REGULAR AND SEMI-REGULAR POLYTOPES

A didactic approach using Boole Stott's methods

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Abstract

Regular and semi-regular polytopes in four dimensions are the generalization of the Platonic Solids and the Archimedean solids. For a better understanding of these four-dimensional objects, we present the method of the amateur mathematician Alicia Boole Stott, who worked on the topic at the end of the 19th century. The methods she introduced in her two main publications are presented in the workshop, together with exercises that help the visualization of these four-dimensional polytopes.

1 Introduction

In the present workshop we intend to make the participant familiar with the notions of regular and semi-regular polytopes in four dimensions using the methodology provided by the amateur mathematician Alicia Boole Stott. The first part of the workshop is devoted to introducing the Platonic Solids (or regular polyhedra) and their analogues in four dimensions: the regular polytopes. We also provide a short biography of Boole Stott. The remaining of the course is organized as follows. First, we discuss the 1900 publication of Boole Stott, where the three-dimensional sections of the four dimensional polytopes are treated. For a better understanding of her method, we first look at the three-dimensional case, and generalize the results to the fourth dimension. Finally, we treat Boole Stott's results in deriving semi-regular polytopes from regular ones. As before, examples in the third dimension will be first given as a preceding step to the four-dimensional case.

2 Platonic and Archimedean solids

The so-called Platonic Solids or regular polyhedra are subsets of the three-dimensional space that are bounded by isomorphic regular polygons and having the same number of edges meeting at every vertex. There are five of them, namely the tetrahedron, cube, octahedron, dodecahedron and icosahedron.

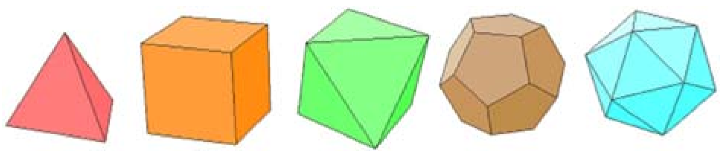

Figure 1 – Platonic Solids

If different types of polygons are allowed as faces, one obtains the semi-regular polyhedra. These are subsets of the three-dimensional space bounded by regular polygons of two or more different types, ordered in the same way around each vertex. This group can be divided into the so-called prisms (constructed from two congruent n-sided polygons and n parallelograms), the antiprisms (constructed from two n-sided polygons and $2n$ triangles) and the Archimedean solids (the remaining ones). There are 13 Archimedean solids, shown in the figure below.

