READING AND DOING MATHEMATICS IN THE HUMANIST TRADITION

ANCIENT AND MODERN ISSUES

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Abstract

L'atelier dont on résume ici le propos est basé sur la lecture de textes mathématiques et rhétoriques anciens, dans le but d'aborder sous un jour original certains débats contemporains touchant la conception moderne de l'apprentissage des mathématiques qui centre le plus souvent ce dernier sur l'activité autonome de l'élève. Ces textes anciens montrent en effet que si la question de la production autonome d'un discours, fût il mathématique, est bien sous-jacente à ces textes, cette production n'est pas cependant opposée à l'apprentissage systématique d'un savoir traditionnel enseigné par un maître. Ce détour historique permet donc d'envisager autrement les conceptions en question et les difficultés qu'elles soulèvent.

1 INTRODUCTION

Partly through the influence of constructivist theories, mathematics education has for many years now tended away from predetermined mathematical material approached in predetermined ways. Argumentation, communication, investigative activities, and student productions — matters which emphasize students' own part in acquiring mathematical understanding — have accordingly become dominant themes in teaching and research. This tendency, on the face of it, seems at odds with historians' disciplined readings of mathematical texts, their distancing themselves from their own modern preconceptions, and their fixed desire to read texts as the authors wrote them. And yet, historically, in the humanist educational tradition, the classical *paideia*, the reading of texts appears to have been more in the spirit of those themes of mathematics education to which we referred just now. Ironically, then, by reading historical texts with such current mathematics education tendencies in mind, we are, as Collingwood might put it, reenacting the historical context of the reading of these texts; we are, in this way, truly engaging in an historical study while developing our own mathematical sensibilities.

The workshop presented at HPM-ESU5, therefore, was meant to give participants a concrete sense of how these modern concerns might arise out of a historical reading of mathematical texts when the education background of those texts, namely, the classical humanist tradition, is taken into account. Our discussion here will run as follows. The first section will describe the historical motivation behind the design of the workshop. It must be understood that neither here nor, for that matter, in the paper as a whole are we trying to prove a historical thesis, but only to provide enough background regarding classical Greek mathematics, mathematics education, and rhetoric to give our approach to reading mathematical texts in educational contexts some historical plausibility. In the second section, we shall give an account of the mathematical and rhetorical texts used in the workshop, how they were chosen and how they were treated. The final section corresponds to the third part of the workshop and comes closest to the main goal of the workshop, namely, to show how these historical readings may provide a platform for discussing modern educational issues. Here, we only give an example of a recent debate that resonates with the ancient rhetorical tradition and which we used as a springboard for discussion.

2 Historical Motivation — Ancient Issues

Given the familiarity of the phrase "Greek mathematics", one might well assume it refers to a perfectly clear and circumscribed notion. In fact, the best one can say is that it refers to a certain kind of intellectual activity that occupied certain thinkers living in a certain region around the Mediterranean Sea from something like 600 B.C.E. to 600 C.E. Even the word "Greek" itself is not unproblematic. Nevertheless, Greeks themselves spoke about "Greeks" — and they spoke about "mathēmatika". Hence, we shall refer as "Greek" the common tradition making it possible for Euclid, Archimedes, Apollonius, Pappus and Proclus, were they brought together in a room, to speak together and understand one another. Clarifying that common tradition is precisely the challenge of the history Greek mathematics.

Indeed, even when looking closely at an individual mathematician, say, Apollonius of Perga, one never drifts far from the tradition which made him — and this is no less true when considering his most idiosyncratic and original work. But getting close to the tradition that made mathematicians like Apollonius or Euclid is not only a matter of surveying their influences, but also, and perhaps primarily, understanding the nature of their education. For this reason, the study of the history of Greek mathematics is an enterprise intimately connected with the history of Greek mathematics education. And that education, in its turn, is must be viewed in light of a more general Greek education, what they called *paideia*.

