Viète and the Advent of Literal Calculus

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Abstract

Survey of Viete's contribution.

The resolution of equations: a change of point of view. Study of the equation $x^2 + ax = b$ *through texts by Al Khwarizmi and Viète. The role of geometry and identities with a use in class.*

The resolution of problems of construction by algebra. Study inscribing a square in a triangle through texts by Al Khwarizmi and Bézout. Putting in equations and literal treatment of geometry with uses in class.

The importance of literal calculus.

1 Viète's contribution

Literal calculus dates from 1591, when François Viète (1540–1603), a jurist born in Fontenay-le-Comte, in Poitou, counsellor to the King and the Court of France, publishes, in Tours, a booklet of 14 pages which will revolutionise the practice of mathematics: *In Artem Analyticem Isagoge (Introduction to the analytic Art).* The proof? Three years later, 1594 October 10^{th}, at Fontainebleau, Viète solves, within three hours, the challenge that Adrien Romain made to all mathematicians of the world: *"Ut legi, ut solvi" (As soon as I read it, as soon as I solved it*). And he added: *"I who do not profess to be a mathematician, but who, whenever there is leisure, delight in mathematical studies."* The problem is to solve an equation of degree forty five. Amazing! Viète gives the 23 positive solutions with 9 digits decimal values and their geometric construction (see annex 5.1). How could Viète, an unknown mathematician, beat all the mathematicians of his time?

1.1 A new algebra

In the context of the Renaissance, Viète rediscovers the works of the great jurists, poets, writers and mathematicians of the Antiquity. Those of the Greek scientists, sometimes uncompleted, deliver a lot of results, but also unsolved problems, lost solutions, and no indications about the method, the analysis, which allowed finding these results. Then, he rediscovers the solution of an Apollonius' problem: how is it possible to draw a circle tangent to three given circles? (See annex 5.1) He will publish his solution in 1595. He will work also about the trisection of the angle, the construction of the regular heptagon, the duplication of the cube, the squaring of the circle. At that time, there are lots of treatises of Algebra, and the necessity of notations appears clearly: they abound, but the method to solve the problems and the equations is always given with numerical examples. So Viète's researches bring him to create a new algebra: *"All the mathematicians knew that under their Algebra or* Almulcabala *that they praised so much and that they called the* Great Art, *were hidden incomparable masses of gold, but they could not find them. So they made great sacrifices to Apollo and the Muses when they reached the solution of a single of these problems that*

I can spontaneously solve in their dozens, which proves that our art is the most certain method of invention in mathematics." (Dedication to Catherine of Parthenay). He names this new algebra the *Analytic Art* which he defines as *"the science of finding correctly in mathematics"* and to which he assigns the aim of solving any problem: *"Analytic Art rightly claims for itself the magnificent problem of problems which is: How to solve any problem."* To perform this, he creates a calculus entirely with symbols instead of numbers, which he names *"Specious logistics"*: *"But how we must approach this research requires that we resort to a special art which does not work which numbers, as the ancient analysts wrongly thought, but with a new logistics... Specious logistics is that which is exposed by signs and symbols for example letters of the alphabet."* This calculus uses only letters: the letter A or any other vowel E, I, O, U, Y for: the magnitudes which are to be found, the letters B, C, D or other consonants for the magnitudes that are given. It is what you are calling now the literal calculus. This calculus obeys the law of homogeneous quantities, that is to say the dimension of magnitudes; for dimension 2: A square, B plane, for dimension 3: D cube F solid... Then we write with letters the relations between magnitudes and we obtain either equations, or formulas.

1.2 The possibility of solving problems in the general case using algebra

To illustrate his new algebra, Viète publishes contemporaneously with *Isagoge*, five books of researches: *The Zetetics*. Most of them are problems from *The Arithmetica* of Diophantus. Thus, he wants to make appreciate by the reader the important change brought by his new calculus.

Let study first, the first problem of the first book: *"Given the difference between two sides and their sum, find the sides."* (Text of *Zetetics* I 1: see annex 5.2). Diophantus treats the problem with an example, as Viète's contemporaries do. He takes 100 for the sum, and 40 for the difference. Choosing for unknown the minus of the two required numbers, he finds 30 and 70. What does Viète? He uses the same way for the resolution, but he notes B the difference, D the sum, A the smallest of the required numbers, and E the greatest. He finds A equal to $\frac{1}{2}D - \frac{1}{2}B$ and E equal to $\frac{1}{2}D + \frac{1}{2}B$. After this he puts the result in words as a general rule and ends by a numerical application. With which numbers? Guess it. Diophantus'ones!

For all the problems Viète takes the same outlines: general solution using literal calculus (his specious logistic), statement of a rule or theorem, numerical application using the numerical algebra of his contemporaries (numerical logistic as he says) with cossic symbols. Thus you preserve the data: you find them in the formulas giving the unknown quantities as a function of known quantities. The problem is solved in the general case. For the particular cases, you just have to do a numerical application. It's a proceeding which became standard, and current in Physics. Without literal calculus, Diophantus or anybody else would have to solve the problem again for other numerical data.