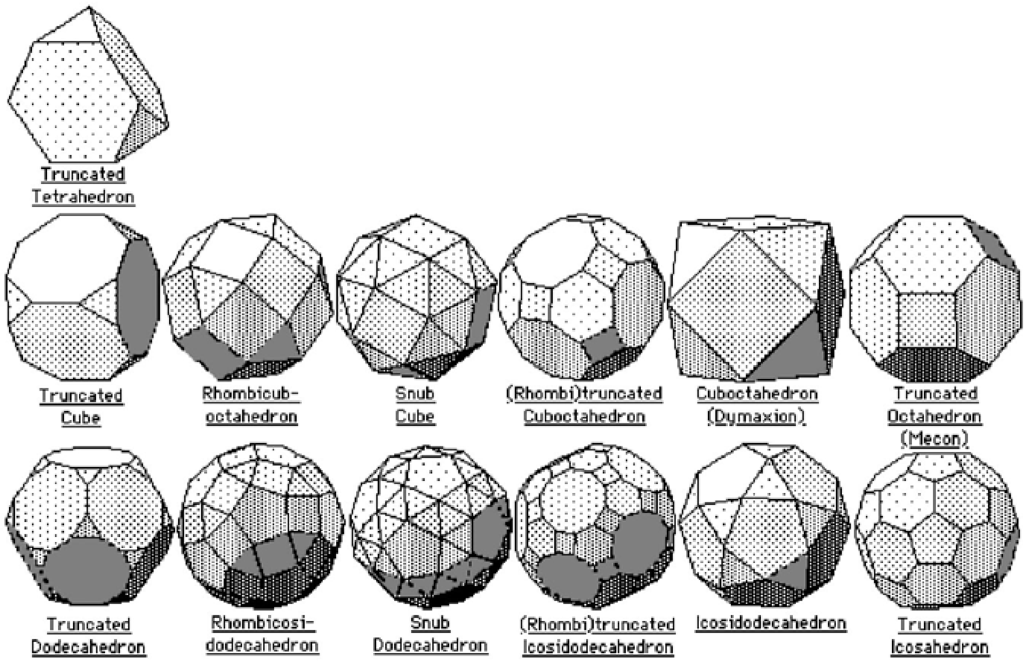

Figure 2 – Archimedean Solids

3 Regular four-dimensional polytopes

The four-dimensional objects analogous to polyhedra are called polytopes. As polydedra are built of two-dimensional polygons, so polytopes are built of three-dimensional polyhedra. The regular polytopes, which are the equivalent of the Platonic solids in the fourth dimension, can be defined as subsets of the four-dimensional space with faces isomorphic to the Platonic solids and with the same number of faces at each vertex. There exist six regular polytopes in four dimensions, namely the hypertetrahedron, the hypercube, the hyperoctahedron, the 24-cell, the 120-cell and the 600-cell. Their number of vertices (v), edges (e), faces (f) and cells (c) and the type of cells are given in the following table 1.

Table 1 – Six regular polytopes

Polytope	***v***	***e***	***f***	***c***	**cell**
Hypertetrahedron or 5-cell	5	10	10	5	tetrahedron
Hypercube or 8-cell	16	32	24	8	cube
Hyperoctahedron or 16-cell	8	24	32	16	tetrahedron
24-cell	24	96	96	24	octahedron
120-cell	600	1 200	720	120	dodecahedron
600-cell	120	720	1 200	600	tetrahedron

These regular polytopes were first discovered by Schläfli between 1850 and 1852 (only published in 1901), and independently rediscovered by several mathematicians like Stringham (1880), Hoppe (1882), Schlegel (1883), Puchta (1884), Cesàro (1887), Curjel (1899), Gosset (1900) and Boole Stott (1900).

Boole Stott found the six regular polytopes using a very intuitive method. In order to give an insight of her proofs, we present a series of exercises that indicate how to use her reasoning in order to find which Platonic Solids can occur.

Exercise: Suppose that P is a Platonic Solid made of n-gons and let a be its inner angle. Note that $a = 180(n-2)/n$. How many n-gons can meet at each vertex? Note the following: suppose there are m, n-gons at a vertex. Then $m > 2$ and $a + \ldots + a = m \cdot a < 360°$. For example, suppose P is made of triangles. Then $a = 180(3-2)/3 = 60°$. How many triangles can meet at a vertex? The same reasoning for squares, pentagons, etc.

Note that this exercise shows that there exist at most five Platonic Solids, but does not prove their existence (the construction of the solids would be needed).

Exercise: Once the number of faces (equivalently edges) in each vertex is known, we can find v, e, and f (here v, e, and f denote the number of vertices, edges and faces of the polyhedron) as follows. Let P be a polyhedron bounded by n-gons. Let s be the number of faces meeting at a vertex (note that this number is the same as the number of edges at a vertex). Write f in terms of s, v, and n and e in terms of f and n. Use this two formulas and Euler's formula $f - e + v = 2$ to find v, e and f.

We proceed to generalize this reasoning to see what polytopes can occur in four dimensions. The idea of Boole Stott's proof is as follows: Let P be a regular polytope made of cubes. Let V be one of the vertices of P. Intersect P with a three-dimensional space H passing near the vertex V such that H intersects all the edges coming from V. In particular, each cube meeting in V is intersected by the three-dimensional space in a triangle. Therefore, the section $H \cap P$ is a Platonic Solid bounded by equilateral triangles. The Platonic Solids bounded by triangles are: the tetrahedron (bounded by 4 triangles), the octahedron (bounded by 8 triangles), and the icosahedron (bounded by 20 triangles). We conclude the following: the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Eight cubes fill up the three-dimensional space, hence eight are too many. So are twenty cubes. We conclude that there exists only one regular polytope made of cubes, namely the hypercube, which has 4 cubes at each vertex.