From a modern perspective, it is natural to expect a continuous educational nexus leading to works as expansive and as deep as Archimedes' *On the Sphere and Cylinder* or Apollonius' *Conics*: a program or at least, a pattern of mathematical education from K-12 to undergraduate to graduate studies. Of course there were educational institutions in the Classical period that supported work in mathematics, the *Museum* in Alexandria and the *Academy* in Athens being famous examples. But between these institutions of advanced learning and very rudimentary mathematical training there appears to be a gap. Indeed, given the sophistication and level of mathematical works such as those of Archimedes, Euclid, and Apollonius, it is surprising to discover that the ordinary education of youth, at least in 4th and 5th century Athens, seems to have included very little mathematics of any weight at all.[1]

What one does find educationally is an emphasis on rhetorical training, beginning with the Sophists in 5th century B. C. E and arriving, finally, to a point of great technical perfection and sophistication by the end of the Hellenistic period. However, it is important to stress

[1] Ian Mueller (1991) observes that despite an apparent common ability to perform calculations such as and 2 000/10 and 3×700 [Mueller is relying here on passages from Aristophanes' *Wasps* and Plato's *Hippias Minor*, respectively], "... it appears that the average Athenian citizen knew remarkable little arithmetic from our point of view and that he did not acquire his knowledge in school. But even if he did learn arithmetic at school, we have no right to assume he learned any geometry, astronomy, or music theory, despite the fact that we have plenty of evidence associating these subjects with the intellectual heights of fifth-century culture" (p. 88). Thomas Heath is more generous in his estimate of children's arithmetical education (see Heath, 1981, vol. I, pp. 18–19). However, whether or not mathematics was included in the basic education of Athenian youth in fact, if we consider the accounts of basic Athenian education by Protagoras in Plato's *Protagoras* (325e–326c) and Glaucon in the *Republic* (522b), we must accept that neither saw mathematics as an *obvious enough* component of elementary education to mention it in their descriptions; for them, it seems, "the three R's" of education were Reading, Rhythm, and wRestling!

that rhetorical education was not a technical education merely, but one also that aspired to genuine knowledge and a perspective on how one should live: the word embracing its educational ideals was *paideia*. *Paideia* entailed knowledge of a certain corpus of literature, but it meant most of all having the skills and presence of mind allowing one to speak and act in an intelligent way, one might say in a *cultured* way. "Culture", in fact, just as "education" itself, is a frequent translation of *paideia*, and the Latin translation of *paideia* came to be, tellingly enough, *humanitas*. In sum, *paideia* is the heart of that common tradition we referred to at the outset.[2]

By the end of the Hellenistic period, and certainly by late antiquity, rhetorical education had become the predominant form of education in the classical world. It is this kind of education, then, which we must imagine as the basic education of citizens in the classical world from the Hellenistic period onward, certainly of the intellectual elite, including mathematicians. The structure and vocabulary of ancient mathematical texts reveal the influence of that education, their authors' *paideia*. In late classical mathematical works such as those of Pappus and Proclus one can see the influence of *paideia* in the particular shape of those works (Bernard, 2003a, 2003b). Such works were written by people trained to write rhetorical texts that inspire rhetorical practice. A text written with this background "...therefore functions as a kind of *trap* for its reader or its listener... Mathematical texts, that is, texts that are *mathemata* in the true sense, *'learning matters'*, also share in this particular form" (Bernard, 2003b, p. 409). Like the rhetorical texts they knew so well, it reasonable to think that writers of these mathematical texts might also have thought of them as models for imitation and sources for invention. Here also an important and subtle point ought to be brought out. The *paideia* of classical times invited reflection on the tradition it represented and engaged the reader to move beyond it.[3] Tradition in this sense ought not be thought of necessarily as a force preserving the *status quo* and stifling invention, but as a foundation on which one may develop ones own creative powers.

3 Reading Ancient Texts: Parts I and II of the Workshop

The historical picture sketched above motivates the workshop we have conceived in two ways. First, assuming Greek mathematical texts were written both as works to be imitated and sources for invention, as we have argued, the workshop begins with reading selections from Euclid and Proclus closely and raising questions meant to clarify the text as a text while, simultaneously, inviting invention based on the text. Second, selections from classical rhetoric are read to give participants a feeling for the cultural background of ancient readers and writers of mathematical texts.

3.1 Euclid's *Elements*, VI.2, 8, 9–12

Although our purposes for this part might have been served by any number of Greek mathematical texts, selections from Euclid's *Elements* seemed to have a certain inevitability. First, it is arguably the most well-known of all Greek mathematical works. Moreover, many propositions in the *Elements*, especially in Books I, III, IV and VI, correspond to those taught in school geometry today. At the same time, the particular form in which Euclid presents and demonstrates these propositions is often quite different from what modern teachers are used to. So, Euclid's *Elements* was chosen for its fame and its fruitful mix of the familiar and unfamiliar.

