) \Rightarrow (6)$ and $(11) \Rightarrow (5)$ can be proved but we cannot do it. The first attempt to prove them uses the following hypothesis.

Hypothesis 1. I will write a proof of $(5) \Rightarrow (11)$ divided into small steps. Then I will work backwards and either find the proof of $(5) \Leftarrow (11)$, or find out why the proof cannot be found. Similarly for $(6) \Rightarrow (10)$.

1. I know that (5) holds, thus for all P and Q, it holds that $s_P(Q) - \circ - Q = P$.
2. I suppose that $s_P(R) = s_Q(R)$.
3. Because the equality remains true if I use the same operation on both sides, it holds $s_P(R) - \circ - R = s_Q(R) - \circ - R$.
4. From (5) it follows that the left side equals P and the right one equals Q, thus $P = Q$.

An attempt to reverse the sequence of steps fails and we cannot see why. Similarly for $(6) \Rightarrow (10)$. What is the reason?

A student might notice that (5) and (6) are statements with the form of equality and statements (11) and (10) are implications. Thus, a new hypothesis is formulated.

Hypothesis 2. From the implication an equality cannot be proved.

This hypothesis proves to be wrong quite quickly. Students notice that they have already proved equality (6) from the implication (7).

A new hope of solution arises if a student notices that in (5) and (6), there are both operations s and $- \circ -$, but (10) only contains $- \circ -$ and (11) only s. A new hypothesis appears.

Hypothesis 3. From the statement which contains only one of the two operations s and $-\circ-$ no statement with both operations can be proved.

The hypothesis usually stimulates a discussion. We will present a hypothetical discussion in which student A defends the hypothesis and student B refutes it. It consists of ideas which come from real discussions among students.

A. "Look. I cannot prove (5) from (11), because (11) does not say anything about the operation $-\circ-$ and (5) works with it."
B. "You are not right. We know the way these two operations are linked. That follows from the way we use the trileg."
A. "But we are not talking about the trileg but about the statements from φ only."
B. "Yes, but all statements from φ were received via the trileg. They would not exist without it."
A. "You are right that the trileg is the starting point for the statements but now we see them as abstract objects. Letters P, Q, R, ... do not have to be points. They can be numbers and in such a case, no trileg can be used."
B. "It's true, yet even for them the operation $-\circ-$ means the middle, that is the arithmetic mean, and the operation s is a little more complicated expression." (He writes $s_a(b) = 2b-a$.) "The trileg is a bit hidden, but it is still there."
A. "You are not right but I do not know how to persuade you."

The discussion ends in a draw. After some time, student A comes with another hypothesis.

Hypothesis 4. What if I took another instrument instead of the trileg for which (10) would hold and (6) would not?

It does not take long to find out such an instrument — it is a "one third trileg", whose inner leg divides the outer legs in the ratio of 1 : 2. If we put the first outer leg into point A and the second outer leg into B, the inner leg points to point X for which $|AB| = 3|XB|$ and $|AX| = 2|XB|$. In this case the symbol $A-\circ-B$ will denote point X. If we put the first outer leg to C and the inner leg to D, the second outer leg will point to Y for which $|CY| = 3|DY|$ and $2|DY| = |CD|$. The symbol $s_D(C)$ will denote Y.

It can easily be shown that with this new instrument, (10) holds but (6) does not (see fig. 4).

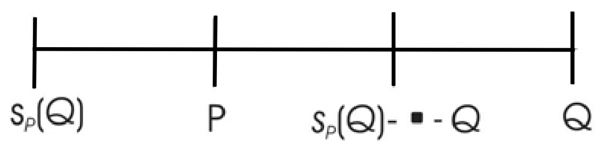

Figure 4

With this picture, student A goes to student B and tries to persuade him. Student B refuses the argument but his reasons are only emotional. For him, the set φ and the original trileg are connected very closely. Only at the end of their discussion, student B is convinced.

B. "OK, keep the new trileg. But the rules which you will deduce will be totally different from φ."
A. "And that's it. The situations are different but (10) is common to both. By the way, there are even more statements common to both. For example, (1), (4), (11), (12) and maybe even others. They will differ in others such as (2), (3) and (6)."
B.(After a longer pause.) "OK, it is true. But what does it mean? We speak about the proof of (10) $\Rightarrow$ (6) from the point of view of situation φ. When you find another context in

which there are different operations of s and $-\circ-$ and you will show that (10) $\Rightarrow$ (6) does not hold, it does not say anything about the situation of φ."
A. "This is your misunderstanding. We speak about the rules of φ in an abstract way, as rules which are related to all situations of the trileg type. If (10) $\Rightarrow$ (6) does not hold in only one of them, it does not hold generally."

Student A created another geometry with a new tool. This geometry answers the question (*). It shows that (10) $\Rightarrow$ (6) cannot be proved. If it were true, it would have to be true in all "geometries of the trileg type". We have found one geometry in which this statement does not hold, thus it cannot hold generally.