Analogously, the remaining polytopes can be obtained. Just like in the three-dimensional case, the argument explains why there are at most six regular polytopes, but the existence of them is yet to be established.

4 A short biography of Boole Stott

Alicia Boole Stott (1860–1940) was born in Castle Road, near Cork (Ireland). She was the third daughter of the famous logician George Boole (1815–1864) and Mary Everest (1832–1916). Boole Stott made a significant contribution to the study of four-dimensional geometry. Although she never studied mathematics, she taught herself to "see" the fourth dimension and developed a new method of visualising four-dimensional polytopes. In particular, she constructed three-dimensional sections of these four-dimensional objects which resulted in a series of Archimedean solids. The presence in the University of Groningen of an extensive collection of these three-dimensional models (see Figure 3), together with related drawings, reveals a collaboration between Boole Stott and the Groningen professor of geometry, P. H. Schoute.

This collaboration lasted for more than 20 years and combined Schoute's analytical methods with Boole Stott's unusual ability to visualize the fourth dimension. After Schoute's

Figure 3 – Models of sections of polytopes, by Boole Stott (courtesy of the University Museum of Groningen, The Netherlands)

death in 1913 Boole Stott was isolated from the mathematical community until about 1930 when she was introduced to the geometer H. S. M. Coxeter with whom she collaborated until her death in 1940.

5 Two-dimensional sections of the Platonic Solids

In Boole Stott's 1900 publication, the three-dimensional sections of the six regular polytopes are computed. These sections are the result of intersecting the four-dimensional object with particular three-dimensional spaces. We will first discuss her methodology in the three-dimensional case. With this purpose, some exercises to calculate the two-dimensional sections of some Platonic Solids are provided. Boole Stott's method consisted mainly on unfolding the object into a dimension lower, and work on the sections in the new picture.

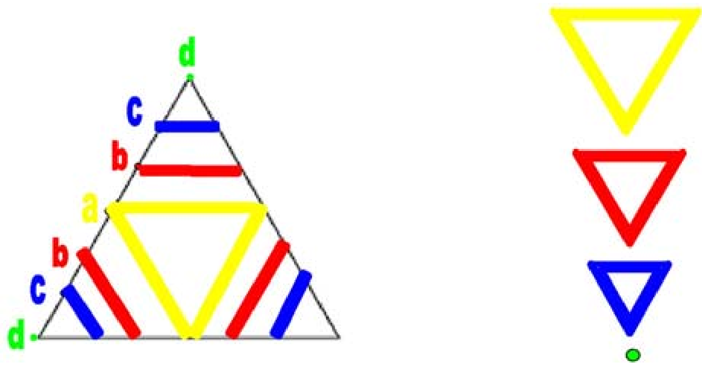

Figure 4 – Unfolded tetrahedron and its parallel sections

Let us begin by calculating the sections of the tetrahedron that are parallel to a face. Consider a plane passing through one of its faces. Clearly, the intersection of the plane and the tetrahedron will be a triangle of the size of the face. In the unfolded tetrahedron (see Figure 4), the section is the triangle with vertex a. For the next section, the plane is moved parallel to this triangle until it passes through the point b. In the unfolded figure, the edges of the triangle are moved parallel at the same distance until passing through b, forming again a triangle of smaller size.

It is then clear that all sections are triangles decreasing in size (ending with the vertex d). One can see that the sections can be computed in the unfolded figure without actually visualizing the three-dimensional object.

In the same manner, the sections of other Platonic solids can be calculated. As an exercise, the sections of the octahedron and the cube were calculated during the course. Their unfoldings (or *nets*) are provided in the following figure.