[2]Thus, Jaeger writes "...it was perfectly natural for the Greeks in and after the fourth century, when the concept finally crystallized, to use the word *paideia* — in English, *culture* — to describe all the artistic forms and the intellectual and aesthetic achievements of their race, in fact the whole content of their tradition" (Jaeger, 1945, vol. I, p. 303).

[3]This is implied in the very word 'tradition', whose root, *tradere*, means both 'to pass down' and 'to betray' (see Brann, 1979, p. 67).

As for the specific propositions chosen, our criteria were 1) that, again, the propositions should treat familiar geometric facts or problems but should display the peculiarities of Euclidean form and concepts; 2) that they should belong to a series of propositions — they should, in a small way, be a text within the text. We also wanted to include problems, since problems, *problēmata* in Greek, were also important in rhetorical training, where they served to call the learner into action: in a way, *theorem*, meaning literally something to look at and *problem*, literally, something thrown out [to do], run parallel to the "reading and doing" in our title. With that, still several choices would have been appropriate, for example, one possibility was II.6 and II.11. We decided, however, on VI.2, 8, 9–13 from the book on geometrical applications of proportion partly because the propositions are seemingly straightforward and partly because proportion, equality of areas and similarity of figures are among those familiar-unfamiliar concepts described above.

Propositions VI.2 and VI.8 are theorems. Proposition VI.2 tells us that a line drawn parallel to one of the sides of a triangle will cut the remaining sides proportionally, and, conversely, a line cutting two sides of a triangle proportionally will be parallel to the remaining side. Proposition VI.8 shows that a perpendicular drawn from the right angle of a right-angled triangle divides the triangle into two triangles similar to one another and the whole triangle itself. Propositions VI.9-13, on other hand, are problems related to VI.2 and VI.8. VI.9 requires cutting a prescribed part from a given line (e.g. a third); VI.10 requires cutting a given line similarly to a given divided line; VI.11–12 requires finding a third proportional and a fourth proportional respectively; VI.13 requires finding a mean proportional.

Having read the theorems and their Euclidean demonstrations, the participants were then asked to consider the following questions regarding VI.2 and 8:

- For each part of the proof, what is being referred to and what is required for that stage of the argument?
- The *porism*, at the end of VI.8, begins with the words 'it is clear'. What do you make of this?
- What is your general impression about these two propositions?

The first question is deceivingly simple. To start, there are many terms, such as "ratio", "proportion" and even "triangle" that need to be placed in their Euclidean setting. This, eventually, we discussed, but not before the participants formulated how they understood these terms from their own knowledge. The same could be said about the stages in the argument, the order of the statements, the warrants for the conclusions. Here, it is important to point out that while we used the standard English translation by Heath (1926), we removed Heath's parenthetical proposition citations. This was done not merely to be faithful to the style of the Greek text, which has no such references, but because that style has the effect, precisely, of forcing readers to look into themselves, to recall or reconstruct the sources of their knowledge: omitting such references is a call to activity: it belongs to the "method" of the text.

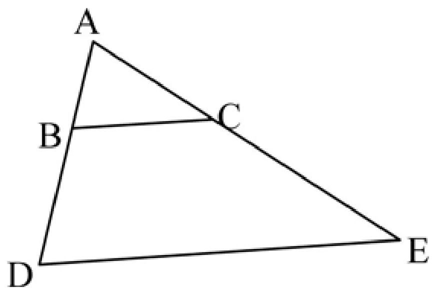
The question about the *porism* in VI.8 was meant to suggest a double perplexity. First, there is the oddness of a *porism* itself — what is its character the makes it worthy of a special name? Proclus, Euclid's 5$^{\text{th}}$ century C. E. commentator, is unclear himself as to what a *porism* is, describing it variously as a "lucky find", a problem requiring discovery rather than construction, an intermediary between a theorem and a problem (*In Eucl.* Friedlein, pp. 301.20–302.10; Morrow, 1970, p. 236). Second, there is apparent superfluity of the specific *porism* here: in the course of the proof of VI.6, Euclid shows that if AD is the perpendicular from the right angle and if AD divides the base into the segments BD and DC, then $BD : AD :: AD : DC$; the *porism* then states that the perpendicular drawn from the right angle is a mean proportional between the segments of the base. Having defined what a "mean proportional" is, it is unclear what the *porism* adds: what is the lucky find? What has been discovered?