For the first problem of his *Zetetics*, Viète follows Diophantus for the proceeding. But for the other problem Viète shows us that his new algebra allows, for the first time, to prove formulas and theorems using calculus, to create new ones, and to use them. Thus Viète is able to create new methods to solve problems. Have a look at the fourth problem of the second book of *Zetetics*: *"Given the product of two numbers and their sum, find the numbers."* (Text of *Zetetics* II 3: see annex 5.2). It's a classical problem: you can find it in Diophantus, but also in Babylonian mathematics. To solve this problem, Viète does not follow at all the Diophantus' proceeding: he uses a formula linking xy, $x + y$ and $x - y$ to reduce the problem to the first of *Zetetics* I. Look at his method with our notations. Translation with letters: find x and y knowing that $x + y = S$ and $xy = P$. Viète uses the formula $(x + y)^2 - 4xy = (x - y)^2$ established in his work *Notae priores*. Then you can find

$x-y$ as a function of S and P. Knowing $x+y$ and $x-y$, you can find x and y by mean of the first problem (*Zetetics* I 1). Viète ends with a numerical application: $S = 12$, $P = 20$, then $(x-y)^2 = 64$, and he lets you finish. The use of remarkable identities, or other identities obtained from them, permits to solve a lot of systems of the first degree in two unknowns. This method also permits to solve the equation of the second degree in a different way from the usual way. Here is how: you write the equation under the form $x^2 + ax = b$, and in the next place as a constant product $x(x+a) = b$. Let $y = x + a$, you then have to find x and y knowing that $xy = b$ and $y - x = a$: it is the problem 3 of *Zetetics* II (see annex 5.2) solved in the same manner as *Zetetics* II 4.

2 Solution of equations: a change of point of view

To appreciate the change due to Viète, we compare the solution of an equation $x^2 + ax = b$ in Al Khwarizmi's work and in Viète's work (texts: see annex 5.3). We shall use present notations to compare the methods, but it is important to be confronted with the original texts. In the present case, algebra is often linked with the use of symbols. However the Al Khwarizmi's text shows that you can practice algebra without any symbol. And even in Viète's text, the language remains to designate equality, powers, dimensions of constants (law of homogeneous quantities), multiplication (in), double (bis)... but without symbols (letters) for known and unknown quantities literal calculus cannot exist: here is Viète's invention.

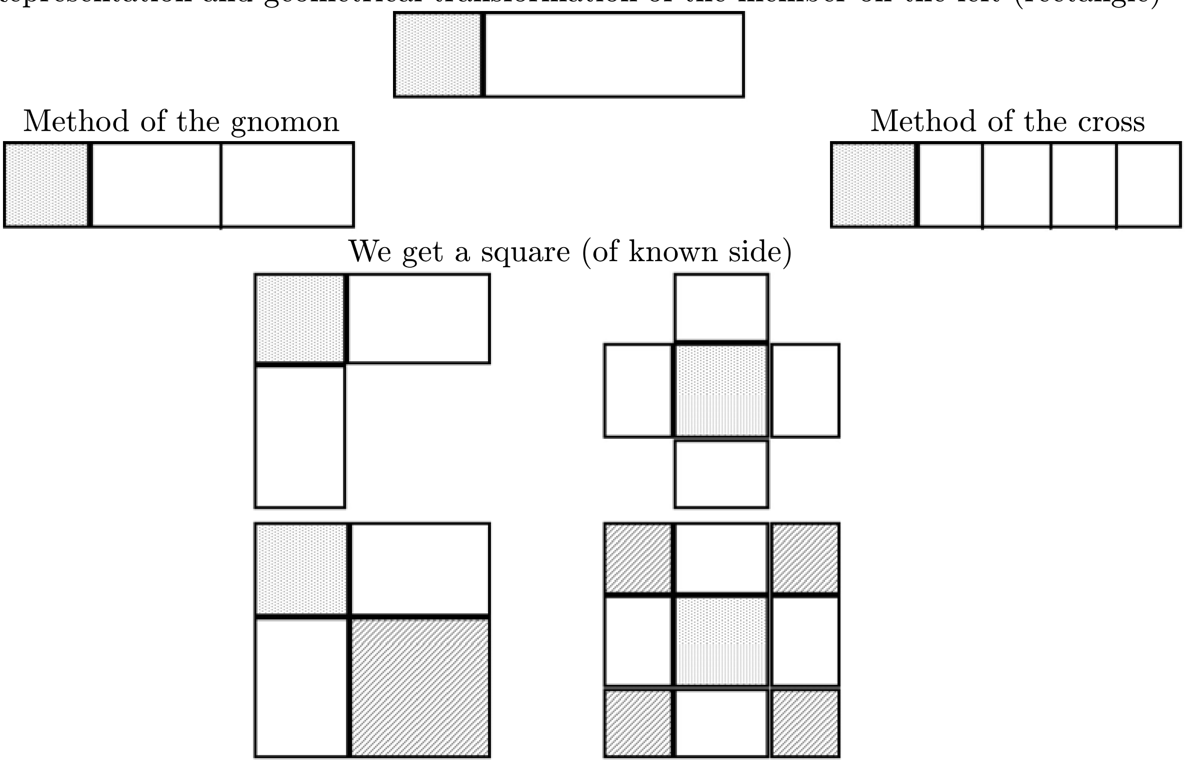
2.1 Before Viète

We transform a geometric figure, a rectangle into a square, *by means of the gnomon or the cross.*

It's the geometric figure of the algebraic expression, the theorems on the transformation of geometric figures of same area, and the aimed geometric figure which are the guides of the algorithmic proceeding and its validation.

Method of resolution of the equation $x^2 + ax = d$ before Viète for $x^2 + 2x = 20$

Representation and geometrical transformation of the member on the left (rectangle)



$(x+1)^2 = 20 + 1 \qquad (x+1)^2 = 20 + 4 \times 0{,}25$

$(x+1)^2 = 21$ then $x + 1 = \sqrt{21}$ and $x = \sqrt{21} - 1$

2.2 With Viète

We transform an algebraic form, an algebraic sum, into a square, *by means of identities and changes of variable.*

It's the form of the algebraic expression, the catalogue of forms (*identities*), and the aimed form (*canonical equation*) which are the guide of the algebraic proceeding.