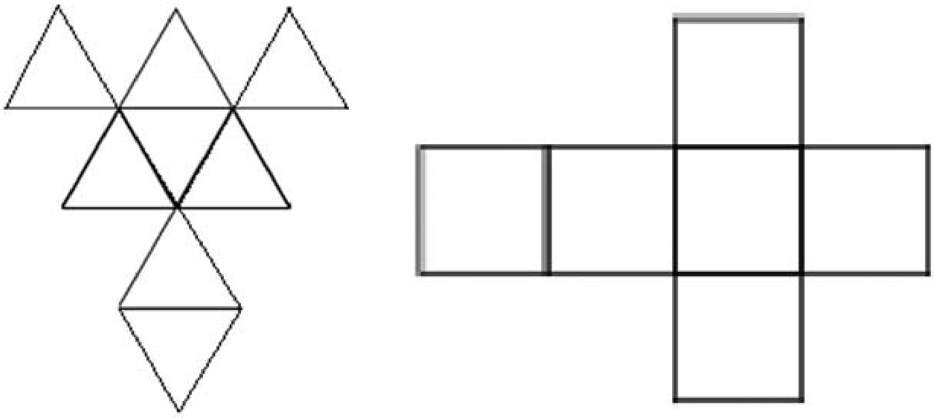

Figure 5 – Unfolded octahedron and unfolded cube

6 Boole Stott's method to calculate sections: four-dimensional case

The same methodology can be used with four-dimensional solids. Boole-Stott's method of computing three-dimensional parallel sections uses the unfolding of the four-dimensional body in a three-dimensional space, as it was done for one dimension lower. Let us now compute the three-dimensional sections of the hypercube parallel to a cell.

Figure 6 shows part of an unfolded cube (original drawing by Boole Stott). We note that some of the two-dimensional faces (i.e., squares) must be identified in order to recover the original hypercube (this identification, of course, can only be understood in four dimensions). The first three-dimensional section is the result of intersecting the polytope P with a three-dimensional space H_1 containing the cube $ABCDEFGH$. To obtain the second section, the space H_1 is moved towards the center of the polytope, until it passes through the point a. Call this new three-dimensional space H_2. The second section is $H_2 \cap P$. Note that the faces of the new section must be parallel to the faces of the cube $ABCDEFGH$. In particular, the section $H_2 \cap P$ contains the squares $abcd$, $abfg$ and $adef$. After the necessary identification of the points, edges and faces that occur more than once in the unfolded polytope, and using the symmetry of the polytope, one can conclude that the section $H_2 \cap P$ is again a cube isomorphic to the original cube-cell $ABCDEFGH$. Analogously, the third section will again be a cube.

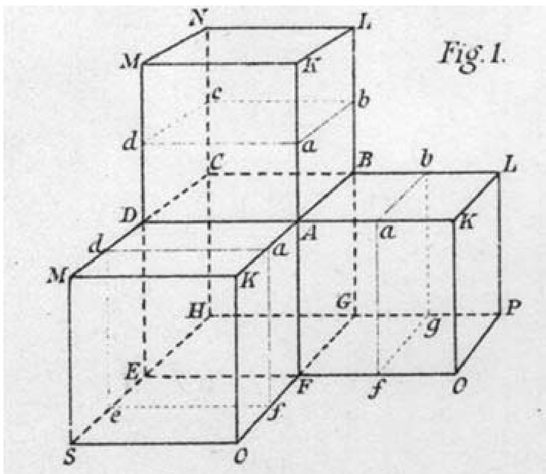

Figure 6 – Part of an unfolded cube (Boole Stott, 1900)

This simple example gives the idea of Boole Stott's method. Following the same reasoning, one can also compute the sections of the 16-cell and 24-cell. This is proposed in the following exercises, where the original drawings by Boole Stott of the unfoldings are displayed. We omit the remaining cases, which are more difficult. For a complete study of these sections and drawings of the results one may look at (Boole Stott, 1900).