3.9 Models

From the previous section we can prove, e.g., that the set $\Gamma = \{(3), (6), (11), (12)\}$ is the basis of our geometry which is given by statements φ. To prove that Γ is the basis of φ means to show that

1. each of statements of φ can be proved from the four statements of Γ (see fig. 3) and
2. none of the four statements of Γ can be proved from the remaining three.

The second point is demanding. We have to find a model $\Gamma(3)$ of the geometry of trileg type in which all statements (6), (11), (12) are true, but (3) is not. If such a model exists, it follows that that $(6) \wedge (11) \wedge (12) \Rightarrow (3)$ cannot be proved.

Similarly, we have to find a model $\Gamma(6)$ in which (3), (11), (12) are true, but (6) is not, and also models $\Gamma(11)$ and $\Gamma(12)$.

The term 'geometry of the trileg type' is understood intuitively as something connected to the trileg. It is an instrument with two outer legs U, V and one inner leg W which keeps two rules:

1. Leg W lies between U a V.
2. The ratio of lengths $|UW| : |WV|$ is a constant positive real number p.

The situation can also be described in an arithmetic way. If all three legs are put on a number line and the coordinate of A (resp. B, resp. C, resp. D) is denoted a (resp. b, resp. c, resp. d), then it is $a - \circ - b = \frac{a+pb}{1+p}$ and $s_d(c) = \frac{-c+(1+p)d}{p}$.

The original TMG has changed into a class of geometries of trileg type. This is a one-parametric class given by the parameter p — we will denote it p-TMG.[4]

The following task will enable us to familiarize ourselves with p-TMG.

T1. Find out for which of p-TMG, statement (1), (2), ..., (12) is/is not true.

Without any help of the teacher, students could look for models for months. That is why we recommend that the teacher shows them at least some models as an inspiration and to ask them to prove whether they really are models of TMG. Some models follow.

M1. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = \frac{1}{2}(p+q) - \frac{1}{6}|p-q|$, $s_p(q) = \frac{1}{4}(9p - 5q + 3|p-q|)$.
M2. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = 1 + \frac{1}{2}(p+q) - \frac{1}{6}\sqrt{(p-q)^2+36}$, $s_p(q) = \frac{1}{4}(9p - 5q - 9 + 3\sqrt{(q+1-p)^2+8})$.
M3. Model $\Gamma(3)$. $p - \bullet - q = 2p - q$, $s_p(q) = \frac{1}{2}p + \frac{1}{2}q$.
M4. Model $\Gamma(12)$. $(\mathbf{R}, -\circ-, s)$ where $p - \circ - q = p + q$, $s_p(q) = p - q$.

[4]Here we can see a parallel with the phylogeny of non-Euclidean geometries where the role of parameter p was played by the curvature of the hyperbolic plane.

4 Conclusion

The length of the article does not allow us to go more deeply into the question of models. We suggest that the reader tries exploring TMG by solving problems him/herself first to see its potential. It is our experience that TMG is very motivating for students as the discoveries of solutions to problems (which cannot be found anywhere) are the source of joy for them. We have elaborated and used with future mathematics teachers another context, this time in algebra, in which they can discover concepts for themselves by solving problems. We called it restricted arithmetic. It is, in fact, congruence modulo 99 in disguise. It is elaborated in great detail in Stehlíková (2004).[5]

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References

– Hejný, M., 1990, *Aj geometria naučila človeka mysliet̆.* Bratislava : Slovenské pedagogické nakladateľstvo.

– Stehlíková, N., 2004, *Structuring mathematical knowledge in advance mathematics.* Praha : PedF UK.

[5] Available in pdf format from the author.

Appendices

Appendix 1

In the equations, X, Y, Z are unknowns.

$A - \bullet - X = s_{X-\bullet-A}(B - \bullet - C)$

$X - \bullet - A = B$

$Y - \bullet - E = E$

$s_X(B - \bullet - E) = B - \bullet - E$

$s_A(Y - \bullet - C) = C - \bullet - Y$

$s_Z(Z) - \bullet - B = s_C(Z - \bullet - B)$

$s_{A-\bullet-Z}(B) = s_{A-\bullet-Z}(Z - \bullet - C)$

$s_{E-\bullet-X}(X) = s_{X-\bullet-E}(E)$

$C - \bullet - s_{Z-\bullet-A}(C) = B$

$s_Z(F - \bullet - Z) = s_Z(s_E(Z) - \bullet - Z)$

$(A - \bullet - C) - \bullet - (X - \bullet - B) = X$

$s_X(C) - \bullet - X = A - \bullet - B$

$s_A(B) - \bullet - [(s_B(X) - \bullet - X) - \bullet - C] = s_A(s_{X-\bullet-A}(X))$