The repetitiousness in the demonstration itself of VI.8, figured in the responses to the third question. Looking closely at the proposition one begins to see its didactic nature, how its very repetitiousness forces one to reconsider over and over the flow of the argument and the necessity of its various phases. How this may bring teachers to reflect on their practice was underlined in the responses of two teachers: one remarked how she would never use VI.8 with her students because of its verboseness, while another teacher said he definitely saw pedagogical benefits in Euclid's demonstration. The point is not that Euclid's demonstration is or is not good for a high school geometry class, but that can force teachers to think about their teaching.

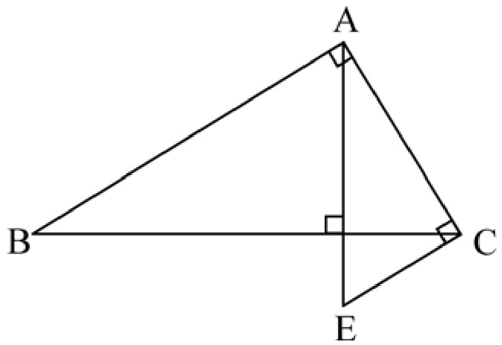
Questions similar to the first and third questions above were also asked with respect to the problems VI.9-13. The second question, however, asked the participants to engage in a process of invention based on the propositions read so far:

- Is Euclid 's solution the solution you would propose? What are your own solutions based on?

In VI.11, for example, Euclid finds the third proportional to two given lines BA and AC by setting BD equal to AC, drawing DE parallel to BC and then applying VI.2 to conclude that $AB : BD :: AC : CE$, that is, $AB : AC :: AC : CE$, so that CE is the required third proportional.



One of the participants provided an alternative inspired by VI.8, drawing what he called a "spiral-like figure":



By the similarity of triangles proven in VI.8, then, we have $BA : AC :: AC : CE$. Again, we avoided the question whether this alternative is better or worse than Euclid's, but emphasized how the participants' activity arose from reflecting on the given propositions and making inventive departures from them.

3.2 PROCLUS ON EUCLID'S *Elements*, I.1

Following Euclid, we turned to Proclus of Lycia (5^{th} cent. C.E.). Among his many other treatises and commentaries, Proclus wrote a detailed commentary on the first book of Euclid's

Elements.[4] The selection chosen is from the very end of his commentary on *Elem.* I.1, where Euclid shows how an equilateral triangle may be constructed on a given straight line.

Participants were first asked to read (or re-read) the definitions, postulates and common notions given at the beginning of *Elem.* I, along with *Elem.* I.1. Having done this, they had to consider the *other* constructions presented by Proclus, namely the construction of the isosceles and scalene triangles.[5] With that, we asked:

- After reading this, do feel you understand what is an isosceles and what is a scalene triangle?

Working through Proclus' constructions and simultaneously going back to Euclid's definitions, it becomes clear that "equilateral", "isosceles" and "scalene" triangles are *not* what they are for modern readers, but are three different species of the genus "triangle." In particular, the two equal sides of an isosceles triangle must be *different* from the base.

The way reading and understanding these constructions lead one to a better understanding of the various species of triangles, recalls our remark above (cf. 3.1) on the "method" of the text: here as before, the text forces one to reconsider previously defined geometrical objects: that they were defined does not mean they were fully understood. This touches on a fundamental difference between modern, "axiomatized" definitions and Euclid's: the former are meant to be a complete, unambiguous and "functional" account of a given object; the latter are more like issues to be re-discussed in order to be better understood.

Proclus's commentary on his own constructions continues with a call for readers to modify the constructions themselves: "And it is possible [for the reader] to train himself by adding or subtracting [conditions] on each of the hypotheses" (the entire passage is Proclus 1 in the appendix). Regarding this, we asked:

- Since this "is possible", according to Proclus, *can* you do it?

The crucial point here is questions posed somewhat artificially in our section on Euclid are, in a sense, *already included in Proclus's text itself.* In other words, Proclus' readers are invited explicitly to practice themselves certain constructions by following the model given by Euclid and Hero-Proclus *and* by supplying new constructions by modifying certain conditions. The readers' activity is fundamental to Proclus' purposes: the concrete geometrical exercises are meant to guide one directly to a theoretical view of the nature of problems and how they depend on their specific enunciations and conditions.[6] That a problem, mathematical or not, ought to induce learning or doing, is acknowledged by Proclus explicitly:

> One should also recognize that one speaks about 'problem' in various senses. Indeed anything propounded, either for the sake of learning [*eite tēs matheseōs heneka*] or also for the sake of doing [*eite tēs poišseōs heneka*], is called a problem.