Exercise: Calculate the three-dimensional sections of the 24-cell (using the unfolded polytope in Figure 7) as follows. Let P be the 24-cell. Let H_1 be a three-dimensional space passing through the octahedron $ABCDEF$. Find

- 1^{st} section: $H_1 \cap P$
- 2^{nd} section: $H_2 \cap P$ where H_2 is parallel to H_1 and passing though the point a
- 3^{rd} section: $H_3 \cap P$ where H_3 is parallel to H_1 and passing though the point AC
- 4^{th} section: $H_4 \cap P$ where H_4 is parallel to H_1 and passing though the point a_1
- 5^{th} section: $H_5 \cap P$ where H_5 is parallel to H_1 and passing though the point A

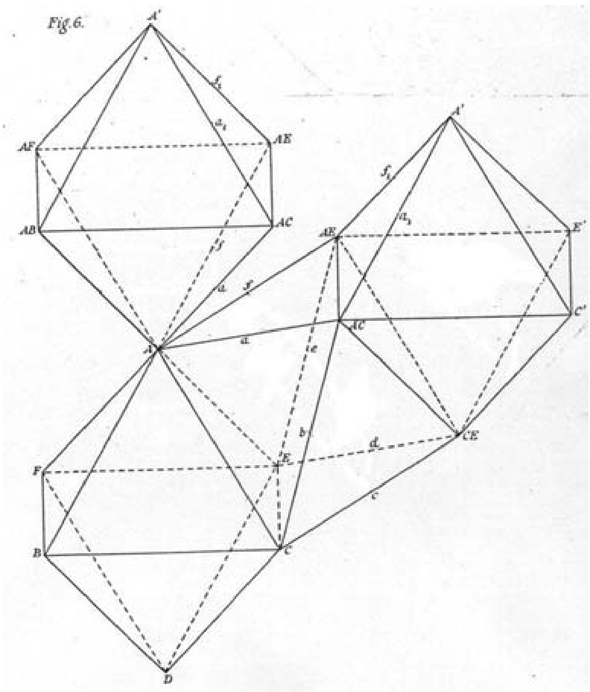

Figure 7 – Part of an unfolded 24-cell (Boole Stott, 1900)

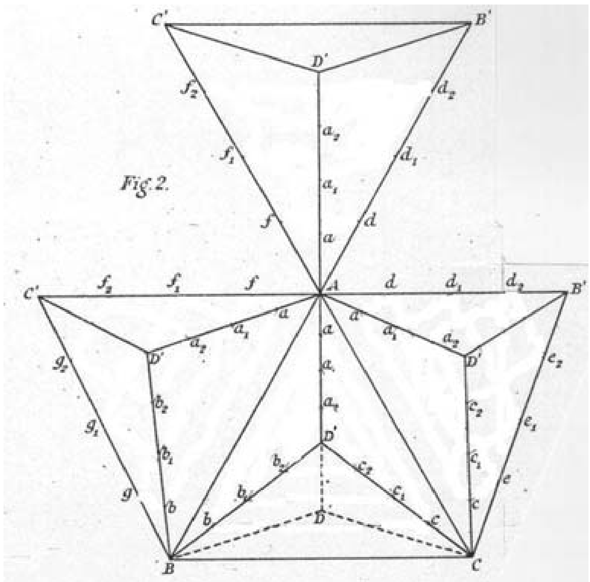

Figure 8 – Part of an unfolded 16-cell (Boole Stott, 1900)

Exercise: Calculate the three-dimensional sections of the 16-cell (see unfolding in Figure 8). Let P be the 16-cell. Let H_1 be a three-dimensional space passing through the tetrahedron $ABCD$. Find

- 1^{st} section: $H_1 \cap P$
- 2^{nd} section: $H_2 \cap P$, H_2 parallel to H_1 and passing though a
- 3^{rd} section: $H_3 \cap P$, H_3 parallel to H_1 and passing though a_1
- 4^{th} section: $H_4 \cap P$, H_4 parallel to H_1 and passing though a_2
- 5^{th} section: $H_5 \cap P$, H_5 parallel to H_1 and passing though D'

7 Deriving semi-regular polyhedra and polytopes from regular ones

As mentioned before, the Archimedean solids are the semi-regular polyhedra that are not a prism (two n-gons and n parallelograms) or an antiprism (two n-gons and 2n triangles). Equivalently, the semi-regular polytopes can be defined. In her 1910 publication, Boole Stott found a method to obtain the semi-regular solids in three and four dimensions. In order to do that, she applied two operations, defined by her as follows:

Definition: The operation expansion with respect to the vertices of a polytope consists of considering the set of its vertices (equivalently edges, faces, cells, ...), and move each element of the set at the same distance away from the center of the polyhedron such that the new (extended) set of vertices (eq. edges, faces. etc) define a semi-regular polytope.