Method of resolution of the equation $x^2 + ax = d$ with Viète

We make a change of variable $a = 2b$ $x^2 + 2bx = d$ (affected form)

We use an identity $x^2 + 2bx + b^2 = (x + b)^2$ $x^2 + 2bx + b^2 = d + b^2$

And a change of variable $x+b = X$ $X^2 = d+b^2$ (pure form) $X = \sqrt{d + b^2}$ $x = \sqrt{d + b^2} - b$

Numerical application: $b = 1$, $d = 20$, *then* $x = \sqrt{21} - 1$

We can notice that Viète, and Al Khwarizmi also, "omit" a solution, the negative one. But it's not an omission. For Al Khwarizmi such a solution cannot appear because the algorithms are based on geometric transformations. And for Viète only positive quantities exist. Nevertheless by the use of literal calculus, little by little, the mathematicians will accept the existence of negative and imaginary quantities.

2.3 Utilization in class

We have seen, in the solution of the equation $x^2 + ax = d$ and in the solving of problems 3 and 4 of *Zetetics* II, the central place of literal formulas in Viète's algebraic method.

In *Zetetics* II, Viète reduces the solution of any problem of 2$^{\text{nd}}$ degree in 2 unknowns to the solution of a system of the first degree in 2 unknowns by using formulas. The elements of these formulas (identities) are x^2, y^2, $x + y$, $x - y$, xy, $x^2 + y^2$, $x^2 - y^2$. I think that the use in class of these problems and of the Viète's method is a good work for pupils for using remarkable identities because frequently the required work on this subject is only technical without problem solving. Examples are given in annex 5.4 (See also [U1]).

3 Solution of geometric problems using algebra

3.1 The section of the angles

By creating his new algebra, Viète intended to solve the famous problems of the Antiquity. And at the end of the *Introduction to the Analytic Art* he emphasises the fact that his new algebra allowed him to penetrate the mystery of angular sections: *"The analyst solves the most famous problems called irrational such as that of the section of an angle into three equal parts, the invention of the side of the heptagon and all others which fall into formulas of equations... the mystery of angular sections that nobody has known up to this day."* In fact, his new algebra allowed him to establish literal formulas of trigonometry and to reduce the division of an angle into n equal parts to an equation of degree n (see annex 5.5). Then for him, to solve Adrien Romain's challenge became easy (see annex 5.6).

3.2 Inscription of a square in a triangle

Literal calculus allows solving geometric problem of construction in the general case: this Viète's aim will be taken again by Descartes in his *Geometry* with an extension to the locus problems. To measure the change, we propose again the same problem treated by Al Khwarizmi and by Bézout : two texts utilized in class with pupils (see annex 5.7).

With Al Khwarizmi's text pupils can discover the solution of a geometric problem by the means of algebra : an algebra with numerical coefficients — *the algebra* before Viète which he called *numerical logistic* —, an isosceles triangle (particular case) and numerical data. But it is an interesting problem, and I utilize it with my pupils 13–14 years old (see annex 5.7.1). For uses in other classes, see [U2].

On the other hand, the literal calculus allows the treatment of this problem in the general case, as shown by Bézout's text. Two centuries after Viète's invention, we see his method working with our notations:

1. Algebraic translation: writing a literal equation (different from analytic geometry).
2. Algebraic solving of the equation: by means of literal calculus, the unknown is written as a formula function of the data.
3. This formula allows to construct geometrically the solution and to do numerical applications.

This problem is a good situation to work with literal calculus in class (see [U2]).

4 Capital importance of the invention of literal calculus

The analysis is issued from "specious" algebra: notion of equations of curve, of variable, of function... The explicitation of properties of operations is issued from literal calculus. The modelisation with the algebraic language allowed the rapid progress of mathematics and of the other sciences.

But now, in France, in secondary school, there is a quasi exclusive use of *"numerical algebra"*, and the learning of algebraic calculus is done formally in the field of numbers. So it is urgent, on our point of view, to rehabilitate *the "specious algebra"* as a *tool for problem solving in the field of quantities* in order to:

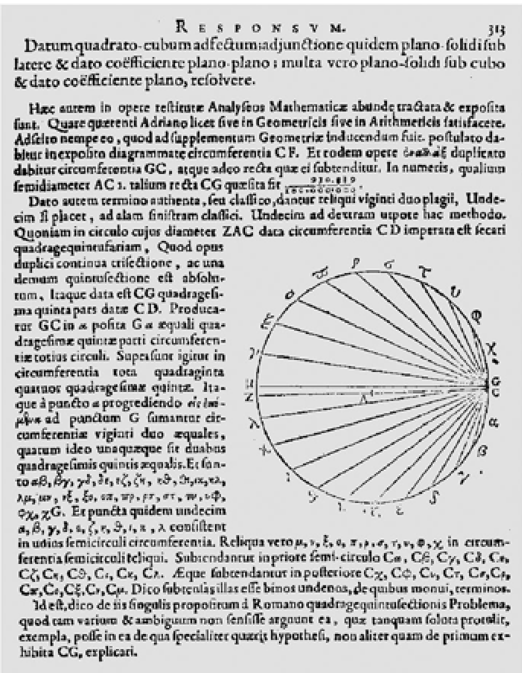
- solve geometric problems: construction, proof, locus of points...
- establish general formulas: perimeters, areas, volumes, number properties...

The utilizations in class that we have presented, borne on historical texts, show examples of this rehabilitation.

5 Annex

This annex contains historical documents illustrating the paper. Three of them point out a utilization with pupils.