The necessity of readers' own activity in producing alternative constructions as well as the general characterization of problems' leading ambiguously to learning and producing, which we have just seen in Proclus, were essential aspects of *paideia* and were, therefore, mentioned in the ancient textbooks of rhetoric. That Proclus himself was aware of the nature

[4] A much-used translation is Morrow (1970). The quotations below owe much to Morrow's translation; however, since Morrow's version is not always completely reliable, we have modified the translation somewhat.

[5] These constructions Proclus' own: they belonged to earlier commentaries, beginning with Hero of Alexandria. This fact is acknowledged by Proclus himself, referring to "all-too-well known commentaries". He does not, however, name Hero explicitly here, as he does in other places.

[6] In ancient terms, this kind of reflection on the 'determination' of problems refers to the *diorismoi* discussions.

and practice of *paideia* is not surprising for from his biography[7] we know that he was not only fully trained in rhetoric but was also a champion and defender of *paideia*.[8]

3.3 ISOCRATES AND AELIUS THEON ON *invention* AND *imitation*

That mathematical studies were situated within this more general context of ancient *paideia*, was discussed in part 2 above. The final set of readings, then, set out some of the key ideas behind the literate and intellectual practices of ancient *paideia*.

The main readings here were by the great 4th cent. B.C.E teacher, Isocrates, whose lessons and philosophy became highly influential in the Hellenistic period. The first reading was a short but famous remark by Isocrates praising the *logos*, which should be ambiguously, but tellingly, understood as "speech" and "reason" (see Isocrates 1 in the appendix). This brief quotation brings out two points surprising for modern students. First, Isocrates makes no sharp distinction between *speaking* and *thinking*: those most able to persuade themselves, and, hence, to think by themselves, are *therefore* those most able to persuade others. Secondly, speaking well directly reflects one's ethical values: speaking and living well are not and should not be distinguished.

The second two texts were from Isocrates' early pamphlet *Against the Sophists*, where he made clear for the first time the fundamental principles of his own school (Isocrates 2 in the appendix). The first quotation, directed against his detractors, reveals the practical aspects of Isocrates' *paideia*:

- That his art is **a creative process**, *poištikon pragma*, literally an "act of production": it should enable students to produce discourses (and thus prepare them, ultimately, to lead their whole lives).

- That the art of discourse is really an art. It requires progressive training and familiarization, like any other apprenticeship.

- Moreover, it aims to develop a **capacity of invention** or *dunamis heuretikē*. (This concept of *heurēsis–inventio* in Latin)

This capacity would be purposeless were the discourse without real content. In the later tradition again, that "purpose" was called a *problem*: it was a *challenge* for the rhetor set either by his teacher (in a scholastic context) or, ultimately, by the circumstances of life.

In the less polemic part of his pamphlet, Isocrates develops his view of roles of teachers and students in the kind of training he has in mind (Isocrates 3 in the appendix). Two key ideas are noted: 1) Although students aim to develop their own capacity for production (in speech and in life), they must do so *thorough the acquisition of knowledge*, namely of the figures, which, combined in practice, provide the *means* to invent something. 2) Teachers should not content themselves with imparting knowledge for students to put into practice: they must also produce their own discourses, supplying students with a pattern to follow or surpass, a practice later epitomized in the crucial notion of **imitation**, *mimēsis*.

Isocratean ideas were incorporated among the many ideas and techniques that later produced the rhetorical tradition proper. Some of the ways these ideas and techniques were translated concretely into exercises (*gymnasmata*) students actually engaged in can be seen in a manual for teachers from about the first century C.E., the *Progymnasmata* of Aelius

[7] Namely Marinos of Neapolis' discourse *On Happiness*, presented on the first anniversary of Proclus's death. An excellent edition with commentary and French translation of Marinus' text is Saffrey (2002).

[8] This should not be taken as self-evident: many of the church fathers — for example, St. Jerome and St. Augustine — were superbly trained in rhetoric and the liberal arts in general and yet their writings are critical of those same arts (see Morrison, 1983).