$(s_A(Y) - \bullet - Y) - \bullet - A = s_C(s_{B-\bullet-Y}(B)) - \bullet - Y$

$(s_A(B) - \bullet - B) - \bullet - (s_A(B) - \bullet - C) = X - \bullet - A$

$(X - \bullet - F) - \bullet - s_G(F - \bullet - X) = s_{F-\bullet-G}(X)$

$X = s_X(s_{A-\bullet-X}(X)) - \bullet - (B - \bullet - C)$

$E = s_F(s_{E-\bullet-X}(E))$

$X = s_A(s_A(X) - \bullet - s_A(s_X(B) - \bullet - B))$

$s_{A-\bullet-B}(X) = s_A(X) - \bullet - s_B(C)$

$C = (A - \bullet - X) - \bullet - (s_X(A) - \bullet - s_B(X))$

$X - \bullet - s_X(A) = s_X(B - \bullet - X) - \bullet - X$

Appendix 2

$$\begin{tikzcd}
\boxed{1} \arrow[d] & 10 & \boxed{3} \arrow[d] \\
4 \arrow[d] & \boxed{6} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & 2 \\
12 & 5 \arrow[l] & \boxed{11} \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
10 \arrow[r] & 3 & \boxed{2} \arrow[l] \\
\boxed{6} \arrow[u] \arrow[r] \arrow[d] & 5 \arrow[u] \arrow[dr] & \boxed{11} \arrow[l] \\
4 & \boxed{1} \arrow[l] \arrow[r] & 12
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{2} \arrow[d] & 11 & 12 \\
3 & \boxed{5} \arrow[l] \arrow[u] \arrow[ur] \arrow[d] & 4 \arrow[u] \\
\boxed{10} \arrow[u] \arrow[r] & 6 \arrow[ur] & \boxed{1} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{3} \arrow[r] & 2 & 11 \arrow[l] \\
\boxed{10} \arrow[r] & 6 \arrow[u] & \boxed{5} \arrow[l] \arrow[u] \\
4 \arrow[ur] & \boxed{1} \arrow[l] & 12 \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
11 & \boxed{6} \arrow[r] & 10 \arrow[d] \\
1 & \boxed{5} \arrow[l] \arrow[ul] \arrow[r] \arrow[d] & 3 \\
\boxed{4} \arrow[u] & 12 \arrow[l] & \boxed{2} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{3} \arrow[r] & 2 & \boxed{6} \arrow[l] \arrow[d] \\
1 & \boxed{5} \arrow[l] \arrow[u] \arrow[d] \arrow[dr] & 10 \\
\boxed{4} \arrow[u] \arrow[r] & 12 & 11
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{2} \arrow[d] & \boxed{11} \arrow[d] & \boxed{4} \arrow[d] \\
3 & 5 \arrow[l] \arrow[r] \arrow[dr] & 1 \\
10 \arrow[u] & \boxed{6} \arrow[l] \arrow[u] & 12
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{10} \arrow[r] \arrow[d] & 3 & \boxed{2} \arrow[l] \\
6 & \boxed{5} \arrow[l] \arrow[u] \arrow[r] \arrow[d] \arrow[dr] & 11 \\
\boxed{4} \arrow[r] & 1 \arrow[r] & 12
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{10} \arrow[d] & 1 & \boxed{4} \arrow[l] \arrow[d] \\
6 \arrow[dr] & \boxed{5} \arrow[u] \arrow[r] \arrow[dr] & 12 \\
\boxed{3} \arrow[r] & 2 & 11 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
10 \arrow[r] & \boxed{6} \arrow[d] \arrow[dr] & 3 \arrow[d] \\
1 & 5 \arrow[l] \arrow[d] \arrow[dr] & 2 \\
\boxed{4} \arrow[u] \arrow[r] & 12 & \boxed{11} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
11 & \boxed{5} \arrow[l] \arrow[d] & \boxed{6} \arrow[ul] \arrow[u] \arrow[d] \\
\boxed{2} \arrow[r] & 3 & 10 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
11 & \boxed{5} \arrow[l] \arrow[r] \arrow[d] & 6 \arrow[ul] \arrow[u] \\
\boxed{2} \arrow[r] & 3 & \boxed{10} \arrow[l] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
\boxed{10} \arrow[r] & 6 \arrow[u] \arrow[ur] \arrow[d] & \boxed{5} \arrow[l] \arrow[d] \\
\boxed{3} \arrow[r] & 2 & 11 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
10 & \boxed{5} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & 6 \arrow[ul] \\
\boxed{3} \arrow[r] & 2 & \boxed{11} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
11 \arrow[r] & 2 & \boxed{6} \arrow[l] \arrow[ul] \arrow[u] \arrow[d] \\
\boxed{5} \arrow[u] & \boxed{3} \arrow[u] & 10
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{12} \arrow[r] & 1 \arrow[r] & 4 \\
\boxed{11} \arrow[r] & 5 \arrow[d] & \boxed{6} \arrow[l] \arrow[ul] \arrow[u] \arrow[d] \\
\boxed{2} \arrow[r] & 3 & 10 \arrow[l]
\end{tikzcd}$$