5.1 Viète's solution of Adrien Romain's challenge and Apollonius' problem



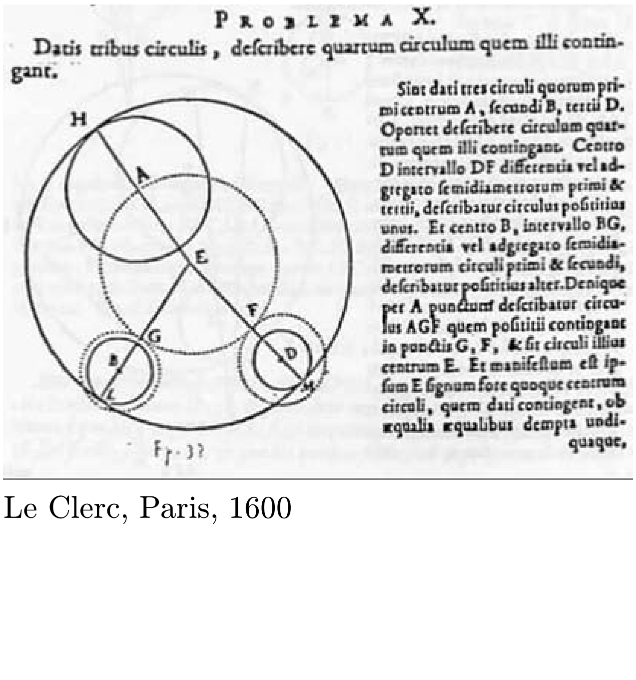
RESPONSVM. 313

Datum quadrato-cubum adfectum adjunctione quidem plano-solidi sub latere & dato coëfficiente plano-plano : multa vero plano-solidi sub cubo & dato coëfficiente plano, resolvere.

Hæc autem in opere restitutæ Analyseos Mathematicæ abunde tractata & exposita sunt. Quare quærenti Adriano licet sive in Geometricis sive in Arithmeticis satisfacere. Adscito nempe eo, quod ad supplementum Geometriæ inducendum fuit postulato dabitur in exposito diagrammate circumferentia CF. Et eodem opere duplicato dabitur circumferentia GC, atque adeo recta quæ ei subtenditur. In numeris, qualium semidiameter AC 1. talium recta CG quæsita fit

Dato autem termino authenta, seu classico, dantur reliqui viginti duo plagii, Undecim si placet, ad alam sinistram classici. Undecim ad dextram utpote hac methodo. Quoniam in circulo cujus diameter ZAC data circumferentia CD imperata est secari quadragequinquefariam, Quod opus duplici continua trisectione, ac una demum quinquesectione est absolutum, Itaque data est CG quadragesima quinta pars datæ CD. Producatur GC in α posita Gα æquali quadragesimæ quintæ parti circumferentiæ totius circuli. Supersunt igitur in circumferentia tota quadraginta quatuor quadragesimæ quintæ. Itaque à puncto α progrediendo ... ad punctum G sumantur circumferentiæ viginti duo æquales, quarum ideo unaquæque sit duabus quadragesimis quintis æqualis. Et sunto αβ, βγ, γδ, δε, εζ, ..., χG. Et puncta quidem undecim α, β, γ, δ, ε, ζ, ..., λ consistent in uno semicirculi circumferentia. Reliqua vero μ, ν, ξ, ο, π, ρ, σ, τ, υ, φ, χ in circumferentia semicirculi reliqui. Subtendantur in priore semi-circulo Cα, Cβ, Cγ, Cδ, Cε, Cζ, ..., Cλ. Æque subtendantur in posteriore Cχ, Cφ, Cυ, Cτ, Cσ, Cρ, Cπ, Co, Cξ, Cν, Cμ. Dico subtensas illas esse binos undenos, de quibus monui, terminos.

Id est, dico de iis singulis propositum à Romano quadragequinquesectionis Problema, quod tam varium & ambiguum non sensisse arguunt ea, quæ tanquam soluta protulit, exempla, posse in ea de qua specialiter quæris hypothesi, non aliter quam de primum exhibita CG, explicari.



PROBLEMA X.

Datis tribus circulis, describere quartum circulum quem illi contingant.

Sint dati tres circuli quorum primi centrum A, secundi B, tertii D. Oportet describere circulum quartum quem illi contingant. Centro D intervallo DF differentia vel adgregato semidiametrorum primi & tertii, describatur circulus posititius unus. Et centro B, intervallo BG, differentia vel adgregato semidiametrorum circuli primi & secundi, describatur posititius alter. Denique per A punctum describatur circulus AGF quem posititii contingant in punctis G, F, & sit circuli illius centrum E. Et manifestum est ipsum E signum fore quoque centrum circuli, quem dati contingent, ob æqualia æqualibus dempta undiquaque,

Le Clerc, Paris, 1600

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FRANCISCI VIETÆ

AD

PROBLEMA, QVOD OMNIBVS MATHEMATICIS TOTIVS ORBIS CONSTRUENDUM PROPOSUIT ADRIANUS ROMANUS;

RESPONSVM.

SI toto terrarum orbe non errat ADRIANVS ROMANVS, dum Mathematicos totius terrarum orbis unius sui Problematis solutioni vix censet idoneos, non ille saltem Gallias, nec Galliarum Lycia suo dimensus est radio. Cedat ROMANO Belga, cedat ROMANVS Belgæ, vix sinet Gallus à ROMANO vel Belga gloriam suam sibi præripi. Ego qui me Mathematicum non profiteor, sed quem, si quando vacat, delectant Mathematices studia, Problema ADRIANICVM ut legi ut solvi, nec me malus abstulit error. Sic trihorio ingens prodii Geometra. Neque vero placet barbarum idioma, id est; Algebricum. Geometrica Geometrice tracto, Analytica Analytice. Curabo tamen ut me, sive quasi Geometram sive novum Analystam, vulgus Algebristarum satis exaudiat.

CAPVT I.

Proponentis Adriani Romani verba.

PRimum igitur Adriani Romani proponentis ipsa verba refero, ne immutato quidem commate.