Theon (see Kennedy, 2003).[9] Aelius Theon describes exercises that provide matter for actual practice such as: anecdote (*chreia*), narration (*diêgêsis*), common-place (*topos*), description (*ekphrasis*), personification (*prosôpopeia*), praise (*engkômion*), comparison (*synkrisis*), thesis (*thesis*) or laws (*nomoi*). Examples of exercises that were themselves practices and of what Theon thought students could gain by them can be seen in the following list:

Type of exercise	**Theon's comments** (excerpts)
• *anagnôsis* reading aloud (a piece of classical discourse)	• *"it is the nourishment of style; for we imitate most beautifully when our mind has been stamped by beautiful models"*
• *akroasis* hearing, listening (for the sake of learning by heart and re-writing)	• it provides us *"what has been created by the toil of others"*
• *paraphrasis* paraphrasing (putting in different words the same thoughts)	• this exercise is useful because *"thought is not moved by any one thing in only one way... but it is stirred in a number of different ways... "*
• *exergasia* elaboration	
• *antirrhêsis* contradiction	

Ideally, one should try some of these exercises oneself, as we intended participants of the workshop to do with *chreia*, had time permitted. But suffice it to say these exercises make concrete Theon's insistence that one read and re-read classical authors, turn their thoughts into different words, and, ultimately, *change* the thoughts. This recalls our discussions on the repetitive structure of the mathematical texts read earlier in the workshop — in Theon, the cognitive and intellectual value of such (apparently formal) exercises is recognized and encouraged explicitly. Even just reading aloud and discussing classical texts, as we have done during the workshop, are deemed important pedagogical exercises for their own sake.[10]

The name *Progymnasmata* refers to *preparatory exercises* to rhetoric proper; teachers' own skill in carrying out such exercises, however, and their own production in rhetoric was an essential aspect of rhetorical teaching. Like Isocrates, Theon regarded teachers' own works and those of other rhetors as models for imitation and sources for students' own invention, their own *heuresis*.[11] This was the content of the last reading from Theon's preface (Aelius Theon 1 in the appendix), and was intended to make clear that, with the central role of teachers' own production, that is, of their *own* learning, rhetorical education blurred the dividing line between teaching and learning.

[9]The complete text may be found in English in Kennedy's translation (Kennedy, 2003, significant parts of which can be read online on 'Google Books'). There is also an excellent edition cum French translation by M. Patillon in the Budé Collection (Patillon & Bolognesi, 1997).

[10]These exercises are also the subject of the ethical reflections contained Plutarch's insightful essay on "Listening to Lectures" (*Peri tou akouein*).

[11]The idea that teachers and their works should be foci of imitation has deep roots in the archaic Greek education. Teachers were mentors whose deeds and lives were to be emulated by the children in their charge: as it was with earlier authorities of the classical period, like Isocrates, they saw themselves inculcating a way of life

4 Modern Issues: Part III of the Workshop

These two parts set the stage for the final part of the workshop dedicated to the modern issues such as active learning, investigative activities, and communication and how they relate to classical humanism, to *paideia*. Rather than provide answers in this part of the workshop, we asked questions (in keeping with the entire spirit of the activity) to prompt participants own ideas. These questions were as follows:

- What light does the humanistic tradition shed on the question of active or student-centered learning?
- Does this tradition provide insight for math teachers considering their own teaching practices?
- How might this approach encourage non-trivial collaborations between teachers of maths and teachers of language, history, philosophy?
- What should lead teaching mathematics, form, argumentation, communication or explicit attention to content?
- Should mathematics be considered an integral part of general education? Or more, generally, should we be concerned with presenting a unified education?
- Is Euclid really so bad? What about Archimedes and Apollonius? What about Proclus?