Definition: The operation contraction consists of taking the set of elements considered in the expansion (i.e., vertices, edges or faces) and moving them uniformly towards the center until they meet.

In the two-dimensional space, one can expand an n-gon with respect to its edges. This results in a $2n$-gon, as shown in Figure 9.

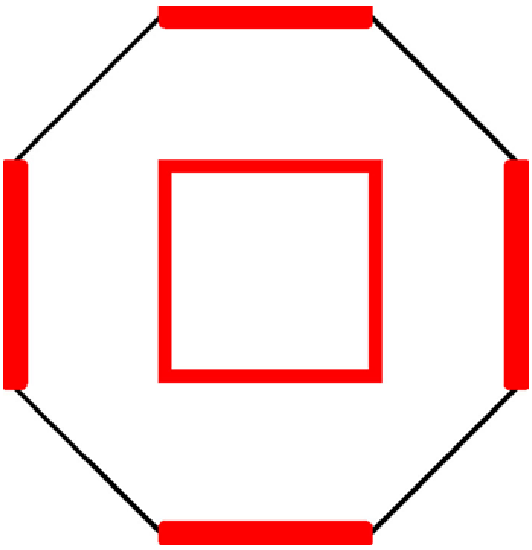

Figure 9 – Expansion (edges) of a regular n-gon gives a $2n$-gon

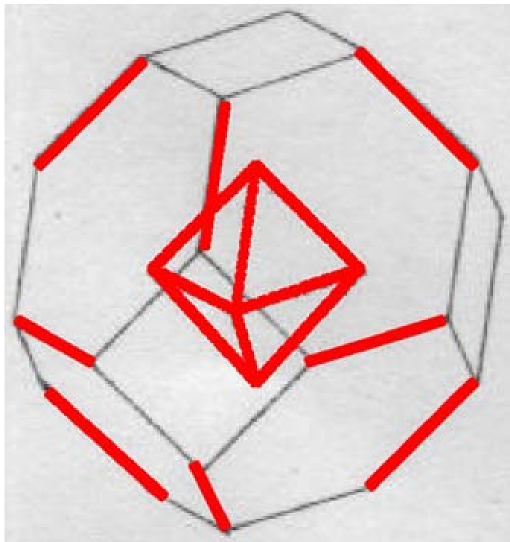

Figure 10 – Expansion (edges) of an octahedron is a truncated octahedron

In the three-dimensional space, a Platonic Solid may be expanded with respect to its edges. The result is the same solid truncated (i.e., all the corners are cut off).

If one applies the operation expansion with respect to the faces to a Platonic Solid, the result is a semi-regular polyhedron where the original faces of the Platonic Solid remain the same, all edges are replaced by squares and all vertices are replaced by n-gons (here n is the number of edges at each vertex). We suggest the following exercise.

Exercise: Calculate the expansion (faces) of the cube and the expansion (faces) of the octahedron. Look at the list of Archimedean polyhedra to identify the new solids. Can you draw any conclusion?

For more information on polyhedra and four-dimensional polytopes we refer to (Cromwell, 1997) and (Coxeter, 1961, Chapter 22) respectively.

8 Conclusions

Four-dimensional polytopes are usually very difficult to visualize. For a better understanding of these objects we propose to follow the methodology used by Boole Stott on the topic. First, exercises for the three-dimensional case have been provided in order to help the participant to get familiar with Boole Stott's method. After that, the method is generalized to the four-dimensional case. New operations are defined and performed on the polytopes to obtain Boole Stott's results.

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