PROBLEMA MATHEMATICVM OMNIBVS ORBIS MATHEMATICIS AD CONSTRVENDVM PROPOSITVM. "

" Si duorum terminorum prioris ad posteriorem proportio sit, ut 1 (1) ad "
" 45 (1) − 3795 (3) + 9,5634 (5) − 113,8500 (7) + 781,1375 (9) − 3451,2075 (11) + 1, "
" 0530, 6075 (13) − 2, 3267, 6280 (15) + 3,8494, 2375 (17) − 4, 8849, 4125 (19) "
" + 4,8384, 1800 (21) − 3, 7865, 8800 (23) + 2, 3603, 0652 (25) − 1, 1767, 9100 "
" (27) + 4695, 5700 (29) − 1494, 5040 (31) + 376, 4565 (33) − 74, 0459 (35) + "
" 11, 1150 (37) − 1, 2300 (39) + 945 (41) − 45 (43) + 1 (45) deturque terminus "
posterior, invenire priorem.

Exem-

314 AD ADR. ROMANI PROBLEMA

				PARTIVM.	SCRVP.
In numeris qualium AC	100,000	000		XC.	∴
Talium data CA sit	41,582	338		XII.	∴
Classica CG quæsita	930	839	*Qualium*	∴	XVI.
Reliquarum Endecas prima			*autem*		
Gα	13,022	572		III.	XLIV.
Cβ	40,671	389	*totacir-*	XI.	XLIV.
Cγ	67,528	585	*culicir-*	XIX.	XLIV.
Cδ	63,071	414	*cumfe-*	XXVII.	XLIV.
Cε	116,802	731	*rentia est partium*	XXXV.	XLIV.
Cζ	136,260	439	*IIICLX. talia ipsa,*	XLIII.	XLIV.
Cη	157,027	354	*quæ are-*	LI.	XLIV.
Cθ	172,737	783	*bus designa-*	LIX.	XLIV.
Cι	185,086	061	*tis subtendunter. cir-*	LXVII.	XLIV.
Cκ	193,831	852	*cumferen-*	LXXV.	XLIV.
Cλ	198,849	238	*tia semissis sunt*	LXXXIII.	XLIV.
Endecas altera.					
Cχ	28,756	098		VIII.	XVI.
Cφ	56,021	654		XVI.	XVI.
Cυ	82,196	811		XXIV.	XVI.
Cτ	106,772	100		XXXII.	XVI.
Cσ	129,269	199		XL.	XVI.
Cρ	149,250	207		XXVIII.	XVI.
Cϖ	166,326	235		LVI.	XVI.
Cο	180,164	914		LXIV.	XVI.
Cξ	190,496	888		LXXII.	XVI.
Cν	197,121	055		LXXX.	XVI.
Cμ	199,908	485		LXXXVIII.	XVI.

CAPVT IX.

Ratio constructionis.

RAtionem constructionis edocet Analyticus angularium sectionum primus, seu catholicus, in quo ordinata sunt Theoremata hæc.

E duobus angulis acutis trianguli, is qui continetur abs hypotenusa & base acuti nomen retineto. Alter qui continetur abs hypotenusa & perpendiculo, esto reliquus è recto.

Mettayer, Paris, 1595

5.2 Two Viète's problems: *Zetetics*, Mettayer, Tours, 1591

Book I, problem 1.

ZETETICVM I.

DAta differentia duorum laterum, & adgregato eorumdem, invenire latera.

Sit data B differentia duorum laterum, & datum quoque D adgregatum eorumdem. Oportet invenire latera.

Latus minus esto A, majus igitur erit A + B. Adgregatum ideo laterum A 2 + B. At idem datum est D. Quare A 2 + B æquatur D. Et per antithesim, A 2 æquabitur D − B, & omnibus subduplatis, A æquabitur D $\frac{1}{2}$ − B $\frac{1}{2}$.

Vel, latus majus esto E. Minus igitur erit E − B. Adgregatum ideo laterum. E 2 − B. At idem datum est D. Quare E 2 − B æquabitur D. & per antithesim, E 2 æquabitur D + B, & omnibus subduplatis E æquabitur D $\frac{1}{2}$ + B $\frac{1}{2}$.

Data igitur differentia duorum laterum & adgregato eorumdem, inveniuntur latera.

Enimvero

Adgregatum dimidium laterum minus dimidia differentia æquale est lateri minori, plus eadem, majori.

Quod ipsum est quod arguit Zetesis.

Sit B 40. D 100 *A fit* 30. E 70.

Given the difference between two sides and their sum, find the sides.

Let B be the given difference of the two sides, and let D be their sum. We have to find the sides. Let A be the smaller side, then the bigger side will be $A + B$. For this reason the sum of the sides will be $2A + B$. This is the same thing as D. That is what $2A + B$ equals D. And by antithesis, $2A$ will equal $D - B$ and all being divided by two, A will equal $\frac{1}{2}D - \frac{1}{2}B$.

Or let E be the bigger side. The smaller will then be $E - B$. For this reason the sum of the sides will be $2E - B$. This is the same thing as D. That is why $2E - B$ will equal D and by antithesis $2E$ will equal $D + B$; and all being divided by two, E will equal $\frac{1}{2}D + \frac{1}{2}B$.

So, given the difference between two sides and their sum, the sides will be found. Indeed:

Half the sum of the sides, minus half the difference, equals the smallest side; the same quantities added give the bigger side.

This was to be done.

Given: B 40. D 100. A equals 30. E equals 70.

Book II, problem 3.

ZETETICVM III.

Dato Rectangulo sub lateribus & differentia laterum inueniuntur latera.

Enimuerò quadratum differentiæ laterum adiunctum quadruplo Rectangulo sub lateribus æquatur quadrato adgregati laterum.

Iam enim ordinatum est, Quadratum adgregati laterum minus quadrato differentiæ æquari quadruplo Rectangulo sub lateribus, adeò vt sola fuit opus anthitesi. Datâ porro differentia duorum laterum & eorum summa dantur latera.

Sit 20 Rectangulum sub duobus lateribus quorum differentia est 8. Summa laterum esto 1 N. 1 Q æquatur 144.

Given the product of two numbers and their difference, find the numbers.

In fact: The square of the difference of the numbers, plus four times their product, equals the square of their sum.