That said, we did provide one concrete example as a focus to keep the discussion from becoming a free-for-all. The example, which referred to the first and second questions, was a piece written by Mary Burgan called "In Defense of Lecturing" (Burgan, 2006). As we mentioned earlier the rhetorical tradition balanced imitation and invention, or, one might say, balanced the role of the teacher with the activity of the student. Behind Burgan's position is the diminished, or at least unclear, role of the teacher in light of greatly emphasized student activity in modern education, especially in constructivist educational settings. The kind of view she questions is seen in this statement by Larry D. Spence (quoted by Burgan): "We won't meet the needs for more and better higher education until professors become designers of learning experiences and not teachers." Against this, Burgan argues, like the teachers in the humanist tradition, that teachers, by their own practice and production, are essential in providing students with a model for imitation. As she puts it, "...students benefit from seeing education embodied in a master learner who teaches what she has learned...", and, finally, "...lecturing should be defended because a narrow view of learning as mainly self-generated misses the fact that the vitality of the educational exchange in college often derives from the engagement of the student with a professor who is himself involved in a lifetime of discovery." We are not necessarily advocating Burgan's views, but we wish to emphasize here, as we have throughout this paper, that this modern debate echoes much more ancient issues and, therefore, can be informed by them. Although we did not have the time for the more lengthy conversation we envisioned, we were pleased to discover that what conversation we had continued after the workshop: nothing could have been a greater fulfillment of our ends.

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Appendix: Text Excerpts for part 3.2–3

Proclus 1

While these matters have been dealt with over and over again, there is something more refined about these [constructions], namely, that the equilateral triangle, which is equal on every side, is constructed in a unique way, whereas the isosceles, to which belongs equality for only two of its sides, is constructed in a double way; for the given straight line is either lesser than either of the two equal sides, as we have done, or is greater than the two. As for the scalene [triangle], since it is wholly unequal, it is constructed in a triple way; for the given [straight line] is either the greatest, or the least of the three, or is greater than one and lesser than the other. **And it is possible [for the reader] to train himself by adding or subtracting [conditions?] to each of the hypotheses**. As for us, we will contend ourselves with what has been presented. In generally then, we shall observe that among the problems some are solved in a unique way [*monachôs*], some in a multiple way [*pleonachôs*], and still others in an indefinite way [*apeirachôs*] [all emphases added] (*In Eucl.* (Friedlein) pp. 219–220)

Isocrates 1

Through [the power of speech = *logos*] we educate the ignorant and appraise the wise; for the power to speak well is taken as the surest index of a sound understanding, and discourse which is true and lawful and just is the outward image of a good and faithful soul. With this faculty we both contend against others on matters which are open to dispute and seek light for ourselves on things which are unknown; for the same arguments which we use in persuading others when we speak in public, we employ also when we deliberate in our own thoughts; and, while we call eloquent those who are able to speak before a crowd, we regard as sage those who most skillfully debate their problems in their own minds. (*Antidosis*, Norlin 255–256)

Isocrates 2

I marvel when I observe these men setting themselves up as instructors of youth who cannot see that they are applying the analogy of an art with hard and fast rules to **a creative process**. For, excepting these teachers, who does not know that the art of using letters remains fixed and unchanged, so that we continually and invariably use the same letters for the same purposes, while exactly the reverse is true of **the art of discourse**? For what has been said by one speaker is not equally useful for the speaker who comes after him; on the contrary, he is accounted **most skilled in this art** who speaks in a manner **worthy of his subject** and yet is **able to invent** [*heuriskein*] from it topics which are nowise the same as those used by others [all emphases added]. (*Against the Sophists*, 12 (Norlin, p. 170))

Isocrates 3

. . . for this, the student must not only have the requisite aptitude but he must learn the different kinds of discourse and practice himself in their use; and the teacher, for his part, must so expound the principles of the art with the utmost possible exactness as to leave out nothing that can be taught, and, **for the rest, he must in himself set such an example of oratory** [*paradeigma*] that the students who have taken form under his 5 instruction and are able to pattern [*mimêsasthai*] after him will, from the outset, show in their speaking a degree of grace and charm which is not found in others. [all emphases added] (*Against the Sophists*, 17–18)

Aelius Theon 1

Now I have included these remarks, not thinking that all are useful to all beginners, but in order that we may know that training in exercises is absolutely useful not only to those who are going to practice rhetoric but also if one wishes to undertake the function of poets or historians or any other writers. These things are, as it were, the foundation of every kind (*idea*) of discourse, and depending on how one instills them in the mind of the young, necessarily the results make themselves felt in the same way later. **Thus, in addition to what has been said, the teacher himself must compose some especially fine refutations and confirmations and assign them to the young to retell, in order that, molded by what they have learned, they may be able to imitate.** When the students are capable of writing, one should dictate to them the order of the headings and epicheiremes and point out the opportunity for digression and amplification and all other treatments, and one must make clear the moral character (*êthos*) inherent in the assignment (*problêma*) [all emphases added] (Kennedy, 2003, p. 13)