Indeed, we have shown before that the square of the sum of two numbers minus the square of their difference equals four times their product, then, by antithesis, we have the first statement. The difference between the two numbers and their sum is yet given, and then we can get the numbers.

Given 20 the product of the two numbers, and 8 their difference. Let 1N be their sum. 1Q (its square) equals 144

5.3 The resolution of equations: *equation* $x^2 + ax = b$

5.3.1 Text of Al Khwarizmi: Algebra, Chapter IV. Squares and roots that are equivalent to numbers

There is equivalence between squares and roots on the one hand and numbers on the other hand if, for example, one says that a square and ten roots are equal to 39 units.

The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39?

The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square.

5.3.2 Text of Viète: Treatise about equations

(De Emendatione aequationum tractatus secundum, Laquehay, Paris, 1615)

De Reductione Quadratorum adfectorum ad pura.
Formulæ tres.

II.

Si A quadratum — B in A bis. } æquetur Z plano.
A — B esto E. igitur E quadratum, æquabitur { Z plano + B quadrato.
Consectarium.
Itaque, l. { Z plani. + B quadrato. } + B, fit A, de qua primum quærebatur.
Sit B 1. *Z planum* 20. *A* 1 N.
1 Q — 2 N. *æquabitur* 20. *& fit* 1 N *l.* 21 + 1.

How to reduce quadratic equation from affected to pure
Three formulas
II.

If $A^2 - 2AB = Zp$. $A - B = E$ then $E^2 = Zp + B^2$,
That is why $\sqrt{Zp + B^2} + B = A$, which was to be found.
Given $B = 1$, $Zp = 20$, $A = 1N$
$1Q - 2N = 20$ and N is equal to $\sqrt{21} + 1$.

5.4 Exercises about Viète's Zetetics

These exercises have been given to pupils 14–17 years old during the theme about remarkable identities. See also [U1].

Demonstrate the following theorems stated and demonstrated in 1591 by the French mathematician Viète (1540–1603), born in Fontenay-le-Comte, which was then the capital of Lower Poitou.

1. Twice the product of two numbers added to the sum of their squares is equal to the square of their sum. If we subtract it from the sum of their squares, we get the square of the difference between the two numbers.
2. The square of the sum of two numbers added to the square of the difference between them is equal to the double of the sum of their squares.
3. The square of the sum of two numbers minus the square of the difference between them is equal to four times their product.
4. If the difference between the squares of two numbers is divided by the difference between the two numbers, the quotient is the sum of the two numbers.
5. If the difference between the squares of two numbers is divided by the sum of the two numbers, the quotient is the difference between the two numbers.

In Book 3 of his research (Zetetics), Viète implements his calculations with letters, that he invented, to find formulas on rectangular triangles. Find his formulas.

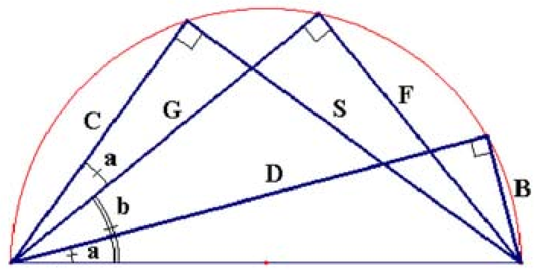
1. Given the perpendicular of a right triangle and the difference between the base and the hypotenuse, find the base and the hypotenuse. Numerical application: 5 and 1.
2. Given the perpendicular of a right triangle and the sum of the base and the hypotenuse, find the base and the hypotenuse. Numerical application: 5 and 25.
3. Given the hypotenuse of a right triangle and the difference between the sides around the right angle, find the sides around the right angle. Numerical application: 13 and 7.
4. Given the hypotenuse of a right triangle and the sum of the sides around the right angle, find the sides around the right angle. Numerical application: 13 and 17.

Use these rules to solve, as Viète did it in Book 2 of his Researches (Zetetics), the following systems of two equations with two unknowns, reducing them to the search of the sum and the product of two numbers.

1. $xy = 20$ and $x^2 + y^2 = 104$.
2. $xy = 20$ and $x - y = 8$.
3. $x - y = 8$ and $x^2 + y^2 = 104$.
4. $x + y = 12$ and $x^2 + y^2 = 104$.
5. $x - y = 8$ and $x^2 - y^2 = 95$.
6. $x + y = 12$ and $x^2 - y^2 = 95$.
7. $xy = 20$ and $x^2 - y^2 = 95$.
8. $xy + x^2 + y^2 = 124$ and $x + y = 12$.
9. $x^3 - y^3 = 316$ and $x^3 + y^3 = 370$.
10. $x^3 - y^3 = 316$ and $xy = 1$.
11. $x - y = 6$ and $x^3 - y^3 = 504$.
12. $(x - y)(x^2 - y^2) = 32$ and $(x + y)(x^2 + y^2) = 272$.
13. $x^2 + y^2 = 20$ and $\dfrac{xy}{(x-y)^2} = 2$.
14. $x^2 + y^2 = 20$ and $\dfrac{xy}{(x-y)^2} = 1$.

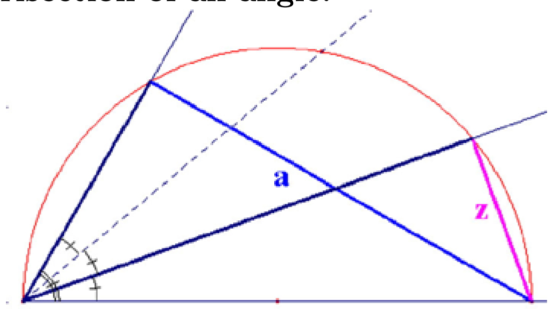
5.5 Trigonometry and trisection

Trigonometry: addition formulas



With Viete's notations
S **is** F **in** $D+B$ **in** G and C **is** G **in** $D-F$ **in** B
With modern notations:
$\sin(a+b) = \sin b \cos a + \sin a \cos b$ and $\cos(a+b) = \cos b \cos a - \sin b \sin a$

Trisection of an angle:



$3z - z^3 = a$

5.6 Viète's method to solve Adrien Romain's challenge

Adrien Romain's equation

Transcription in present notations.

$$45x - 3\,795x^3 + 95\,634x^5 - 1\,138\,500x^7 + 7\,811\,375x^9 - 34\,512\,075x^{11} + 10\,5306\,075x^{13} - 232\,676\,280x^{15} + 384\,942\,375x^{17} - 488\,494\,125x^{19} + 483\,841\,800x^{21} - 378\,658\,800x^{23} + 236\,030\,652x^{25} - 117\,679\,100x^{27} + 46\,955\,700x^{29} - 14\,945\,040x^{31} + 3\,764\,565x^{33} - 740\,259x^{35} + 111\,150x^{37} - 12\,300x^{39} + 945x^{41} - 45x^{43} + x^{45} = \mathbf{a}.$$

With **a** equal to: $\sqrt{\frac{7}{4} - \sqrt{\frac{5}{16}} - \sqrt{\frac{15}{8} - \sqrt{\frac{45}{64}}}}$

The principle of Viète's reconstruction.

To divide an angle into n equal parts can be reduced to an equation of degree n.

If the given equation is that of the division of an angle into 45 equal parts, $45 = 3 \times 3 \times 5$, we can make three steps.

First step: Let z such as

$$3z - z^3 = a \tag{1}$$

Equation corresponding to the division of an angle into 3 equal parts.

Second step: Let y such as

$$3y - y^3 = z \tag{2}$$

The same; then the given angle is divided into 9 equal parts.

Third step: Let x such as

$$5x - 5x^3 + x^5 = y \tag{3}$$

Equation corresponding to the division of an angle into 5 equal parts; then the given angle is divided into 45 equal parts.

By using (3), equation (2) becomes:

$$3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3 = z.$$

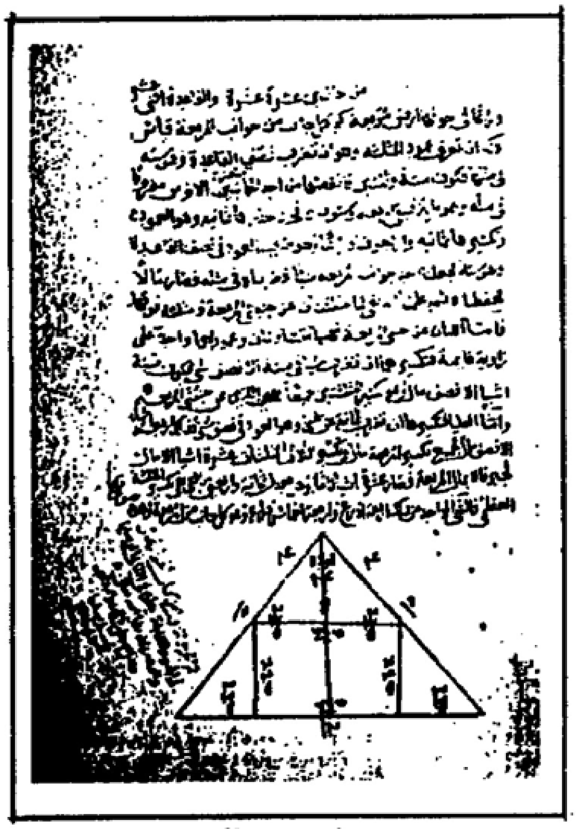
And equation (1) becomes:

$$3[3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3] - [3(5x - 5x^3 + x^5) - (5x - 5x^3 + x^5)^3]^3 = a.$$

Expanding, we find Romain's equation.

5.7 Inscription of a square in a triangle

5.7.1 A problem by Al Khwarizmi (extract from Kitab al-Jabr wal Muqabala)



Une page de
L'abrégé du calcul par l'algèbre et la muqâbala
d'al-Khwârizmî (IX^e s.)

Given a triangular plot of land with sides of 10, 10 and 12 cubits, and inside it a square piece of land, what is the side of this piece of land?

Multiply half the base by itself, subtract it from one of the smaller sides multiplied by itself and that is 100. The remainder is 64. Take the root of this number, 8 and that is the height. And the area is 48 and that is the product of the height by half the base which is 5.

We state that one of the sides of the square plot of land is a thing, we multiply it by itself, it becomes "the capital" and we keep it. Then we notice that we are left with 2 triangles on the vertical sides of the square and a triangle on top of it. As for the two triangles that are on the vertical sides of the square plot of land, they are equal and their height is the same and they have a right angle. So their area is calculated by multiplying a thing by 6 minus half a thing, which makes 6 things minus half a square; and that is the area of the two triangles together which are on the vertical sides of the square plot. As for the area of the triangle at the top, we get it by multiplying 8 minus one thing, which is the height, by half a thing, that makes 4 things minus half a square. This is the area of the square plot and the three triangles, and that is 10 things and equal to 48 which is the area of the big triangle. Thus the thing is 4 cubits and $\frac{4}{5}$ and that is each side of the square plot and here is its figure.

(From an oral translation by Ahmed Djebbar during a talk.)

This text has been utilized with pupils 13–14 years old with the following instructions (see also [U2]).

1. What does Al Khwarizmi try to find in his problem? Draw a figure and note the data and colour in red what must be found.
2. What does Al Khwarizmi find? Check if his result is correct.
3. Explain Al Khwarizmi's method.
4. Here is an extract from the introduction to Al Khwarizmi's book:

   "I wrote, in the field of calculus by al jabr, an epitome including the finest and noblest operations of calculus which the men needed to do their heritages and donations, their partitions and judgments, their commercial transactions and all the operations which interest them, as land-surveying, distribution of river waters, architecture and other things."

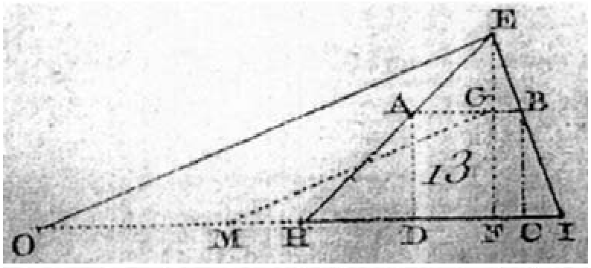
   Explain why Al Khwarizmi invented algebra.
5. In order to know Al Khwarizmi better.

   Try and find where he lived, what he did, what are the rules of al-jabr and al-muqabala, what word was created after his name… (note the references of the documents in which you found pieces of information: books, websites, …)

5.7.2 The problem by Bézout

(*Cours de Mathématiques à l'usage des gardes du Pavillon de la Marine*, volume 3, 1766, or *du Corps Royal de l'Artillerie*, tome 2, 1788)

251. Propofons-nous donc pour premiere queftion, de *décrire un quarré* ABCD (Fig. 13) *dans un triangle donné* EHI.



Par ces mots, *un triangle donné*, nous entendons un triangle dans lequel tout est connu, les côtés, les angles, la hauteur, &c.

Avec un peu d'attention, on voit que cette question se réduit à trouver sur la hauteur EF un point G par lequel menant AB parallèle à HI, cette ligne AB soit égale à GF ; ainsi l'équation se présente tout naturellement, il n'y a qu'à déterminer l'expression algébrique de AB, & celle de FG, & ensuite les égaler.

Nommons donc a la hauteur connue EF ; b, la base connue HI, & x la ligne inconnue GF ; alors EG vaudra $a - x$.

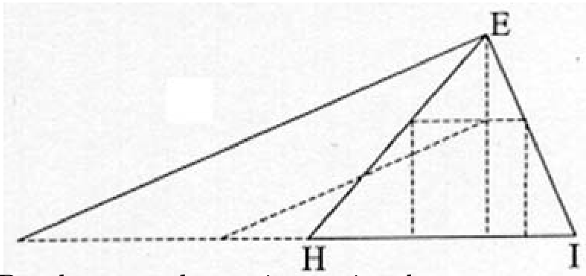
Or puisque AB est parallèle à HI, on doit (*Géom.* 109) avoir $EF : EG :: FI : GB :: HI : AB$; c'est-à-dire, $EF : GE :: HI : AB$, ou $a : a - x :: b : AB$, donc (*Arith.* 169) $AB = \frac{ab - bx}{a}$; puis donc que AB doit être égal à GF, on aura $\frac{ab-bx}{a} = x$; d'où, par les règles de la première Section, on tire $x = \frac{ab}{a+b}$.

Pour construire cette quantité, il faut, conformément à ce que nous avons dit (184), trouver une quatrième proportionnelle à $a + b$, b, & a, ce que l'on exécutera en cette manière. On portera de F en O une ligne FO égale à $a + b$, c'est-à-dire égale à $EF + HI$, & l'on tirera EO ; puis ayant pris FM égale à $HI = b$, on mènera, parallèlement à EO, la ligne MG, qui par sa rencontre avec EF, déterminera GF pour la valeur de x; car les triangles semblables EFO, GFM, donnent $FO:FM::FE:FG$, ou $a + b : b :: a : FG$; FG vaudra donc $\frac{ab}{a+b}$.

(Géom. 109)

Deux triangles qui ont les angles égaux chacun à chacun, ont les côtés homologues proportionnels, & sont, par conséquent, semblables.

251. For the first question, we propose to describe a square $ABCD$ (Fig. 13) in a given triangle EHI.



By these words, a given triangle, we mean a triangle in which everything is known, the sides, the angles, the height etc.

With a little attention, we see that this question amounts to finding, on the height EF, a point G through which, drawing AB parallel to HI, this line AB should be equal to GF; thus the equation is quite natural. We only have to determine the algebraic expression of AB and that of GF and then equal them.

So let's name a the known height EF, b the known base HI, and x the unknown line GF; then EG will equal $a - x$.

Now, since AB is parallel to HI, we must (Geom. 109) have $EF : EG :: FI : GB :: HI : AB$; that is to say, $EF : EG :: HI : AB$; or $a : a-x :: b : AB$; so (Arith. 169) $AB = ab - \frac{bx}{a}$; and therefore that AB must be equal to GF, we will have $ab - \frac{bx}{a} = x$; whence, by the rules of the first section, we derive the $x = \frac{ab}{a+b}$.

To construct this quantity, we must, in accordance with what we have said earlier (184), find a fourth proportional to $a + b$, b, and a, which we will do like this. We will draw from F to O a line FO equal to $a + b$, that is to say, equal to $EF + HI$, and we will draw EO; then taking FM equal to $HI = b$, we will draw, parallel to EO, a line MG, which when meeting EF will determine GF for the value of x; because the similar triangles EFO, GFM give $FO : FM :: FE : FG$, or $a + b : b :: a : FG$; so FG is equal to $\frac{ab}{a+b}$

(Géom.109)

The homologous sides of two triangles whose angles are equal each to each, are proportional, and thus these triangles are similar.

This text has been utilized with pupils 15–16 years old (see [U2]).

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- [U2] **A square in a triangle**
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  Text by Bézout: IREM de Dijon "Pot pourri: activités historico-mathématiques", 2004, and Repères-IREM No 63, 2006 (*).

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- [W] http://www.cc-parthenay.fr/parthenay/creparth/GUICHARDJp/VIETEaccueill.html
- A diaporama is avaible on the site "CultureMath": http://www.dma.ens.fr/culturemath/.